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Received January 7, 1992 (2884)
Revised version March 16, 1993

STUDIA MATHEMATICA 107 (1) (1993)

Factorization of Montel operators

by

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Abstract. Consider the following conditions. (a) Every regular LB-space is complete; (b) if an operator T between complete LB-spaces maps bounded sets into relatively compact sets, then T factorizes through a Montel LB-space; (c) for every complete LB-space E the space $C(\beta\mathbb{N}, E)$ is bornological. We show that (a)⇒(b)⇒(c). Moreover, we show that if E is Montel, then (c) holds. An example of an LB-space E with a strictly increasing transfinite sequence of its Mackey derivatives is given.

0. Introduction. In Banach space theory there is a famous result [11] (see also [13] and [15, Theorem 6.3.4]) that every weakly compact operator between Banach spaces factorizes through a reflexive Banach space. The ideal of operators mapping bounded sets into relatively (weakly) compact sets seems to be the proper analogue in the Fréchet setting of the ideal of (weakly) compact operators in the Banach case. This leads to the following factorization problem: Does every Montel operator (i.e., an operator mapping bounded sets into relatively compact sets) between Fréchet spaces factorize through a Fréchet–Montel space? Surprisingly enough, it seems that not much is known about it as well as about the dual problem concerning factorization of all Montel maps between (complete) LB-spaces through a Montel LB-space.

The best result which could be derived from the known facts (see Corollary 3.2 below) says that if E is a quasinormable Fréchet space, F is an arbitrary Fréchet space and $T : E \to F$ is a Montel map, then T factorizes through a Fréchet–Schwartz space.

1991 *Mathematics Subject Classification*: Primary 46A04, 46A13, 46A50, 46E40; Secondary 46A11, 46A17, 46A45, 47B07.

Key words and phrases: Fréchet space, Fréchet–Montel space, complete LB-space, Montel LB-space, regular LB-space, Mackey completion of an LB-space, bornologicity of $C(K, E)$.

The paper was written while the second author held the A. von Humboldt Fellowship at Bergische Universität Wuppertal. He is especially grateful for the additional support given by the Foundation to that research.

Since a positive solution to our factorization problem for complete LB-spaces would imply a positive solution to the Fréchet space counterpart (Proposition 3.7 below), we concentrate on the first question. We were unable to solve it but we discovered a surprising fact: a negative solution to our problem would solve in the negative the old and honourable question of Grothendieck [1, p. 78, Problem 1] of whether every regular LB-space is complete, while a positive solution would solve in the affirmative a more recent open problem [20, Ch. IV] of whether $C(\beta\mathbb{N}, E)$ is bornological for complete LB-spaces ($\beta\mathbb{N}$ denotes the Čech–Stone compactification of the natural numbers). In the course of our study, we get plenty of new information on the structure of compact sets in complete LB-spaces, which in turn allows us to construct probably the first example of an LB-space E such that all its Mackey derivatives of countable order are different. We define the *Mackey derivative* $E^{(1)}$ of E to be the set of all local limits in the completion $\widetilde{E}$ of local Cauchy sequences in E; inductively, we define the Mackey derivatives of higher orders:

$$E^{(\alpha+1)} := (E^{(\alpha)})^{(1)} \quad \text{and} \quad E^{(\beta)} := \bigcup_{\alpha<\beta} E^{(\alpha)} \quad \text{for limit ordinals } \beta\,.$$

As a second by-product, we show that if E is a Montel LB-space, then $C(\beta\mathbb{N}, E)$ is bornological.

We hope that the presented results will convince the reader that the factorization problems (the one mentioned above and their "relatives") are essential in the study of Fréchet and LB-spaces.

1. Preliminaries. Let E be an lcs. Then $\mathcal{K}(E)$ denotes the family of all compact absolutely convex sets in E. If B is an absolutely convex bounded set, then E_B denotes $\operatorname{lin} B$ equipped with the gauge functional of B as a norm. By a *bornivorous set*, we always mean an absolutely convex set absorbing all bounded sets. A sequence (x_n) in E is called *local Cauchy* (*locally convergent to* x) if there is an absolutely convex closed bounded set B in E such that (x_n) is a Cauchy sequence (resp. is convergent to x) in E_B. We call a subspace $F \subset E$ *Mackey closed* if it contains all local limits of sequences in F. By the *Mackey completion* $\breve{E}$ of E we define the smallest Mackey closed space F contained in the completion of E and containing E. The space E is called *Mackey complete* (or sometimes *locally complete*, see [1] or [16]) if $E = \breve{E}$. It is clear that $E^{(\Omega)} = \breve{E}$ (see the definition of $E^{(\beta)}$ in the introduction), where Ω denotes the first uncountable ordinal. By E'_{b} and E'_{i} we denote the strong and the inductive dual, resp. The latter is the dual E' equipped with the bornological topology associated with the equicontinuous bornology on E'.

If (B_j) is a sequence of sets containing 0, we define

$$\sum_{j\in\mathbb{N}} B_j := \Big\{x : \exists n \in \mathbb{N},\ x \in \sum_{j\le n} B_j\Big\}\,.$$

Moreover, if $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$, then $\mathcal{S}(E)$ denotes the family of all absolutely convex sets which are compact in some step spaces of E. A space is called *countable quasibarrelled* if every strongly bounded sequence in the dual is equicontinuous. For the definition and basic properties of coechelon and echelon spaces, see [16, 30.8], [1]. We will use the notation $k_\infty(a_{in})$ for the coechelon space of type ∞ defined by the matrix $(a_{in})_{i,n\in\mathbb{N}}$. LB-spaces are always assumed to be separated. For other undefined notions from functional analysis see [16] and [14].

If K is a topological completely regular Hausdorff space, then βK denotes its Čech–Stone compactification. If A, B are sets, then A^B denotes the family of *all* functions $f : B \to A$.

We will need three known results on LB-spaces. The first is due to Mujica [18] (see also [1, Theorem 3.12]).

THEOREM 1.1. *Let $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$ and let E_n be Banach spaces. If there exists a Hausdorff locally convex topology τ on E such that the unit ball in each E_n is τ-compact, then E is complete. In fact, there is a Fréchet space Y such that $E \simeq Y'_{\mathrm{i}}$.*

The second result is due to Pfister [19]; it was then improved by Cascales and Orihuela [9], [10] (see also [21]).

