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Calderón couples of rearrangement invariant spaces

by

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Abstract. We examine conditions under which a pair of rearrangement invariant function spaces on $[0,1]$ or $[0,\infty)$ form a Calderón couple. A very general criterion is developed to determine whether such a pair is a Calderón couple, with numerous applications. We give, for example, a complete classification of those spaces X which form a Calderón couple with L_∞. We specialize our results to Orlicz spaces and are able to give necessary and sufficient conditions on an Orlicz function F so that the pair (L_F, L_∞) forms a Calderón pair.

1. Introduction. Suppose (X,Y) is a compatible pair of Banach spaces (see [4] or [5]). We denote by $K(t,f) = K(t,f;X,Y)$ the Peetre K-functional on $X+Y$, i.e.

$$K(t,f) = \inf\{\|x\|_X + t\|y\|_Y : x+y = f\}.$$

Then (X,Y) is called a *Calderón couple* (or a *Calderón–Mityagin couple*) if whenever f, g satisfy

$$K(t,f) \le K(t,g)$$

for all t then there is a bounded operator $T : X+Y \to X+Y$ such that $\|T\|_X, \|T\|_Y < \infty$ and $Tg = f$. We will say that (X,Y) is a *uniform* Calderón couple (with constant C) if we can further insist that $\max(\|T\|_X, \|T\|_Y) \le C$. Calderón couples are particularly important in interpolation theory because it is possible to give a complete description of all interpolation spaces for such a couple. Indeed, for such a couple, it is easy to show that a space Z is an interpolation space if and only if it is K-monotone, i.e. if $f \in Z$ and $g \in X+Y$ with $K(t,g) \le K(t,f)$ imply $g \in Z$. It follows from the K-divisibility theorem of Brudnyĭ and Kruglyak [7] that if Z is a normed K-monotone space then $\|f\|_Z$ on Z is equivalent to a norm $\|K(t,f)\|_\Phi$ where Φ is an appropriate lattice norm on functions on $(0,\infty)$. Thus, for Calderón couples, one has a complete description of all interpolation spaces.

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We also remark at this point that there are apparently no known examples of Calderón couples which are not uniform.

There has been a considerable amount of subsequent effort devoted to classifying Calderón couples of rearrangement-invariant spaces on $[0,1]$ or $[0,\infty)$. It is a classical result of Calderón and Mityagin ([9], [32]) that the pair (L_1, L_∞) is a uniform Calderón couple with constant 1. It is now known that any pair (L_p, L_q) is a Calderón couple (and, indeed, weighted versions of these theorems are valid); we refer the reader to Lorentz and Shimogaki [27], Sparr [36], Arazy and Cwikel [1], Sedaev and Semenov [35] and Cwikel [13], [15]. Subsequent work has shown that under certain hypotheses pairs of Lorentz spaces or Marcinkiewicz spaces are Calderón couples; see Cwikel [14], Merucci [30], [31] and Cwikel–Nilsson [16], [17]. For further positive results on Calderón couples see [18] (for weighted Banach lattices), and [21] and [38] (for Hardy spaces).

On the negative side, Ovchinnikov [34] showed that on $[0,\infty)$ the pair $(L_1+L_\infty, L_1\cap L_\infty)$ is not a Calderón couple; indeed, Maligranda and Ovchinnikov show that if $p \neq 2$ then $L_p \cap L_q$ and $L_p + L_q$ $(1/p + 1/q = 1)$ are interpolation spaces not obtainable by the K-method [29].

The general problem we consider in this article is that of providing necessary and sufficient conditions on a pair (X,Y) of r.i. spaces (always assumed to have the so-called Fatou property) on either $[0,1]$ or $[0,\infty)$ so that (X,Y) is a Calderón couple. Although we cannot provide a complete answer to this problem, we can resolve it in certain cases and this enables us to settle some open questions in the area (see e.g. Maligranda [28], Problems 1–3, or Brudnyĭ–Kruglyak [8], p. 685, [g], [i]). For example, we give a complete classification of all r.i. spaces X so that (X, L_∞) is a Calderón couple and hence give examples of r.i. spaces (even Orlicz spaces) X so that (X, L_∞) is not a Calderón couple. Our methods give fairly precise information in the problem of classifying pairs of Orlicz spaces which form Calderón couples. It should also be mentioned that our results apply equally to symmetric sequence spaces.

We now describe our results in more detail. Let X be an r.i. space on $[0,1]$ or $[0,\infty)$ or a symmetric sequence space. Let $e_n = \chi_{[2^n,2^{n+1})}$ for $n \in \mathbb{J}$ where $\mathbb{J} = \mathbb{Z}_- = -\mathbb{N}$ or $\mathbb{J} = \mathbb{Z}$ or $\mathbb{J} = \mathbb{N}\cup\{0\}$. We associate with X a Köthe sequence space E_X on $\mathbb{J}$ by defining

$$\|\xi\|_{E_X} = \Big\| \sum_{n\in\mathbb{J}} \xi(n) e_n \Big\|_X .$$

We then say that X is *stretchable* if the sequence space E_X has the *right-shift property* (RSP), i.e. there is a constant C so that if $(x_n, y_n)_{n=1}^N$ is any pair of finite normalized sequences in E_X so that $\operatorname{supp} x_1 < \operatorname{supp} y_1 < \operatorname{supp} x_2 < \ldots < \operatorname{supp} y_n$ then for any $\alpha_1, \ldots, \alpha_N$ we have

$$\Big\| \sum_{n=1}^N \alpha_n y_n \Big\|_{E_X} \le C \Big\| \sum_{n=1}^N \alpha_n x_n \Big\|_{E_X} .$$

Thus E_X has (RSP) if the right-shift operator is uniformly bounded on the closed linear span of every block basic sequence with respect to the canonical basis. We similarly say that X is *compressible* if E_X has the corresponding *left-shift property* (LSP). Finally, we say that X is *elastic* if it is both stretchable and compressible. It is easy to see that L_p-spaces and more generally Lorentz spaces with finite Boyd indices are elastic because E_X in this case is a weighted ℓ_p-space (in fact, this property characterizes Lorentz spaces when the Boyd indices are finite). On the other hand, it is not difficult to give examples of r.i. spaces which are neither compressible nor stretchable. Curiously, however, we have no example of a space which is either stretchable or compressible and not elastic.

The significance of these ideas is illustrated by Theorem 5.4. The pair (X, L_∞) is a Calderón couple if and only if X is stretchable. Dually, if we assume that X has nontrivial concavity then (X, L_1) is a Calderón couple if and only if X is compressible (Theorem 5.5). More generally, if (X,Y) is any pair of r.i. spaces such that either the Boyd indices satisfy $p_Y > q_X$ or there exists p so that X is p-concave and Y is p-convex and has nontrivial concavity then (X,Y) is a Calderón couple if and only if X is stretchable and Y is compressible.