THEOREM 1.2. *If E is a DF-space or an LF-space, then every precompact set in E is metrizable.*

Remark. As easily seen, if (A_n) is a sequence of absolutely convex precompact sets in E as above, then $E_0 := \operatorname{lin}\{A_n : n \in \mathbb{N}\}$ has a weaker metrizable topology. Indeed, for every n there is a sequence $(U_{mn})_{m\in\mathbb{N}}$ of 0-neighbourhoods in E such that $(U_{mn} \cap A_n)_{m\in\mathbb{N}}$ forms a 0-neighbourhood basis in A_n. The sequence $(U_{mn} \cap E_0)_{m,n\in\mathbb{N}}$ gives a 0-neighbourhood basis for the required metric topology on E_0.

We finish this section by a useful "density condition" result proved implicitly in [2] and [3]:

THEOREM 1.3. *Let $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$ be an LB-space and let B be a metrizable bounded set in E. Then for every sequence (a_j) of positive real numbers there is an $m \in \mathbb{N}$ such that $\overline{\sum_{j\le m} a_j B_j}$ contains a 0-neighbourhood in B, where B_n is the unit ball in E_n.*

2. Montel operators. In order to study the factorization problem, we must first study Montel operators. We call an operator $T : E \to F$

between lcs a *bpc-operator* if it maps every bounded set into a precompact set. Obviously, every Montel operator is bpc.

PROPOSITION 2.1. *Let E, F be lcs.*

(a) *An operator $T : E \to F$ is a bpc-operator iff T maps bounded sequences into precompact ones.*

(b) *Let F be a countably quasibarrelled space. An operator $T : E \to F$ is a bpc-operator iff $T' : F'_{\mathrm{b}} \to E'_{\mathrm{b}}$ is bpc as well.*

(c) *Let F be a metrizable space and let E satisfy one of the following conditions: (i) E has a fundamental sequence of bounded sets; (ii) $E = \mathrm{ind}_{n\in\mathbb{N}} E_n$, where the E_n are metrizable spaces; (iii) $E = (\mathrm{ind}_{n\in\mathbb{N}} E_n)'_{\mathrm{b}}$, where the E_n are metrizable spaces. Then every bpc-operator $T : E \to F$ has a separable range.*

Remark. Part (c) almost immediately implies the assertion of Theorem 1.2, and that proof seems to be more elementary than the original one [19].

Proof. (a) Obvious.

(b) Let B be an absolutely convex bounded subset in E. If T is a bpc-operator, then $T(B)$ is precompact in F. This means that

$$T'|_{U^\circ} : (U^\circ, \sigma(F', F)|_{U^\circ}) \to E'_{\mathrm{b}}$$

is continuous for any 0-neighbourhood U in E [14, 8.5.1]. In particular, T' maps all equicontinuous sequences into relatively compact ones. By (a), if F is countably quasibarrelled, then $T' : F'_{\mathrm{b}} \to E'_{\mathrm{b}}$ is a bpc-operator.

Let $T' : F'_{\mathrm{b}} \to E'_{\mathrm{b}}$ be a bpc-operator. Then as above we show that $T'' : (E'_{\mathrm{b}})'_{\mathrm{b}} \to (F'_{\mathrm{b}})'_{\mathrm{b}}$ maps equicontinuous sets into relatively compact ones. Since every bounded set B in E is equicontinuous in $(E'_{\mathrm{b}})'_{\mathrm{b}}$, the set $T''(B) \subseteq F$ is relatively compact in $(F'_{\mathrm{b}})'_{\mathrm{b}}$, and thus also relatively compact in the (weaker) topology of uniform convergence on equicontinuous subsets of F'_{b}. The latter topology induces on F the original topology and therefore T is a bpc-operator.

(c) If F is metrizable and $T(E)$ is not separable, then there exists a 0-neighbourhood U in F and an uncountable family $(x_i)_{i\in I}$ in F such that $T(x_i) - T(x_j) \notin U$ for $i, j \in I$, $i \neq j$. We achieve a contradiction by proving in all cases that $(x_i)_{i\in I}$ contains an infinite bounded subset.

This is trivial in case (i). If (ii) is satisfied, then, without loss of generality, we may assume that $(x_i)_{i\in I}$ is contained in some E_n. Let (U_m) be a countable 0-neighbourhood basis in E_n. Then there is a decreasing sequence of uncountable sets (I_m), $I_m \subseteq I$, and a sequence of finite constants b_m such that

$$x_i \in b_m U_m \quad \text{for all } i \in I_m\,.$$

Now, we find an infinite sequence of pairwise different elements (i_m) such that $i_m \in I_m$. Obviously, $(x_{i_m})_{m\in\mathbb{N}}$ is the sequence we are looking for.

If (iii) holds, then for the restriction maps $r_n : E \to (E_n)'_{\mathrm{b}}$ we can find a decreasing sequence of uncountable sets (I_n), $I_n \subseteq I$, such that $\{r_n(x_i) : i \in I_n\}$ is bounded in $(E_n)'_{\mathrm{b}}$. Taking a sequence of pairwise different elements (i_n), $i_n \in I_n$, we find that $(x_{i_n})_{n\in\mathbb{N}}$ is bounded in $\mathrm{proj}_{n\in\mathbb{N}}(E_n)'_{\mathrm{b}}$. As all E_n are quasibarrelled the identity map $\mathrm{proj}_{n\in\mathbb{N}}(E_n)'_{\mathrm{b}} \to E$ maps bounded sets into bounded sets.

Now, we list consequences for Montel maps.

COROLLARY 2.2. *If either both E and F are Fréchet spaces or both are LB-spaces, then the range of every Montel operator $T : E \to F$ is separable and submetrizable.*

Proof. Separability follows from 2.1(c) and 1.2. If E and F are LB-spaces, then the submetrizability follows from the remark after Theorem 1.2.

COROLLARY 2.3. *Let E and F be Fréchet spaces and let $T : E \to F$ be an operator. Then the following assertions are equivalent:*

(a) *T is a Montel map;*

(b) *$T' : F'_{\mathrm{b}} \to E'_{\mathrm{b}}$ is a Montel map;*

(c) *$T' : F'_{\mathrm{i}} \to E'_{\mathrm{i}}$ is a Montel map.*

Proof. By Proposition 2.1(b), it is enough to show (b)$\Leftrightarrow$(c). Since the topology of E'_{i} is stronger than that of E'_{b} and since F'_{b} and F'_{i} have the same bounded sets, we have (c)$\Rightarrow$(b). On the other hand, since every compact set in E'_{b} is separable and metrizable (Theorem 1.2), the topology of E'_{i} coincides on these sets with $\beta(E', E)$, by [16, 29.3(8)]. This completes the proof.

COROLLARY 2.4. *Let E and F be complete LB-spaces and let $T : E \to F$ be an operator. Then the following assertions are equivalent:*

(a) *T is a Montel map;*

(b) *$T' : F'_{\mathrm{b}} \to E'_{\mathrm{b}}$ is a Montel map.*

3. Factorizable operators. Let us first observe that under some additional assumptions Montel operators factorize through Montel spaces of a suitable type. The first part of 3.1 is due to Grothendieck.

PROPOSITION 3.1. *Let E be a quasinormable lcs and let F be a Banach space. Then every Montel operator $T : E \to F$ is compact and, in particular, it factorizes through a Fréchet–Schwartz space.*

Proof. By quasinormability of E, there exists a 0-neighbourhood V in E such that for every $\varepsilon > 0$ there exists a bounded subset B of E satisfying

$V \subseteq B+\varepsilon T^{-1}(B_F)$, where B_F is the unit ball in F. Thus $T(V) \subseteq T(B)+\varepsilon B_F$ and $T(V)$ is relatively compact in F.

Now, since T is compact, it factorizes through a compact map S between Banach spaces. By [15, 6.3.10], it follows easily that S factorizes through a Fréchet–Schwartz space.

By embedding the space F as a closed subspace in a product of Banach spaces, we obtain easily:

COROLLARY 3.2. *Let E be a quasinormable Fréchet space and let F be a Fréchet space. Then every Montel map $T : E \to F$ factorizes through a Fréchet–Schwartz space.*

COROLLARY 3.3. *Let E be an LB-space and let F be a complete LB-space. Then every Montel operator $T : E \to F$ factorizes through a complete semi-Montel lcs.*

There also exists a Fréchet analogue of Corollary 3.3.

PROPOSITION 3.4. *If E, F are Fréchet spaces and $T : E \to F$ is a Montel map, then T factorizes through a semi-Montel space.*

Proof. The same reason as in 3.2 allows us to restrict ourselves to Banach spaces F. For G an lcs, denote by $\tau_k(G)$ the topology in G' of uniform convergence on compact sets in G. Now, by Corollary 2.3 and Proposition 3.1, $T' : F'_b \to E'_b$ is compact and it is continuous as a map $T' : (F', \tau_k(F)) \to E'_b$. Thus $T'' : (E'_b)'_b \to F = (F', \tau_k(F))'$. On the other hand, by Theorem 1.2 and [16, 29.3(8)], every compact set in E'_b is also compact in E'_i. Therefore $T' : F'_b \to E'_i$ is compact. Thus T'' extends to a continuous map $T_1 : ((E'_i)', \tau_k(E'_i)) \to (F'_b)'_b$. Since $(E'_b)'$ is dense in the domain of T_1 and F is closed in $(F'_b)'_b$, we have $\operatorname{Im} T_1 \subseteq F$. Every $\tau_k(E'_i)$-bounded set in $(E'_i)'$ is bounded in $(E'_i)'_b$ and thus it is also equicontinuous. On equicontinuous sets in $(E'_i)'$, the topology $\tau_k(E'_i)$ coincides with $\sigma((E'_i)', E'_i)$, and $((E'_i)', \tau_k(E'_i))$ is a semi-Montel space. This completes the proof because T factorizes through T_1.

We are interested in a much stronger question if every Montel map between Fréchet (complete LB-) spaces factorizes through a Fréchet–Montel (Montel LB-) space. First we reduce the problem a little.

PROPOSITION 3.5. *The following assertions are equivalent*:

(a) *Every Montel operator between Fréchet spaces factorizes through a Fréchet–Montel space*;

(b) *Every Montel operator $T : E \to F$, where E is a Fréchet space and F is a separable Banach space, factorizes through a Fréchet–Montel space.*

Proof. It is enough to show (b)⇒(a). Let $T : E \to G$ be a Montel map, where E and G are arbitrary Fréchet spaces. By Corollary 2.2, the closure H of the range of T is a separable Fréchet space. Obviously, H can be embedded in the product of a sequence of separable Banach spaces H_n; we define $i_n : H \to H_n$ to be the respective projection. By (b), the map $i_n \circ T$ factorizes through a Fréchet–Montel space G_n and $\prod_{n\in\mathbb{N}} i_n \circ T : E \to \prod_{n\in\mathbb{N}} H_n$ factorizes through the space $\prod_{n\in\mathbb{N}} G_n$ and a map $U : \prod_{n\in\mathbb{N}} G_n \to \prod_{n\in\mathbb{N}} H_n$. Now, $T : E \to H$ factorizes through the Fréchet–Montel space $U^{-1}(H)$.

PROPOSITION 3.6. *The following assertions are equivalent*:

(a) *Every Montel operator $T : E \to F$, where E and F are LB-spaces, factorizes through a Montel LB-space.*

(b) *Every Montel operator $T : E \to F$, where E is a Banach space and F is an LB-space, factorizes through a Montel LB-space.*

A similar equivalence holds whenever we assume in (a) *and* (b) *that F is a complete LB-space.*

Proof. It is enough to show (b)⇒(a). Let $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$, where the E_n are Banach spaces. Since $T_n := T|_{E_n}$ is a Montel operator, it factorizes through a Montel LB-space H_n. Thus $\bigoplus T_n : \bigoplus E_n \to F$ factorizes through the Montel LB-space $\bigoplus H_n$. Obviously, $E = \bigoplus E_n/Y$ for some closed subspace Y of $\bigoplus E_n$ and T factorizes through $\bigoplus H_n/\overline{(\bigoplus T_n)(Y)}$, which is a Montel LB-space as a quotient of a Montel LB-space (it is complete by Theorem 1.1, and thus all its bounded sets can be lifted by the Grothendieck Factorization Theorem).

Now, we explain the relation between the factorization problems for Fréchet and LB-spaces.