In Section 6 we study these concepts for Orlicz spaces. We show that for an Orlicz space to be compressible it is necessary and sufficient that it is stretchable; thus we need only consider elastic Orlicz spaces. We show for example that $L_F[0,1]$ (where F satisfies the Δ_2-condition) is elastic if and only if there is a constant C and a bounded monotone increasing function $w(t)$ so that for any $0 < x \le 1$ and any $1 \le s \le t < \infty$ we have

$$\frac{F(tx)}{F(t)} \le C\frac{F(sx)}{F(s)} + w(t) - w(s) .$$

This condition implies that the Boyd indices (or Orlicz–Matuszewska indices) p_F and q_F of L_F coincide. In fact, it implies the stronger condition that F must be equivalent to a function which is regularly varying in the sense of Karamata (see [6]). We give examples to show that F can be regularly varying with L_F inelastic and that L_F can be elastic without coinciding with a Lorentz space (cf. [26], [33]).

Brudnyĭ (cf. [8]) has conjectured that if a pair of (distinct) Orlicz spaces $(L_F[0,1], L_G[0,1])$ is a Calderón couple then $p_F = q_F$ and $p_G = q_G$. We show by example that this is false. However, we also show that either $p_F = p_G$ and $q_F = q_G$ or both L_F and L_G are elastic, in which case $p_F = q_F$ and $p_G = q_G$.

Let us now introduce some notation and conventions. Let Ω be a Polish space and let μ be a σ-finite Borel measure on Ω. Let $L_0(\mu)$ denote the space of all real-valued Borel functions on Ω (where functions differing on a set of measure zero are identified), equipped with the topology of convergence in μ-measure on sets of finite measure. By a *Köthe function space* on Ω we shall mean a Banach space X which is a subspace of L_0 containing the characteristic function χ_B whenever $\mu(B) < \infty$ and such that the norm $\|\ \|_X$ satisfies the following conditions:

(a) $\|f\|_X \le \|g\|_X$ whenever $|f| \le |g|$ a.e.,
(b) $B_X = \{f : \|f\|_X \le 1\}$ is closed in L_0.

Condition (b) is usually called the *Fatou property*; note here that we include the Fatou property in our definition and so it is an implicit assumption throughout the paper. It is sometimes convenient to extend the definition of $\|f\|_X$ by setting $\|f\|_X = \infty$ if $f \notin X$. We will also write $P_B f = f_B = f\chi_B$ when B is a Borel subset of Ω. We let $\operatorname{supp} f = \{\omega : f(\omega) \neq 0\}$.

If X is a Köthe function space then we say that X is *p-convex* ($1 \le p \le \infty$) if there is a constant M so that for any $f_1, \dots, f_n \in X$ we have

$$\Big\| \Big(\sum_{k=1}^n |f_k|^p \Big)^{1/p} \Big\|_X \le M \Big(\sum_{k=1}^n \|f_k\|_X^p \Big)^{1/p},$$

and *p-concave* if there exists M so that

$$\Big(\sum_{k=1}^n \|f_k\|_X^p \Big)^{1/p} \le M \Big\| \Big(\sum_{k=1}^n |f_k|^p \Big)^{1/p} \Big\|_X.$$

Similarly, we say that X has an *upper p-estimate* if there is a constant M so that if $f_1, \dots, f_n$ are disjoint in X then

$$\Big\| \sum_{k=1}^n f_k \Big\|_X \le M \Big(\sum_{k=1}^n \|f_k\|_X^p \Big)^{1/p},$$

and X has a *lower p-estimate* if there exists M so that if $f_1, \dots, f_n$ are disjoint then

$$\Big(\sum_{k=1}^n \|f_k\|_X^p \Big)^{1/p} \le M \Big\| \sum_{k=1}^n f_k \Big\|_X.$$

See Lindenstrauss–Tzafriri [25] for a fuller discussion.

We will sometimes use $\langle f, g \rangle$ for $\int_\Omega f g \, d\mu$. With this notion of pairing we will also use X^* for the Köthe dual of X (which will coincide with the full dual if X is separable).

If (X, Y) are two Köthe function spaces on (Ω, μ) then the pair (X, Y) is necessarily Gagliardo complete (cf. [4]). We denote by $\mathcal{A}(X, Y)$ the space of admissible operators, i.e. operators $T : X + Y \to X + Y$ such that $\|T\|_X = \sup\{\|Tf\|_X : \|f\|_X \le 1\} < \infty$ and $\|T\|_Y = \sup\{\|Tf\|_Y : \|f\|_Y \le 1\} < \infty$. We norm $\mathcal{A}(X, Y)$ by $\|T\|_{(X,Y)} = \max(\|T\|_X, \|T\|_Y)$.

In the special case when $\Omega = \mathbb{J}$ is a subset of $\mathbb{Z}$ and μ is counting measure we write $\omega(\mathbb{J}) = L_0(\mu)$ and a Köthe function space X is called a *Köthe sequence space* modelled on $\mathbb{J}$. An operator T on X is then called a *matrix* if it takes the form

$$Tx(n) = \sum_{k \in \mathbb{J}} a_{nk} x(k)$$

for some $(a_{nk})_{n,k \in \mathbb{J}}$. We remark here that the assumption that T is a matrix forces the existence of an adjoint operator $T^* : X^* \to X^*$ even in the nonseparable situation, when X^* is not the full dual of X.

If $\Omega = [0, 1]$ or $[0, \infty)$ (with μ Lebesgue measure) or if $\Omega = \mathbb{N}$ (with μ counting measure) then for $f \in L_0(\mu)$ we define the *decreasing rearrangement* f^* of f by

$$f^*(t) = \sup_{B : \mu(B) \le t} \inf_{s \in B} |f(s)|,$$

for $0 < t < \infty$. We say that X is a *rearrangement invariant space* (or a *symmetric sequence space* if $\Omega = \mathbb{N}$) if $\|f\|_X = \|f^*\|_X$ for all $f \in L_0$. If we define

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds$$

then it is well-known that if $f, g \in L_0$ with $f^{**} \le g^{**}$ then $\|f\|_X \le \|g\|_X$.

If X is an r.i. space on $[0, 1]$ or $[0, \infty)$ then the dilation operators D_a on X are defined by $D_a f(t) = f(t/a)$ (where we regard f as vanishing outside $[0, 1]$ in the former case). We can then define the *Boyd indices* p_X and q_X of X by

$$p_X = \lim_{a \to \infty} \frac{\log a}{\log \|D_a\|_X}, \quad q_X = \lim_{a \to 0} \frac{\log a}{\log \|D_a\|_X}.$$

In the case when X is a symmetric sequence space we define p_X and q_X in the same way but we define D_a by the nonlinear formula

$$D_a f(n) = f^*(n/a)$$

where f^* is well-defined on $[0, \infty)$.

Finally, let us mention two special classes of r.i. spaces. If $1 \le p < \infty$ we will say that an r.i. space X on $\Omega = [0, 1]$ or $[0, \infty)$ is a *Lorentz space of order p* if there is a positive increasing weight function $w : \Omega \to (0, \infty)$ such that $\sup_{t, 2t \in \Omega} w(2t)/w(t) < \infty$ and $\|f\|_X$ is equivalent to the quasinorm

$$\|f\|_{w,p} = \Big(\int_\Omega f^*(t)^p w(t)^p \, \frac{dt}{t} \Big)^{1/p}.$$

We can then write $X = L_{w,p}$. If we take $w(t) = t^{1/q}$ we obtain the standard Lorentz spaces $L(q,p)$. It is easy to compute that the Lorentz space $X = L_{w,p}$ has Boyd indices p_X, q_X where

$$\frac{1}{p_X} = \lim_{a\to\infty} \sup_{t,at\in\Omega} \frac{\log w(at) - \log w(t)}{\log a},$$

$$\frac{1}{q_X} = \lim_{a\to\infty} \inf_{t,at\in\Omega} \frac{\log w(at) - \log w(t)}{\log a}.$$

If we impose the additional restriction that $q_X < \infty$ then it can easily be seen that we may suppose that w satisfies $\inf_{t,2t\in\Omega} w(2t)/w(t) > 1$.