PROPOSITION 3.7. *Let E and F be Fréchet spaces and let $T : E \to F$ be an operator. Then the following assertions are equivalent*:

(a) *T factorizes through a Fréchet–Montel space*;
(b) *$T' : F'_b \to E'_b$ factorizes through a Montel LB-space*;
(c) *$T' : F'_i \to E'_i$ factorizes through a Montel LB-space.*

Proof. We have (a)⇒(b)⇒(c), by Corollary 2.3.

(c)⇒(a). If $T' : F'_i \to E'_i$ factorizes through a Montel LB-space G, then $T'' : (E'_i)'_b \to (F'_i)'_b$ factorizes through the Fréchet–Montel space G'_b and an operator $U : G'_b \to (F'_i)'_b$. Obviously, $T''|_E = T$ and T factorizes through the Fréchet–Montel space $U^{-1}(F)$.

Unfortunately, in the dual situation we only have

PROPOSITION 3.8. *If E and F are LB-spaces and $T : E \to F$ is an operator factorizing through a Montel LB-space, then $T' : F'_b \to E'_b$ factorizes through a Fréchet–Montel space.*

Remark. We do not know if the converse to the implication in 3.8 holds; if the factorization problem for LB-spaces has a positive solution, then, by Corollary 2.4, the solution to the above problem is also in the affirmative.

The above results (Corollary 2.3 and Proposition 3.7) imply immediately that a positive solution to the factorization problem for complete LB-spaces gives a positive solution to the corresponding Fréchet space problem. We do not know if the converse holds as well.

4. Various classes of compact sets. As was explained in the previous section, in order to solve the "Montel factorization problem" for complete LB-spaces or Fréchet spaces it is essential to study absolutely convex compact subsets in LB- or dual-LB-spaces. This is the aim of the present section.

Let us introduce first some definition. We call a subset $K_0 \subseteq E$ *factorizable* if there is a $K \in \mathcal{K}(E)$ containing K_0 such that the embedding $i_K : E_K \to E$ factorizes through a Montel LB-space. By 3.6 and 3.7, it is clear that both factorization problems would have positive solutions whenever every absolutely convex compact set in a complete LB-space were factorizable. We denote the family of all absolutely convex factorizable subsets of E by $\mathcal{K}_{\mathrm{f}}(E)$.

Let $\mathcal{L}$ be any family of closed absolutely convex sets in an LB-space E. Then we define inductively an increasing transfinite scale of families of absolutely convex sets. First, $\mathcal{L}^{(1)}$ denotes the family of all closed absolutely convex subsets of E such that there exists a closed bounded set $B \subseteq E$ such that for every $\varepsilon > 0$ there exists $L \in \mathcal{L}$ satisfying

$$K \subseteq \varepsilon B + L .$$

If α is an arbitrary ordinal number, then

$$\mathcal{L}^{(\alpha+1)} := (\mathcal{L}^{(\alpha)})^{(1)} ,$$

and if β is a limit ordinal, then

$$\mathcal{L}^{(\beta)} := \bigcup_{\alpha<\beta} \mathcal{L}^{(\alpha)} .$$

LEMMA 4.1. *If $\mathcal{L}$ consists of precompact sets, then the same holds for $\mathcal{L}^{(\alpha)}$. Hence, if E is complete and $\mathcal{L}$ consists of compact sets, then $\mathcal{L}^{(\alpha)}$ also consists of compact sets.*

Proof. Obvious.

It is clear from the above definition that whenever $\mathcal{L}$ is closed under taking closed absolutely convex subsets, then $\mathcal{L}^{(1)} \supseteq \mathcal{L}$. Similarly, it is easily seen that the procedure of creating $\mathcal{L}^{(\alpha)}$ stops at most at the level $\mathcal{L}^{(\Omega)}$, where Ω is the first uncountable ordinal. We call a family $\mathcal{L}$ *stable* whenever it is closed under taking finite sums and absolutely convex closed subsets.

For our purpose the scale $\mathcal{S}^{(\alpha)}(E)$ will play a central role. For the sake of convenience we define $\mathcal{S}^{(0)}(E) := \mathcal{S}(E)$ and $\mathcal{S}^{(-1)}(E)$ to be the family of all finite-dimensional absolutely convex compact sets. If E is an LB-space, then $(\mathcal{S}^{(-1)}(E))^{(1)} = \mathcal{S}^{(0)}(E)$. By an easy inductive procedure, we can prove:

PROPOSITION 4.2. *If E, F are LB-spaces, F is complete and $T : E \to F$ is an operator, then for every ordinal α,*

$$T(\mathcal{S}^{(\alpha)}(E)) \subseteq \mathcal{S}^{(\alpha)}(F) \quad \text{and} \quad T(\mathcal{K}_{\mathrm{f}}(E)) \subseteq \mathcal{K}_{\mathrm{f}}(F) .$$

We now present the main result:

THEOREM 4.3. *Let E be a complete LB-space. For every ordinal α, $\mathcal{S}^{(\alpha)}(E)$ consists only of factorizable sets, i.e., $\mathcal{S}^{(\alpha)}(E) \subseteq \mathcal{K}_{\mathrm{f}}(E)$.*

The proof is based on the following lemmas.

LEMMA 4.4. *Let $\mathcal{L}$ be a family of closed absolutely convex compact sets in an LB-space E. Suppose that for every $K \in \mathcal{L}$ there is an increasing sequence of sets $K_n \in \mathcal{L}$ for $n \in \mathbb{N}$ such that K is compact (equivalently, precompact) in $\operatorname{ind}_{n\in\mathbb{N}} E_{K_n}$. Then $\mathcal{L} \subseteq \mathcal{K}_{\mathrm{f}}(E)$.*

Proof. This is an easy diagonal argument: we define inductively $(K_{ij}) \subseteq \mathcal{L}$ such that K is compact in $\operatorname{ind}_{j\ge 0} E_{K_{ij}}$ and

$$K \subseteq K_{00} \subseteq K_{01} \subseteq K_{02} \subseteq \ldots$$

Now, if (K_{ij}) is defined for $i < l$, then we find an increasing sequence $(K_{lj})_{j\ge l}$ such that $K_{l-1,l}$ is (pre-)compact in $\operatorname{ind}_{j\ge l} E_{K_{lj}}$ and

$$K_{l-1,j} \subseteq K_{l,j} \quad \text{for } j \ge l .$$

By the Mujica Theorem 1.1, $F := \operatorname{ind}_{i\ge 0} E_{K_{i,i}}$ is a complete space and every bounded set in it is contained in some multiple of some $K_{i,i}$. On the other hand, the construction implies that $K_{i,i}$ and K are compact in F. This completes the proof, since the embedding $i_K : E_K \to E$ factorizes through F.