We will also be interested in Orlicz function spaces and sequence spaces. By an *Orlicz function* we shall mean a continuous strictly increasing convex function $F : [0,\infty) \to [0,\infty)$ such that $F(0) = 0$. F is said to satisfy the *Δ_2-condition* if there is a constant Δ such that $F(2x) \le \Delta F(x)$ for all $x \ge 0$.

The *Orlicz function space* $L_F(\Omega,\mu)$ is defined by

$$\|f\|_{L_F} = \inf\Big\{\alpha > 0 : \int_\Omega F(\alpha^{-1} f(t))\, dt \le 1\Big\}$$

so that $L_F = \{f : \|f\|_{L_F} < \infty\}$.

In this case the Boyd indices $p_F = p_{L_F}$ and $q_F = q_{L_F}$ are closely related to the Orlicz–Matuszewska indices of F (see Lindenstrauss–Tzafriri [25], p. 139). More precisely, let $\alpha^\infty(F)$ (resp. $\alpha^0(F)$) be the supremum of all p so that for some C we have $F(st) \le Cs^p F(t)$ for all $0 \le s \le 1$ and all $t \ge 1$ (resp. $t \le 1$). Similarly, let $\beta^\infty(F)$ (resp. $\beta^0(F)$) be the infimum of all q so that for some C we have $s^p F(t) \le CF(st)$ for all $0 \le s \le 1$ and all $t \ge 1$ (resp. $t \le 1$). Then if $\Omega = [0,1]$ we have $p_F = \alpha^\infty(F)$ and $q_F = \beta^\infty(F)$. If $\Omega = [0,\infty)$ then $p_F = \min(\alpha^\infty(F), \alpha^0(F))$ and $q_F = \max(\beta^\infty(F), \beta^0(F))$. If we assume the Δ_2-condition (and we always will) then $q_F < \infty$.

2. The shift properties. Let $\mathbb{J}$ be one of the three sets $\mathbb{Z}$, $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \ge 0\}$ or $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{Z}_+$. Let $\omega(\mathbb{J})$ denote the space of all sequences modelled on $\mathbb{J}$. If $x = \{x(k)\}_{k\in\mathbb{J}}$ is a sequence (modelled on $\mathbb{J}$) we write $\operatorname{supp} x = \{k : x(k) \ne 0\}$. If A, B are subsets of $\mathbb{J}$ we write $A < B$ if $a < b$ for every $a \in A$, $b \in B$. If I is any interval of $\mathbb{Z}$ and $(x_n, y_n)_{n\in I}$ is a pair of sequences in $\omega(\mathbb{J})$ we say (x_n, y_n) is *interlaced* if each x_n, y_n has finite support and $\operatorname{supp} x_n < \operatorname{supp} y_n$ $(n \in I)$ and $\operatorname{supp} y_n < \operatorname{supp} x_{n+1}$ whenever $n, n+1 \in I$.

Let E be a Köthe sequence space modelled on $\mathbb{J}$. We will say that E has the *right-shift property* (RSP) if there is a constant C such that whenever $(x_n, y_n)_{n\in I}$ is an interlaced pair with $\|y_n\|_E \le \|x_n\|_E = 1$ $(n \in I)$ then for

every finitely nonzero sequence of scalars $(\alpha_n)_{n\in I}$ we have

$$\Big\|\sum_{n\in I} \alpha_n y_n\Big\|_E \le C\Big\|\sum_{n\in I} \alpha_n x_n\Big\|_E.$$

Conversely, we will say that E has the *left-shift property* (LSP) if there is a constant C' so that for every interlaced pair $(x_n, y_n)_{n\in I}$ with $\|x_n\|_E \le \|y_n\|_E = 1$, and every finitely nonzero $(\alpha_n)_{n\in I}$ we have

$$\Big\|\sum_{n\in I} \alpha_n x_n\Big\|_E \le C'\Big\|\sum_{n\in I} \alpha_n y_n\Big\|_E.$$

PROPOSITION 2.1. *E has* (LSP) *if and only if E^* has* (RSP).

Proof. We will only prove one direction. Let us assume E^* has (RSP) with constant C. Let $(x_n, y_n)_{n\in I}$, be an interlaced pair with $\|y_n\|_E \le \|x_n\|_E = 1$. We may assume each x_n, y_n is positive (i.e. $x_n(k), y_n(k) \ge 0$ for every k). Suppose $(\alpha_n)_{n\in I}$ is a finitely nonzero sequence of nonnegative reals. Let $f = \sum \alpha_n y_n$. Then there exists positive $g \in E^*$ with $\operatorname{supp} g \subset \operatorname{supp} f$ and so that $\langle f, g\rangle = \|f\|_E$ while $\|g\|_{E^*} = 1$. We can write

$$g = \sum \beta_n v_n$$

where each v_n is positive, $\|v_n\|_{E^*} = 1$ and $\operatorname{supp} v_n \subset \operatorname{supp} y_n$.

Next pick positive u_n with $\operatorname{supp} u_n \subset \operatorname{supp} x_n$, $\langle x_n, u_n\rangle = 1$ and $\|u_n\|_{E^*} = 1$. We conclude from the fact that E^* has (RSP) that

$$\Big\|\sum_{n\in I} \beta_n u_n\Big\|_{E^*} \le C.$$

Thus

$$\Big\|\sum_{n\in I} \alpha_n y_n\Big\|_E = \sum_{n\in I} \alpha_n\beta_n\langle y_n, v_n\rangle \le \sum \alpha_n\beta_n$$
$$\le \Big\langle\Big(\sum \alpha_n x_n\Big), \Big(\sum \beta_n u_n\Big)\Big\rangle \le C\Big\|\sum \alpha_n x_n\Big\|_E.$$

Thus the proposition is proved. ∎

PROPOSITION 2.2. *Suppose E is a Köthe sequence space modelled on $\mathbb{Z}$. Define $E_+ = E(\mathbb{Z}_+)$ and $E_- = E(\mathbb{Z}_-)$. Then E has* (RSP) *(resp.* (LSP)*) if and only if both E_+ and E_- have* (RSP) *(resp.* (LSP)*).*