LEMMA 4.5. *Let $\mathcal{L}$ be a stable family of absolutely convex compact sets in an LB-space E. Then the following assertions are equivalent:*

(a) $K \in \mathcal{L}^{(1)}$;

(b) *There exists a sequence $(K_n) \subseteq \mathcal{L}$, a vector (a_n) in l_1 and a closed absolutely convex bounded set B in E such that*

$$K \subseteq \overline{\sum_{n\in\mathbb{N}} a_n (B \cap K_n)}^{E_B} ;$$

(c) *There exists a sequence $(K_n) \subseteq \mathcal{L}$, a vector (a_n) in l_1 and a closed* *absolutely convex bounded set B in E such that*

$$K \subseteq \overline{\sum_{n\in\mathbb{N}} a_n(B\cap K_n)}^E ;$$

(d) *There exists a set $C \in \mathcal{L}^{(1)}$ such that for every $\varepsilon > 0$ there is a set $K_\varepsilon \in \mathcal{L}$ satisfying*

$$K \subseteq \varepsilon C + K_\varepsilon .$$

Proof. (a)$\Rightarrow$(b). Assume that

$$K \subseteq \bigcap_{n\in\mathbb{N}} (1/2^n)B + K_n$$

for some closed absolutely convex bounded set B in E and $K_n \in \mathcal{L}$. Then every $x \in K$ splits as

$$x = x_n + y_n, \quad y_n \in (1/2^n)B,\ x_n \in K_n .$$

Putting $x_0 := 0$, $K_0 := \{0\}$ we obtain

$$x - \sum_{j=1}^{n}(x_j - x_{j-1}) = y_n \to 0 \quad \text{in } E_B \text{ as } n\to\infty .$$

On the other hand,

$$x_j - x_{j-1} \in (K_j + K_{j-1}) \cap (1/2^j + 1/2^{j-1})B ,$$

and this completes the proof for $a_j := 1/2^j + 1/2^{j-1}$.

(b)$\Rightarrow$(c). Obvious.

(c)$\Rightarrow$(d). Let $(K_n) \subseteq \mathcal{L}$ and $(a_n) \in l_1$ satisfy

$$K \subseteq \overline{\sum_{n\in\mathbb{N}} a_n(B\cap K_n)}^E$$

for some closed absolutely convex bounded set B. Let $(b_n) \in l_1$ be chosen in such a way that $0 \le b_n \le 1$ for $n \in \mathbb{N}$ and $a_n/b_n \to 0$ as $n \to \infty$. We define

$$C := \overline{\sum_{n\in\mathbb{N}} b_n(B\cap K_n)}^E .$$

For each $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that $\sum_{n>n_\varepsilon} b_n \le \varepsilon$; hence (since K_n are compact in E)

$$C \subseteq \overline{\sum_{n=1}^{n_\varepsilon} b_n(B\cap K_n) + \varepsilon B}^E \subseteq \sum_{n=1}^{n_\varepsilon} b_n(B\cap K_n) + \varepsilon B$$

and $C \in \mathcal{L}^{(1)}$. Moreover, for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that

$$a_n/b_n < \varepsilon \quad \text{for } n > m_\varepsilon .$$

Thus, since the K_n are compact in E,

$$K \subseteq \overline{\sum_{n\in\mathbb{N}} a_n(B\cap K_n)}^E \subseteq \sum_{n\le m_\varepsilon} a_n(B\cap K_n) + \overline{\sum_{n>m_\varepsilon} (a_n/b_n)b_n(B\cap K_n)}^E$$

and the second summand on the right hand side is contained in εC.

(d)$\Rightarrow$(a). Obvious.

LEMMA 4.6. *If $\mathcal{L}$ is a family of closed absolutely convex compact sets in a complete LB-space E satisfying the assumptions of Lemma 4.4, then $\mathcal{L}^{(1)}$ also satisfies the same assumptions and, in particular, $\mathcal{L}^{(1)} \subseteq \mathcal{K}_f(E)$.*

Proof. Let $K \in \mathcal{L}^{(1)}$. Then, by 4.5, there exists $C \in \mathcal{L}^{(1)}$ such that

$$K \subseteq \bigcap_{n\in\mathbb{N}} ((1/2^n)C + K_n) ,$$

where $K_n \in \mathcal{L}$ for $n \in \mathbb{N}$. By the assumption and using a diagonal process, we can find a sequence (L_n) in $\mathcal{L}$ such that, for every $n \in \mathbb{N}$, K_n is compact in $\operatorname{ind}_{n\in\mathbb{N}} E_{L_n}$. We will show that K is precompact in $\operatorname{ind}_{n\in\mathbb{N}} E_{C_n}$, where

$$C_n := C + L_n \in \mathcal{L}^{(1)} .$$

Let (a_j) be any sequence of positive numbers. Choose $n \in \mathbb{N}$ such that $1/2^n < \sum_{j\in\mathbb{N}} a_j$. Then

$$K \subseteq (1/2^n)C + K_n$$

and, by compactness of K_n in $\operatorname{ind}_{n\in\mathbb{N}} E_{L_n}$, there is a finite set $A \subseteq K_n$ such that

$$K_n \subseteq A + \sum_{j\in\mathbb{N}} a_j L_j .$$

Consequently,

$$K \subseteq A + \sum_{j\in\mathbb{N}} a_j C + \sum_{j\in\mathbb{N}} a_j L_j \subseteq A + \sum_{j\in\mathbb{N}} a_j C_j .$$

Since E is complete, C_n is compact and, by Theorem 1.1, $\operatorname{ind}_{n\in\mathbb{N}} E_{C_n}$ is complete. Therefore, K is compact.

The last lemma implies immediately:

COROLLARY 4.7. *If E is a complete LB-space, then $\mathcal{K}_f^{(1)}(E) = \mathcal{K}_f(E)$.*

Proof of Theorem 4.3. By Lemma 4.6, it is enough to observe that for every $K \in \mathcal{S}^{(-1)}(E)$, the map $i_K : E_K \to E$ factorizes through a finite-dimensional Banach (Montel!) space.