Proof. One direction is obvious. For the other, suppose both E_+ and E_- have (RSP) with constant C, say. Suppose $(x_n, y_n)_{n\in I}$ is an interlaced pair of sequences with $\|y_n\|_E \le \|x_n\|_E = 1$ and that $(\alpha_n)_{n\in I}$ is finitely nonzero. Then there exists $m \in I$ so that $\operatorname{supp}(x_n + y_n) \subset \mathbb{Z}_-$ for $n < m$ and $\operatorname{supp}(x_n + y_n) \subset \mathbb{Z}_+$ for $n > m$. Now

$$\Big\|\sum_{n\in I} \alpha_n y_n\Big\|_E \le |\alpha_m| + \Big\|\sum_{n<m} \alpha_n y_n\Big\|_E + \Big\|\sum_{n>m} \alpha_n y_n\Big\|_E$$

$$\leq (2C+1)\Big\|\sum_{n\in I}\alpha_n x_n\Big\|_E.$$

Thus E has (RSP) with constant at most $2C+1$. ∎

To simplify our discussion we introduce the idea of an order-reversal. Let $E = E(\mathbb{J})$ be a Köthe sequence space. We let $\widetilde{\mathbb{J}} = \{-(n+1) : n \in \mathbb{J}\}$ and if $x \in \omega(\mathbb{J})$ we set $\widetilde{x}(n) = x(-(n+1))$ for $n \in \mathbb{J}$. Let $\widetilde{E}(\widetilde{\mathbb{J}})$ be defined by $\|x\|_{\widetilde{E}} = \|\widetilde{x}\|_E$; then $\widetilde{E}$ is the order-reversal of E. Clearly (LSP) (resp. (RSP)) for E is equivalent to (RSP) (resp. (LSP)) for $\widetilde{E}$.

Next observe that if $(w_n)_{n\in\mathbb{J}}$ satisfy $w_n > 0$ for all n then the weighted sequence space $E(w) = \{x : xw \in E\}$ normed by $\|x\|_{E(w)} = \|xw\|_E$ satisfies (LSP) (resp. (RSP)) if and only if E satisfies (LSP) (resp. (RSP)).

PROPOSITION 2.3. *Let $E = E(\mathbb{J})$ be a symmetric sequence space. Suppose E has either* (LSP) *or* (RSP). *Then $E = \ell_p(\mathbb{J})$ for some $1 \leq p \leq \infty$.*

Proof. For convenience of notation we consider only the case $\mathbb{J} = \mathbb{Z}_+$ and (LSP) and leave the reader to make the minor adjustments necessary for the other cases. Let $(u_n)_{n\in\mathbb{N}}$ be any normalized positive block basic sequence in $E(\mathbb{J})$. Select $a_n \in \operatorname{supp} u_n$. Then $(u_{2n}, e_{a_{2n+1}})_{n\in\mathbb{N}}$ is an interlaced pair. Thus

$$\Big\|\sum_{n\in\mathbb{N}}\alpha_n e_n\Big\|_E \leq C\Big\|\sum_{n\in\mathbb{N}}\alpha_n u_{2n}\Big\|_E.$$

Similarly $(e_{a_{2n-1}}, u_{2n})_{n\in\mathbb{N}}$ is an interlaced pair and so

$$\Big\|\sum_{n\in\mathbb{N}}\alpha_n u_{2n}\Big\|_E \leq C\Big\|\sum_{n\in\mathbb{N}}\alpha_n e_n\Big\|_E.$$

Thus (u_{2n}) is C^2-equivalent to (e_n) and similarly so is $(u_{2n-1})_{n\in\mathbb{N}}$. It then follows that the basis (or basic sequence) (e_n) is perfectly homogeneous and by a theorem of Zippin [39] (see Lindenstrauss–Tzafriri [24]) this implies that it is equivalent to the ℓ_p-basis for some p or the c_0-basis; in the latter case we deduce that $E = \ell_\infty(\mathbb{J})$. The result then follows. ∎

PROPOSITION 2.4. *Let $E = E(\mathbb{Z}_+)$ be a Köthe sequence space with* (LSP) *or* (RSP). *If E contains a symmetric basic sequence then there exists $1 \leq p \leq \infty$ and an increasing sequence $(a_k)_{k\geq 0}$ with $a_0 = 0$ so that $E = \ell_p(E[a_k, a_{k+1}))$. In particular, when E is separable, we have $p < \infty$ and any symmetric basic sequence in E is equivalent to the canonical ℓ_p-basis.*

Remark. Of course there is a similar result if $\mathbb{J} = \mathbb{Z}_-$. However, in the two-ended setting $\mathbb{J} = \mathbb{Z}$ we recall that E has (RSP) (resp. (LSP)) if and only if both $E(\mathbb{Z}_+)$ and $E(\mathbb{Z}_-)$ have (RSP) (resp. (LSP)). In particular, $\ell_p(\mathbb{Z}_-) \oplus \ell_r(\mathbb{Z}_+)$ has (LSP) and (RSP) even if $r \neq p$.

Proof of Proposition 2.4. We can suppose that (u_n) is a normalized symmetric block basic sequence. By an interlacing argument as in Proposition 2.3 it will follow that a subsequence (e_{a_k}) of the unit vectors is symmetric. Since the restriction of E to this subsequence has (LSP) or (RSP) it follows that it is equivalent to the ℓ_p-basis for some $1 \leq p < \infty$ or to the c_0-basis by Proposition 2.3. For convenience we suppose the former case and fix p. Let $I_k = [a_k, a_{k+1})$. Then, for suitable C, by an interlacing argument any normalized sequence (v_k) supported on I_{2k} is C-equivalent to the ℓ_p-basis; similarly, any normalized sequence supported on I_{2k+1} is C-equivalent to the ℓ_p-basis and the first part of the result follows. For the last part, if E is separable then obviously $p < \infty$ and a simple blocking argument gives the result. ∎

Remark. It is possible that E contains no symmetric basic sequence. Indeed, Tsirelson space T [37] and its convexifications provide examples of such spaces with (RSP) and (LSP) (see [10] and [12]). It is not difficult to use Krivine's theorem [22] to show that if $E = E(\mathbb{Z}_+)$ has (LSP) (or (RSP)) then there is a subsequence (e_{a_n}) of the unit vector basis so that for some C, p we have for all k and every k vectors $x_1, x_2, \ldots, x_k$ with $\operatorname{supp} x_1 < \operatorname{supp} x_2 < \ldots < \operatorname{supp} x_k$ and $\operatorname{supp}(x_1 + \ldots + x_k) \subset \{a_n\}_{n\geq 1}$,

$$C^{-1}\Big(\sum_{n=1}^{k}\|x_n\|_E^p\Big)^{1/p} \leq \Big\|\sum_{n=1}^{k}x_n\Big\|_E \leq C\Big(\sum_{n=1}^{\infty}\|x_n\|_E^p\Big)^{1/p}$$

with appropriate modifications when $p = \infty$. Thus any space E having either (LSP) or (RSP) and no symmetric basic sequence has a "Tsirelson-like" subspace.

PROBLEM. Does there exist a Köthe sequence space with (LSP) and not (RSP)?

Let us remark that this is probably nontrivial. Indeed, the corresponding question for simple shifts has been considered [3] and Bellenot has only recently given an example [2].

LEMMA 2.5. *Let E be a Köthe sequence space on $\mathbb{J}$ with* (RSP); *then there is a constant C so that whenever $(x_n, y_n)_{n\in I}$ is an interlaced pair of sequences with $\|y_n\|_E \leq \|x_n\|_E = 1$ and $(x_n^*)_{n\in I}$ is a sequence in E^* with $\operatorname{supp} x_n^* \subset \operatorname{supp} x_n$ and $x_n^*(x_n) = \|x_n\|_{E^*} = 1$ then the operator T defined by $Tx = \sum_{n\in I}\langle x, x_n^*\rangle y_n$ is bounded on E with $\|T\|_E \leq C$.*

Proof. For any $x \in E$ with finite support, Tx has finite support and we can define $g_n^* \in E^*$ and a finitely nonzero sequence $(\alpha_n)_{n\in I}$ so that $\|g_n^*\| = 1$ $(n \in I)$, $\operatorname{supp} g_n^* \subset \operatorname{supp} y_n$, $\|\sum \alpha_n g_n^*\|_{E^*} = 1$ and

$$\Big\langle Tx, \sum_{n\in I}\alpha_n g_n^*\Big\rangle = \|Tx\|_E\,.$$

Thus

$$\|Tx\|_E = \sum_{n\in I}\alpha_n x_n^*(x) = \Big\langle x, \sum_{n\in I}\alpha_n x_n^*\Big\rangle$$
$$\le \|x\|_E\Big\|\sum_{n\in I}\alpha_n x_n^*\Big\|_{E^*} \le C\|x\|_E\Big\|\sum_{n\in I}\alpha_n g_n^*\Big\|_{E^*} \le C\|x\|_E$$

where C is the (LSP) constant of E^* (which actually is the (RSP) constant of E by Proposition 2.1 and its proof). The result follows. ∎

LEMMA 2.6. *Under the hypotheses of Lemma 2.5, there is a constant C_1 so that if $(J_n)_{n\in I}$ is a sequence of intervals in $\mathbb{J}$ with $J_n < J_{n+1}$ whenever $n, n+1 \in I$, $(x_n)_{n\in I}, (y_n)_{n\in I}$ are two normalized sequences with* $\operatorname{supp} x_n$, $\operatorname{supp} y_n \subset J_n$ *and (x_n^*) is any sequence with* $\operatorname{supp} x_n^* \subset \operatorname{supp} x_n$ *and $x_n^*(x_n) = 1 = \|x_n^*\|_{E^*}$ then the operator*

$$Tx = \sum_{n\in I}\langle x, x_n^*\rangle y_{n+1}$$

(where $y_{n+1} = 0$ if $n+1 \notin I$) is bounded on E with $\|T\|_E \le C_1$.

Proof. The sequence pairs $(x_{2n}, y_{2n+1})_{2n,2n+1\in I}$, $(x_{2n-1}, y_{2n})_{2n-1,2n\in I}$ are interlaced and the lemma follows from 2.5 with $C_1 = 2C$ by simply adding. ∎

Remark. If E is separable and has both (LSP) and (RSP) then Lemma 2.6 quickly shows that every normalized block basic sequence in E spans a complemented subspace; this property is, of course, enjoyed by Tsirelson space [12] (see also Casazza–Lin [11] for an earlier similar example). If this property holds for a symmetric sequence space then it is isomorphic to ℓ_p for some $1 \le p < \infty$ (see Lindenstrauss–Tzafriri [23]).

3. The shift properties for pairs. We next consider a pair of Köthe sequence spaces (E, F) modelled on $\mathbb{J}$. We will say that (E, F) has (RSP) if there is a constant C so that whenever $\{x_n, y_n\}_{n\in I}$ is an interlaced pair with $\|y_n\|_E \le \|x_n\|_E = 1$ and $x_n, y_n \ge 0$ then there is a positive matrix T with $\|T\|_{(E,F)} \le C$ and $Tx_n = y_n$. We will say that the pair (E, F) has (LSP) if $(\widetilde{F}, \widetilde{E})$ has (RSP). If (E, F) has both (LSP) and (RSP) then we say that it has the *shift property* (SP).

We first note that if (E, F) has (RSP) then E has (RSP). Conversely, it follows from Lemma 2.5 that if E has (RSP) then (E, E) has (RSP).

In this section, we show that, under certain hypotheses, one can deduce (RSP) for the couple (E, F) from the property (RSP) for E alone. We will need some definitions. We define the shift operators τ_n for $n \in \mathbb{Z}$ on $\omega(\mathbb{J})$ by $\tau_n(x)(k) = x(k-n)$, where we interpret $x(j) = 0$ when $j \notin \mathbb{J}$. We define $\kappa_+(E) = \lim_{n\to\infty}\|\tau_n\|_E^{1/n}$ (which can be ∞ in the case when τ_1 is unbounded on E) and $\kappa_-(E) = \lim_{n\to\infty}\|\tau_{-n}\|_E^{1/n}$. We will also let $\varrho(n) = \varrho(n; E, F) = \|e_n\|_E/\|e_n\|_F$. We will say that (E, F) is *exponentially separated* if there exist $\beta > 0$ and C_0 so that if $m, m+n \in \mathbb{J}$ then $\varrho(m+n) \ge C_0^{-1}2^{n\beta}\varrho(m)$.

LEMMA 3.1. *If $\kappa_-(E)\kappa_+(F) < 1$ then (E, F) is exponentially separated.*

Proof. Here we have $\varrho(m+n)/\varrho(m) \ge (\|\tau_{-n}\|_E\|\tau_n\|_F)^{-1}$. The hypothesis then implies that for some $\beta > 0$ we have $\|\tau_{-n}\|_E\|\tau_n\|_F \le C2^{-n\beta}$ for some $C > 0$. The result then follows. ∎

LEMMA 3.2. *Suppose (E, F) is exponentially separated. Then there is a constant C_1 so that if* $\operatorname{supp} x \subset [a, b]$ *then*

$$C_1^{-1}\varrho(a)\|x\|_F \le \|x\|_E \le C_1\varrho(b)\|x\|_F\,.$$

Proof. Suppose $x = \sum_{k=a}^b x(k)e_k$. Then

$$\|x\|_E \le \sum |x(k)|\|e_k\|_E \le \sum |x(k)|\varrho(k)\|e_k\|_F$$
$$\le C_0\sum \varrho(b)2^{-\beta(b-k)}|x(k)|\|e_k\|_F$$
$$\le C_0\varrho(b)\Big(\sum_{k=0}^{\infty}2^{-\beta k}\Big)\max_j|x(j)|\|e_j\|_F \le C\varrho(b)\|x\|_F$$

for a suitable constant C. The other inequality is similar. ∎

LEMMA 3.3. *Let E, F be a pair of Köthe sequence spaces satisfying $\kappa_-(E)\kappa_+(F) < 1$. Suppose E has* (RSP). *Then (E, F) has* (RSP). *Similarly, if F has* (LSP) *then (E, F) has* (LSP).

Proof. We first note that it is only necessary to prove the first statement since $(\widetilde{F}, \widetilde{E})$ will satisfy the same hypothesis $\kappa_-(\widetilde{F})\kappa_+(\widetilde{E}) < 1$ and $\widetilde{F}$ will have (RSP) if F has (LSP).