Summarizing, for all complete LB-spaces E we have

$$\mathcal{S}^{(-1)}(E) \subseteq \mathcal{S}(E) \subseteq \mathcal{S}^{(1)}(E) \subseteq \ldots$$
$$\subseteq \mathcal{S}^{(\alpha)}(E) \subseteq \mathcal{S}^{(\alpha+1)}(E) \subseteq \ldots \subseteq \mathcal{S}^{(\Omega)}(E) \subseteq \mathcal{K}_f(E) \subseteq \mathcal{K}(E) .$$

The factorization problem for complete LB-spaces is equivalent to the question of whether

$$\mathcal{K}_{\mathrm{f}}(E) = \mathcal{K}(E)$$

always holds. The authors do not know if

$$\mathcal{S}^{(\Omega)}(E) = \mathcal{K}(E) \quad \text{or even} \quad \mathcal{S}^{(\Omega)}(E) = \mathcal{K}_{\mathrm{f}}(E)\,.$$

We will show in the next section that all the other inclusions can be strict.

5. Examples. The first result contains the most important example.

THEOREM 5.1. *For every countable ordinal α there is a Montel coechelon LB-space E_α of type ∞ such that every absolutely convex compact set in E_α belongs to $\mathcal{S}^{(\alpha+1)}(E_\alpha)$. Moreover, let $B := \{(x_n) : |x_n| \le 1\}$; then every bounded set C such that $C + aB \supseteq B$ for some $a < 1$ satisfies $C \notin \mathcal{S}^{(\alpha)}(E)$.*

COROLLARY 5.2. *For every countable ordinal α there is a Montel LB-space F_α such that $\mathcal{K}(F_\alpha) \subseteq \mathcal{S}^{(\alpha)}(F_\alpha)$ but $\mathcal{K}(F_\alpha) \not\subseteq \mathcal{S}^{(\beta)}(F_\alpha)$ for any $\beta < \alpha$.*

Proof. For successors α the result follows directly from Theorem 5.1. Now, let α be a limit ordinal and let $\alpha_i \nearrow \alpha$. We define $F_\alpha := \bigoplus_{i\in\mathbb{N}} E_{\alpha_i}$, where the E_{α_i} are defined according to Theorem 5.1.

COROLLARY 5.3. *There exists a complete LB-space E such that for every countable ordinal α, there exists an absolutely convex compact set C_α in E such that $C_\alpha \notin \mathcal{S}^{(\alpha)}(E)$ but $C_\alpha \in \mathcal{S}^{(\Omega)}(E)$.*

Remark. Obviously, it is impossible to find a *Montel* LB-space with such properties.

Proof. Denote by $E_{n,\alpha}$ the nth step in the space E_α constructed in Theorem 5.1. Now, we define a Banach space E_n to be the l_1 direct sum

$$E_n := \Big(\bigoplus_{\alpha<\Omega} E_{n,\alpha}\Big)_{l_1}$$

and set $E := \operatorname{ind}_{n\in\mathbb{N}} E_n$. The required property of E follows from the fact that E contains all E_α as complemented subspaces (use Proposition 4.2).

Proof of Theorem 5.1. Set $E_{-1} := s'_{\mathrm{b}}$, where s is the space of rapidly decreasing sequences. Then E_{-1} satisfies the corresponding condition for the family $\mathcal{S}^{(-1)}(E)$ of finite-dimensional compact sets. Indeed, by the Riesz Lemma [12, Lemma, p. 2], for any $a < 1$ and any finite-dimensional subspace Y in l_∞ there exists a vector x in the unit ball of l_∞ such that $d(Y,x) > a$ but all compact sets are in $\mathcal{S}^{(0)}$.

Now, let $\alpha_i^+ = \alpha$ if α is nonlimit, and let (α_i) be a strictly increasing sequence of nonlimit ordinals with $\alpha_i \nearrow \alpha$ if α is a limit ordinal. Let

$$E_{\alpha_i} := k_\infty(a^{(i)}_{j,n})\,,$$

and define a coechelon space $E = k_\infty(a_{ij,n})$ of double sequences by

$$a_{ij,n} := \begin{cases} i^{-n} & \text{for } i \ge n,\\ a^{(i)}_{j,n} & \text{for } i < n. \end{cases}$$

Assume that

$$C \subseteq \bigcap_{b>0} bB_p + K_b\,,$$

where $K_b \in \mathcal{S}^{(\beta)}(E)$ for some fixed $\beta < \alpha$ and B_p is the unit ball in the pth step space of E. In particular, for $b := \frac{1}{2} r^{-p}(1-a)$, $r > p$, $\alpha_r \ge \beta$, we have

$$B \subseteq aB + bB_p + K_b\,.$$

Let $P : E \to E_{\alpha_r}$ be the projection, $P((x_{ij})) := (x_{rj})$. Then

$$P(B) \subseteq cP(B) + P(K_b)\,,$$

where $c = a + (1-a)/2 < 1$. Since $P(B)$ is of the form B but in E_{α_r}, it follows by the inductive hypothesis that $P(K_b)$ is not of the form $\mathcal{S}^{(\alpha_r)}(E_{\alpha_r})$; a contradiction by Proposition 4.2.

Moreover, $B_p \subseteq (1/n)B_{p+1} + K_n$, where K_n is a bounded set in $\bigoplus_{i\le n} E_{\alpha_i}$ and, by the inductive hypothesis, compact of type $\mathcal{S}^{(\alpha)}$.

Remarks. (a) It is clear from the proof that E_0 is the dual of the famous example of a Montel non-Schwartz Fréchet space due to Grothendieck and Köthe [16, 31.5].

(b) All absolutely convex compact sets in Moscatelli type LB-spaces are of type $\mathcal{S}^{(1)}$ but modifying the Moscatelli construction (for the original construction see [6], [17], and also [5]) we can also obtain examples of the type given in Theorem 5.1.

(c) The authors have recently proved that $\mathcal{K} = \mathcal{S}^{(\Omega)}$ always holds in coechelon spaces of type ∞ [22].