We refer back to Lemma 2.5; it is clear that there exists C_0 so that if $\{x_n, y_n\}_{n\in I}$ is a positive interlaced pair with $\|y_n\|_E \le \|x_n\|_E = 1$ then if we pick $x_n^* \ge 0$ with $\operatorname{supp} x_n^* \subset \operatorname{supp} x_n$ and $\langle x_n, x_n^*\rangle = \|x_n^*\|_{E^*} = 1$ for $n \in I$ then $\|T\|_E \le C_0$ where

$$Tx = \sum_{n\in A}\langle x, x_n^*\rangle y_n\,.$$

Obviously T is a positive matrix. We now compute $\|T\|_F$. Suppose $k \in \operatorname{supp} y_n$ where $n \in I$. Then, since $\operatorname{supp} x_n^* \subset (-\infty, k)$ and $y_n(k)e_k \le y_n$,

$$|Tx(k)| = |x_n^*(x)|y_n(k) \le \|x_{(-\infty,k)}\|_E\|e_k\|_E^{-1}\,.$$

Now

$$\|x_{(-\infty,k)}\|_E \le \sum_{j<k} |x(j)|\|e_j\|_E \le \|e_k\|_E \sum_{j<k} |x(j)|\|\tau_{j-k}\|_E .$$

We have

$$|Tx(k)| \le \sum_{j<k} |\tau_{k-j}x(k)|\|\tau_{j-k}\|_E$$

and hence, since $Tx(k)$ vanishes for all coordinates not of this form,

$$|Tx| \le \sum_{j=1}^{\infty} \|\tau_{-j}\|_E |\tau^j x| .$$

The hypothesis $\kappa_-(E)\kappa_+(F) < 1$ implies that there exist M and $0 < \delta < 1$ so that $\|\tau_{-j}\|_E\|\tau_j\|_F \le M\delta^j$ for $j > 0$. Hence

$$\|Tx\|_F \le \sum_{j=1}^{\infty} M\delta^j \|x\|_F$$

so that $\|T\|_F \le C_1$ for some constant C_1 depending only on E, F. ∎

Although Lemma 3.3 is enough for most of our purposes, there are some possible modifications. First we give a simple argument in the case $F = \ell_\infty$, which will be useful later.

LEMMA 3.4. *Suppose E is a Köthe sequence space with* (RSP) *and that $F = \ell_\infty(\mathbb{J})$. Suppose (E, F) is exponentially separated. Then (E, F) has* (RSP).

Proof. We may assume that for some $C_0, \beta > 0$ we have $\|e_m\|_E \le C_0 2^{-\beta n}\|e_{m+n}\|_E$. In this case we proceed as in Lemma 3.3 but note that

$$\|Tx\|_F = \sup_{k\in J} |Tx(k)| .$$

If $k \in \operatorname{supp} y_n$,

$$|Tx(k)| = |\langle x, x_n^*\rangle| y_n(k) \le \|x_{(-\infty,k)}\|_E \|e_k\|_E^{-1}$$
$$\le \sum_{j<k} |x_j|\|e_j\|_E\|e_k\|_E^{-1} \le C_0 \sum_{j<k} |x_j| 2^{-\beta(k-j)} \le C_1\|x\|_F$$

for a suitable C_1. ∎

Another version of Lemma 3.3, which actually generalizes Lemma 3.4, is given by

LEMMA 3.5. *Suppose (E, F) is exponentially separated, E has* (RSP) *and that either* (a) *there exists $1 \le p \le \infty$ so that E has a lower p-estimate and F has an upper p-estimate or* (b) *E is r-concave for some $r < \infty$ and there exists $1 < q < \infty$ so that E has an upper q-estimate and F has a lower q-estimate. Then (E, F) has* (RSP).

Proof. For (a) we note that the case $p = \infty$ is essentially covered in Lemma 3.4. Suppose $p < \infty$. By Lemma 3.2 there is a constant C_1 so that if $\operatorname{supp} x < \operatorname{supp} y$ then $\|x\|_E\|y\|_F \le C_1\|x\|_F\|y\|_E$. There is also a constant C_2 so that if $u_1, \dots, u_n$ are disjoint vectors in E or F,

$$\Big(\sum_{j=1}^n \|u_j\|_E^p\Big)^{1/p} \le C_2\Big\|\sum_{j=1}^n u_j\Big\|_E ,$$
$$\Big\|\sum_{j=1}^n u_j\Big\|_F \le C_2\Big(\sum_{j=1}^n \|u_j\|_F^p\Big)^{1/p} .$$

We suppose $(x_n, y_n)_{n\in I}$ is a positive interlaced pair with $\|y_n\|_E \le \|x_n\|_E = 1$. Define T as in Lemma 3.3, and set $J_k = \operatorname{supp} x_k^*$. Now if $x \in F$,

$$\|Tx\|_F = \Big\|\sum_{k\in A} x_k^*(x) y_{k+1}\Big\|_F \le C_2\Big(\sum_{k\in A} \|x_{J_k}\|_E^p \|y_{k+1}\|_F^p\Big)^{1/p}$$
$$\le C_2 C_1\Big(\sum_{k\in A} \|x_{J_k}\|_F^p\Big)^{1/p} \le C_2^2 C_1\|x\|_F$$

and (a) follows.

We turn to the proof of (b). Let $1 < p < \infty$ be conjugate to q. Then $\widetilde{E}^*$ has (RSP) by Proposition 2.1. Also, $(\widetilde{E}^*, \widetilde{F}^*)$ is exponentially separated, and $\widetilde{E}^*$ has a lower p-estimate while $\widetilde{F}^*$ has an upper p-estimate; thus by (a) the couple $(\widetilde{E}^*, \widetilde{F}^*)$ has (RSP). It follows that (F^*, E^*) has (LSP). We further can assume, by renorming, that E has an upper p-estimate with constant 1 and an r-concavity constant 1 (apply Lindenstrauss–Tzafriri [25], p. 88, Lemma 1.f.11, to E^*). Let C_1 be the associated (LSP) constant for this couple. We first prove a claim:

CLAIM. *There exist constants C_2 and $\delta < 1$ depending only on (E, F) with the following property. Suppose $\{x_n, y_n\}_{n\in I}$ is a positive interlaced pair of sequences with $\|x_n\|_E = \|y_n\|_E = 1$. Then there is a subset D of $\mathbb{J}$ and a positive matrix operator S with $\|S\|_{(E,F)} \le C_2$ so that $Sx_n = P_D y_n$ and $\|y_n - P_D y_n\|_E \le \delta$, whenever $n \in I$.*

Choose $x_k^* \ge 0$ with $\operatorname{supp} x_k^* \subset \operatorname{supp} x_k$ and $\|x_k^*\|_E = \|x_k\|_E = \langle x_k, x_k^*\rangle = 1$. Similarly, choose $y_k^* \ge 0$ with $\operatorname{supp} y_k^* \subset \operatorname{supp} y_k$ and $\|y_k^*\|_E = \|y_k\|_E = \langle y_k, y_k^*\rangle$. We begin by using the (LSP) property of (F^*, E^*) to produce a positive matrix V on E with $\|V\|_{(E,F)} \le C_1$ and $V^* y_k^* = x_k^*$ whenever $k \in I$. Thus $\langle Vx_k, y_k^*\rangle = 1$.

Fix $\tau > 0$ small enough so that $\frac{1}{2}\tau - \frac{1}{p}C_1^p\tau^p = \gamma > 0$. Let D_k be the set of $j \in \operatorname{supp} y_k$ so that $2Vx_k(j) \ge y_k(j)$. Let $D = \bigcup_{k\in I} D_k$. Clearly there is a positive matrix S with $\|S\|_{(E,F)} \le 2C_1 = C_2$ and $Sx_k = P_D y_k$. Now

observe that $\langle Vx_k - P_D V x_k, y_k^* \rangle \le \frac{1}{2}$ so that

$$\langle \tau P_D V x_k + y_k - P_D y_k, y_k^* \rangle \ge 1 + \frac{1}{2}\tau - \|P_D y_k\|_E .$$

Thus

$$1 + \frac{1}{2}\tau - \|P_D y_k\|_E \le (C_1^p \tau^p + 1)^{1/p} \le 1 + \frac{1}{p} C_1^p \tau^p .$$

Upon reorganization this yields

$$\|P_D y_k\|_E \ge \frac{1}{2}\tau - \frac{1}{p} C_1^p \tau^p = \gamma .$$

This in turn implies

$$\|y_k - P_D y_k\|_E \le (1 - \gamma^r)^{1/r} = \delta < 1 .$$

This establishes the claim.

To complete the proof from the claim is quite easy by an inductive argument. We may clearly construct a disjoint sequence of subsets $(D_n)_{n\ge1}$ of $\mathbb{J}$ and a sequence of positive matrix operators $(S_n)_{n\ge1}$ with $\|S_n\|_{(E,F)} \le 2C_1\delta^{n-1}$ and so that $S_n x_k = P_{D_n} y_k$ and $\|y_k - \sum_{j=1}^n P_{D_j} y_k\|_E \le \delta^n$. The operator $T = \sum_{n=1}^\infty S_n$ is a positive matrix and $Tx_k = y_k$; further $\|T\|_{(E,F)} \le 2C_1(1-\delta)^{-1}$. ∎

PROPOSITION 3.6. *Suppose (E,F) is a pair of Köthe sequence spaces. Suppose either*

(a) *(E,F) is exponentially separated, F is r-concave for some $r < \infty$, and there exists $1 < p < \infty$ so that E has a lower p-estimate and F has an upper p-estimate, or*

(b) $\kappa_-(E)\kappa_+(F) < 1$.

Then (E,F) has (SP) *if and only if E has* (RSP) *and F has* (LSP).

Proof. (a) We use Lemma 3.5 to show that (E,F) has (RSP) and $(\widetilde{F},\widetilde{E})$ has (RSP) and the result follows.

(b) This is immediate from Lemma 3.3. ∎

4. Calderón couples of sequence spaces. We now turn to calculating the K-functional for an exponentially separated pair.

LEMMA 4.1. *Suppose (E,F) is exponentially separated. Then there is a constant C_2 so that if $\varrho(a) \le t \le \varrho(a+1)$,*

$$K(t,x) \le \|x_{(-\infty,a]}\|_E + t\|x_{(a,\infty)}\|_F \le C_2 K(t,x) .$$

In particular,

$$\|x_{(-\infty,a]}\|_E + \varrho(a)\|x_{[a,\infty)}\|_F \le C_2 K(\varrho(a), x) .$$

Similarly, if $t \le \varrho(a)$ for all a (in the case $\mathbb{J} = \mathbb{Z}_+$) then

$$t\|x\|_F \le C_2 K(t,x)$$

while if $t \ge \varrho(a)$ for all a (when $\mathbb{J} = \mathbb{Z}_-$) then

$$\|x\|_E \le C_2 K(t,x) .$$

Proof. If $\operatorname{supp} x \subset (-\infty, a]$ then it follows from Lemma 3.2 that $C_1 K(\varrho(a), x) \ge \|x\|_E$. Similarly, if $\operatorname{supp} x \subset [a,\infty)$ then $C_1 K(\varrho(a), x) \ge \varrho(a)\|x\|_F$. Combining these statements gives the results. ∎

THEOREM 4.2. *Suppose (E,F) is exponentially separated and forms a Calderón pair. Then E satisfies* (RSP) *and F satisfies* (LSP).

Proof. First we remark that it suffices to prove the result for E. Once this is established we can apply an order-reversal argument to get the result for F. Indeed, $(\widetilde{F},\widetilde{E})$ is also exponentially separated and a Calderón pair; thus $\widetilde{F}$ has (RSP) and F has (LSP).

We will suppose that $\varrho(m) \le C_0 2^{-n\beta}\varrho(m+n)$ for $m, m+n \in \mathbb{J}$ and that C_1 and C_2 are the constants given in Lemmas 3.2 and 4.1.

We now introduce a notion which helps in the argument. An *admissible pair* is a pair (x, I) where I is a finite interval in $\mathbb{J}$ and x is a positive vector with $\operatorname{supp} x \subset I$, $\max(\operatorname{supp} x) < \max I$, and $\|x\|_E = 1$. An *admissible family* is a finite collection $\mathcal{F} = (x_k, I_k)_{k=1}^n$ of admissible pairs so that (I_k) are pairwise disjoint. We define $\operatorname{supp}\mathcal{F} = \bigcup I_k$. If $\mathcal{F}$ is an admissible family then we define $\Gamma(\mathcal{F})$ to be the least constant M so that if $(y_k)_{k=1}^n$ satisfy $\|y_k\|_E \le 1$, $\operatorname{supp} y_k \subset I_k$ and $\operatorname{supp} x_k < \operatorname{supp} y_k$, then there exists $T \in \mathcal{A}(E,F)$ with $\|T\|_{(E,F)} \le M$ and $Tx_k = y_k$ for $1 \le k \le n$. Notice that since $\max(\operatorname{supp} x_k) < \max I_k$ there is "room" for some y_k satisfying our hypotheses. It is not difficult to show that such a $\Gamma(\mathcal{F})$ is well-defined since we can restrict the problem for each such family to a finite-dimensional space.

We next make the remark that if T is such an optimal choice of operator then T can be replaced without altering its norm by $\sum_{k=1}^n P_{I_k} T P_{I_k}$. Thus it can be assumed that $Tx = 0$ for any x whose support is disjoint from $\bigcup I_k$. Now suppose $\mathcal{F}$ and $\mathcal{G}$ are two admissible families with disjoint supports so that their union $\mathcal{F} \cup \mathcal{G}$ is also admissible. Then using the above remark it is clear that we can simply add optimal operators to obtain

$$\Gamma(\mathcal{F} \cup \mathcal{G}) \le \Gamma(\mathcal{F}) + \Gamma(\mathcal{G}) . \tag{1}$$

Next suppose $\mathcal{F}$ is a single admissible pair (x, I). Suppose y is supported on I and satisfies $\|y\|_E \le 1$, and $\operatorname{supp} x < \operatorname{supp} y$. Then we can choose $x^* \in E^*$ with $\|x^*\|_{E^*} = 1$, $\operatorname{supp} x^* \subset \operatorname{supp} x$ and $\langle x, x^* \rangle = 1$. Consider the operator S defined by $S\xi = \langle \xi, x^* \rangle y$. Of course $\|S\|_E \le 1$. Now suppose the

maximum of $\operatorname{supp} x$ is a. Then

$$\|S\xi\|_F \le \|y\|_F \|\xi_{(-\infty,a]}\|_E \le C_1^2 \|\xi\|_F$$

where C_1 is the constant of Lemma 3.2. Hence $\Gamma(\mathcal{F}) \le C_1^2$. It then follows by the addition principle (1) that if $|\mathcal{F}| = n$ then $\Gamma(\mathcal{F}) \le nC_1^2$.