6. Relation to the classical problems. The aim of this section is to show the following main result:

THEOREM 6.1. (a) *Let E be an LB-space (equivalently, a Banach space) and let F be a complete LB-space, $F = \operatorname{ind}_{n\in\mathbb{N}} F_n$. If there exists a Montel operator $T : E \to F$ which does not factorize through a Montel LB-space, then the Mackey completion of $\operatorname{ind}_{n\in\mathbb{N}} C(\beta\mathbb{N}, F_n)$ is a Mackey complete (i.e., regular) noncomplete LB-space.*

(b) *Let F be a complete LB-space. If for every LB-space (equivalently, Banach space) E, every Montel map $T : E \to F$ factorizes through a Montel LB-space, then $C(\beta\mathbb{N}, F)$ is bornological.*

(c) *There exists an LB-space $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$ such that the space $G := \operatorname{ind}_{n\in\mathbb{N}} C(\beta\mathbb{N}, E_n)$ has a strictly increasing transfinite sequence of Mackey derivatives $G^{(\alpha)}$ of countable order.*

Remarks. (a) It is an open problem posed by Grothendieck [1] if every regular LF-space (or LB-space) is necessarily complete.

(b) The problem of under what conditions $C(K,F)$ is bornological for a completely regular Hausdorff topological space K and an LB-space F was studied in [4], [7], [8] and [20]. Up to now the problem remains open even for simple compact spaces K like $\beta\mathbb{N}$ or the Alexandrov compactification of the natural numbers (i.e., for $c_0(F)$). The solution has been known only for compactly regular LB-spaces F (i.e., when $\mathcal{K}(F)=\mathcal{S}^{(0)}(F)$). As we know from the previous section, this case is far from being general. The proof below implies a positive solution for $K=\beta\mathbb{N}$ and complete LB-spaces F satisfying $\mathcal{S}^{(\Omega)}(F)=\mathcal{K}(F)$. Quite recently L. Frerick and S. Dierolf solved the problem for Montel DF-spaces F. For some new surprising connections to other problems see [23].

(c) It seems that up to now, no example has been known of an LB-space G for which $G^{(1)}$ is not Mackey complete (as is the case in Theorem 6.1(c)).

(d) In Theorem 6.1 we can substitute $\beta\mathbb{N}$ by βI for any infinite discrete set I.

(e) If F is a Fréchet space, then the space $M(F,C(K))$ of Montel operators with the strong topology (K is an arbitrary compact set) is topologically isomorphic to $C(K,F'_{\rm b})$. Unfortunately, this observation does not help in proving Theorem 6.1(b).

Before we start the proof we first define, for K compact and E an LB-space,

$$C_{\rm f}(K,E):=\{f:K\to E \text{ continuous}: f(K) \text{ is factorizable}\},$$

and we equip this space with the uniform topology induced from $C(K,E)$.

THEOREM 6.2. *For every infinite discrete set I and any LB-space E, the space $C_{\rm f}(\beta I,E)$ is bornological.*

COROLLARY 6.3 (cf. [3, 1.5(b)(2)]). *For every infinite discrete set I and every Montel LB-space E, the space $C(\beta I,E)$ is bornological.*

Remark. By Theorem 6.2 and the remark (b) after the proof of Theorem 5.1, the space $C(\beta I,E)$ is bornological for every LB-space E of Moscatelli type.

Proof of Theorem 6.2. Let U be an arbitrary bornivorous set in $C_{\rm f}(\beta I,E)$. For some sequence of positive numbers (a_n) we have

$$U\supseteq\sum_{n\in\mathbb{N}}(a_nB_n^{\beta I}\cap C_{\rm f}(\beta I,E))=:U_0,$$

where B_n is the unit ball in the Banach space E_n, $E=\operatorname{ind}_{n\in\mathbb{N}}E_n$. We will

show that U_0 contains the 0-neighbourhood

$$V:=\Big(\sum_{n\in\mathbb{N}}(a_n/2)B_n\Big)^{\beta I}\cap C_{\rm f}(\beta I,E).$$

Let $f\in V$. By the definition of $C_{\rm f}(\beta I,E)$, there exists an increasing sequence of compact absolutely convex sets (K_n) in E such that

(i) $K_n\subseteq B_n$;
(ii) $F=\operatorname{ind}_{n\in\mathbb{N}}E_{K_n}$ is a Montel space;
(iii) $\operatorname{absconv} f(\beta I)$ is relatively compact in F.

In particular, by (iii), $f\in C(\beta I,F)$. Since $f(\beta I)$ is compact in F, by Theorems 1.2 and 1.3, there exists a finite set $A\subseteq f(\beta I)$ and $m\in\mathbb{N}$ such that

$$f(\beta I)\subseteq A+\sum_{n\le m}(a_n/2)K_n.$$

Thus for any $i\in I$ we find $a(i)\in A$ and $b_n(i)\in(a_n/2)K_n$ for $n\le m$ such that

$$f(i)=a(i)+\sum_{n\le m}b_n(i).$$

Moreover, $A\subseteq f(\beta I)\subseteq\sum_{n\in\mathbb{N}}(a_n/2)B_n$, and because A is finite, there exists $l\in\mathbb{N}$ such that

$$A\subseteq\sum_{n\le l}(a_n/2)B_n.$$

Thus, there are functions $c_n:I\to(a_n/2)B_n$ of finite range such that

$$a(i)=\sum_{n\le l}c_n(i).$$

Extending c_n for $n\le l$ and b_n for $n\le m$ continuously to βI we obtain

$$f(i)=\sum_{n\le l}c_n(i)+\sum_{n\le m}b_n(i)$$

for all $i\in\beta I$ and

$$c_n:\beta I\to(a_n/2)B_n,\qquad b_n:\beta I\to(a_n/2)K_n.$$

Since the c_n have finite range and K_n are factorizable sets, we conclude that $c_n,b_n\in C_{\rm f}(\beta I,E)$. Finally, $f\in U_0$.

Let $\mathcal{L}$ be a family of absolutely convex compact subsets in an LB-space E. Let K be a compact space. Then we define

$$C_{\mathcal{L}}(K,E):=\{f\in C(K,E):\exists K_0\in\mathcal{L},\ f(K)\subseteq K_0\}.$$

THEOREM 6.4. *Let E be a complete LB-space. With the above assumptions,*

$$C_{\mathcal{L}}(K,E)^{(1)} \subseteq C_{\mathcal{L}^{(1)}}(K,E).$$

If $K = \beta I$ for some infinite discrete set I, then the converse inclusion holds as well.

By Corollary 4.7, we get immediately

COROLLARY 6.5. *For every compact set K and every complete LB-space E, $C_{\mathfrak{f}}(K,E)$ is Mackey complete.*

Proof of Theorem 6.4. Let $f \in C_{\mathcal{L}}(K,E)^{(1)} \subseteq C(K,E)$. Then there exists a bounded absolutely convex set B in E such that for every $a > 0$,

$$(f + a(B^K \cap C(K,E))) \cap C_{\mathcal{L}}(K,E) \neq \emptyset.$$

Thus for every a we find $f_a \in C_{\mathcal{L}}(K,E)$ such that $f - f_a \in aB^K \cap C(K,E)$ and $f(K) \subseteq f_a(K) + aB$. Since $f_a \in C_{\mathcal{L}}(K,E)$, it is obvious that $f \in C_{\mathcal{L}^{(1)}}(K,E)$.

Now, let $f \in C_{\mathcal{L}^{(1)}}(\beta I,E)$. Then, by 4.5, there exists a compact set $K_0 \in \mathcal{L}^{(1)}$ and $K_a \in \mathcal{L}$ such that

$$f(\beta I) \subseteq \bigcap_{a>0} (aK_0 + K_a).$$

For every $i \in I$ there exist $g_a(i) \in aK_0$ and $h_a(i) \in K_a$ such that $f(i) = g_a(i) + h_a(i)$. We can extend both functions continuously to $g_a : \beta I \to aK_0$ and $h_a : \beta I \to K_a$ such that

$$f(i) = g_a(i) + h_a(i) \quad \text{for } i \in \beta I.$$

Moreover, $g_a \in a(K_0^{\beta I} \cap C(\beta I,E)) =: aL$, and $h_a \in C_{\mathcal{L}}(\beta I,E)$. Hence $h_a \to f$ in $(C_{\mathcal{L}}(\beta I,E))_L$ as $a \to 0$, i.e., h_a tends to f locally since L is a closed absolutely convex bounded set in $C(\beta I,E)$.

LEMMA 6.6. *For every infinite discrete set I and every compact set K in an LB-space E there is an $f \in C(\beta I,E)$ such that $f(\beta I) = K$.*

Proof. Obvious, by Theorem 1.2.

Proof of Theorem 6.1. (a) Obviously,

$$\operatorname*{ind}_{n\in\mathbb{N}} C(\beta\mathbb{N},F_n) = C_{\mathcal{S}}(\beta\mathbb{N},F).$$

By Theorem 6.4, the Mackey completion of $\operatorname{ind}_{n\in\mathbb{N}} C(\beta\mathbb{N},F_n)$ is equal to $C_{\mathcal{S}^{(\alpha)}}(\beta\mathbb{N},F)$. On the other hand, by Theorem 4.3,

$$C_{\mathcal{S}^{(\alpha)}}(\beta\mathbb{N},F) \subseteq C_{\mathfrak{f}}(\beta\mathbb{N},F).$$

By Lemma 6.6 and Proposition 3.6, if there exists a nonfactorizable Montel map $T : E \to F$ for some LB-space E, then

$$C_{\mathfrak{f}}(\beta\mathbb{N},F) \neq C(\beta\mathbb{N},F).$$

This completes the proof, since $C(\beta\mathbb{N},F)$ is the completion of the space $\operatorname{ind}_{n\in\mathbb{N}} C(\beta\mathbb{N},F_n)$ [20, I.7.2].

(b) If the assumptions are satisfied, then, by Proposition 3.6, $C_{\mathfrak{f}}(\beta\mathbb{N},F) = C(\beta\mathbb{N},F)$. By Theorem 6.2, $C(\beta\mathbb{N},F)$ is bornological.

(c) Let us take for E the LB-space constructed in Theorem 5.1. Then Lemma 6.6 and Theorem 6.4 imply that E is the example we are looking for.

Added in proof. After the paper was submitted the authors solved in the affirmative the Montel factorization problem for operators with the domain being a Fréchet Köthe space of type 1 [22].

Acknowledgements. The authors are very much indebted to Prof. J. Bonet for providing plenty of information related to the subject of the paper.

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Received August 21, 1992 (2988)

STUDIA MATHEMATICA 107 (1) (1993)

Integrability theorems for trigonometric series

by

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Abstract. We show that, if the coefficients (a_n) in a series $a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nt)$ tend to 0 as $n \to \infty$ and satisfy the regularity condition that

$$\sum_{m=0}^{\infty} \left\{ \sum_{j=1}^{\infty} \left[\sum_{n=j2^m}^{(j+1)2^m-1} |a_n - a_{n+1}| \right]^2 \right\}^{1/2} < \infty,$$

then the cosine series represents an integrable function on the interval $[-\pi, \pi]$. We also show that, if the coefficients (b_n) in a series $\sum_{n=1}^{\infty} b_n \sin(nt)$ tend to 0 and satisfy the corresponding regularity condition, then the sine series represents an integrable function on $[-\pi, \pi]$ if and only if $\sum_{n=1}^{\infty} |b_n|/n < \infty$. These conclusions were previously known to hold under stronger restrictions on the sizes of the differences $\Delta a_n = a_n - a_{n+1}$ and $\Delta b_n = b_n - b_{n+1}$. We were led to the mixed-norm conditions that we use here by our recent discovery that the same combination of conditions implies the integrability of Walsh series with coefficients (a_n) tending to 0.

We also show here that this condition on the differences implies that the cosine series converges in L^1-norm if and only if $a_n \log n \to 0$ as $n \to \infty$. The corresponding statement also holds for sine series for which $\sum_{n=1}^{\infty} |b_n|/n < \infty$. If either type of series is assumed *a priori* to represent an integrable function, then weaker regularity conditions suffice for the validity of this criterion for norm convergence.

1. Introduction. We outline one proof of the integrability results in this section, and comment further on that proof in Section 2. We present another proof of the integrability results in Sections 5 and 6. We also state two theorems about L^1-norm convergence in Section 1, and show in Section 3 how these statements follow from the integrability results. We begin this section by recalling some earlier work on these questions, and we say more in Sections 4 and 5 about how our results compare with other work.

About eighty years ago, W. H. Young [36] related integrability of series to properties of differences of coefficients by showing that if the coefficients in a cosine series tend to 0 and form a convex sequence, then the series

1991 *Mathematics Subject Classification*: Primary 42A32, 42A16; Secondary 46E39, 46E40, 46E30, 46E35.

Research of the second author partially supported by NSERC grant number 4822.