Now we seek to prove that $\Gamma(\mathcal{F})$ is bounded over all admissible families. Let us suppose on the contrary that it is not. We then can construct inductively a sequence of admissible families $(\mathcal{F}_n)$ for $n \in \mathbb{N}$ and an increasing sequence of integers (m_n) so that $\operatorname{supp} \mathcal{F}_n \subset [-m_n, m_n]$ and $\Gamma(\mathcal{F}_{n+1}) \ge n(n+1) + C_1^2(2m_n + 2n + 1)$.

Now refine $\mathcal{F}_n$ by deleting all pairs (x, I) so that I intersects $[-m_n - n, m_n + n]$. This removes at most $2m_n + 1$ pairs and creates a new admissible family $\mathcal{F}_n'$ so that $\Gamma(\mathcal{F}_n') \ge n(n+1)$. The families $\mathcal{F}_n'$ are now disjoint. If we write the members of $\mathcal{F}_n'$ in increasing order of support as $(x_k, I_k)_{k=1}^N$ then we can define $\mathcal{F}_{n,r}$ for $0 \le r \le n$ to be the family of all (x_k, I_k) where $k \equiv r \bmod (n+1)$. At least one of $\mathcal{F}_{n,r}$ satisfies $\Gamma(\mathcal{F}_{n,r}) \ge n$ by (1). Call this family $\mathcal{G}_n$. We note that if (x, I) and (y, J) are two consecutive members of $\mathcal{G}_n$ then $I + n < J$ (since n nontrivial intervals in $\mathcal{F}_n$ lie between I and J). Furthermore, there is a gap of at least n between any interval represented in $\mathcal{G}_n$ and any interval represented in $\mathcal{G}_k$ for some $k < n$.

Finally, let us consider the union of all $\mathcal{G}_n$ for $n \ge 1$. This may be written as a sequence of admissible pairs $(x_k, I_k)_{k\in A}$ where A is one of the sets $\mathbb{Z}, \mathbb{Z}_-, \mathbb{Z}_+$ and $I_k < I_{k+1}$ for all $k, k+1 \in A$. Write $I_k = [a_k, b_k]$. Then $b_k < a_{k+1}$ whenever $k, k+1 \in A$. Furthermore, the gaps between the intervals tend to infinity as $|k| \to \infty$. Precisely, if $\sigma_k = (a_{k+1} - b_k)$ then $\lim_{|k|\to\infty} \sigma_k = \infty$. Now let $d_k = \max(\operatorname{supp} x_k)$ so that $a_k \le d_k < b_k$. Let $J_k = (d_k, b_k]$ for $k \in A$.

We now claim:

CLAIM. *There exists a finite subset A_0 of A and a constant M so that if $A_1 = A \setminus A_0$, and if $(y_k)_{k\in A_1}$ is any sequence satisfying $\|y_k\|_E = 1$ and $\operatorname{supp} y_k \subset J_k$, then there exists $T \in \mathcal{A}(E,F)$ with $\|T\|_{(E,F)} \le M$ and $\|Tx_k - y_k\|_E \le \frac{1}{2}$.*

Let us first assume the claim is established. We consider the space $\mathcal{Y} = \ell_\infty(E(J_k))_{k\in A_1}$ and the map $\mathcal{S} : \mathcal{A}(E,F) \to \mathcal{Y}$ defined by $\mathcal{S}(T) = (P_{J_k} T x_k)_{k\in A_1}$. Clearly $\|\mathcal{S}\| \le 1$ and it follows from the claim that if $\mathbf{y} = (y_k)_{k\in A_1} \in \mathcal{Y}$ there exists $T \in \mathcal{A}(E,F)$ with $\|T\|_{(E,F)} \le M\|\mathbf{y}\|$ and $\|\mathcal{S}(T) - \mathbf{y}\| \le \frac{1}{2}\|\mathbf{y}\|$. By a well-known argument from the Open Mapping Theorem this is enough to show that $\mathcal{S}$ is onto and indeed if $\|\mathbf{y}\| \le 1$ then there exists T with $\|T\|_{(E,F)} \le 2M$ and $\mathcal{S}(T) = \mathbf{y}$.

Now suppose $\mathcal{G}_n = \{(x_k, I_k)\}_{k\in B_n}$ where $B_n \subset A_1$. Then if $(y_k)_{k\in B_n}$ satisfy $\|y_k\|_E \le 1$, and $\operatorname{supp} x_k < \operatorname{supp} y_k \subset I_k$ it follows that there is

an operator $T \in \mathcal{A}(E,F)$ with $\|T\|_{(E,F)} \le 2M$ and $P_{J_k} T x_k = y_k$. If we set $T_0 = \sum_{k\in B_n} P_{J_k} T P_{I_k}$ then $\|T_0\|_{(E,F)} \le 2M$ and $T_0 x_k = y_k$. Thus $\Gamma(\mathcal{G}_n) \le 2M$. Now since A_0 is finite we conclude that $\Gamma(\mathcal{G}_n) \le 2M$ for all but finitely many n. This contradicts the original construction of $\mathcal{G}_n$. The contradiction shows that there is a constant M_0 so that $\Gamma(\mathcal{F}) \le M_0$ for all admissible families $\mathcal{F}$. In particular, if we have a finite set of finitely supported vectors $x_1, x_2, \dots, x_n, y_1, \dots, y_n$ so that $\operatorname{supp} x_1 < \operatorname{supp} y_1 < \dots < \operatorname{supp} x_n < \operatorname{supp} y_n$ and $\|x_k\|_E = 1$ for all k and $\|y_k\|_E \le 1$ then there is an operator $T : E \to E$ with $\|T\|_E \le 2M_0$ and $Tx_k = y_k$. Hence for any $\alpha_1, \dots, \alpha_n$ we would have

$$\Big\| \sum_{k=1}^n \alpha_k y_k \Big\|_E \le 2M_0 \Big\| \sum_{k=1}^n \alpha_k x_k \Big\|$$

and this means that E has (RSP).

Thus it only remains to prove the claim. We start by defining a sequence $(\lambda_k)_{k\in A}$ such that $\lambda_{k+1} - \lambda_k = \frac{1}{2}\beta\sigma_k$. We next make some initial observations. Let us suppose that $\operatorname{supp} u_k \subset I_k$ for $k \in A$ and $\|u_k\|_E = 1$ for all k. We claim that there exists a constant C_3 independent of the choice of (u_k) so that if $k \in A$ then

$$\Big\| \sum_{j\le k} 2^{\lambda_j} u_j \Big\|_E \le C_3 2^{\lambda_k} \tag{2}$$

and

$$\Big\| \sum_{j\ge k} 2^{\lambda_j} u_j \Big\|_F \le C_3 2^{\lambda_k} \|u_k\|_F \,. \tag{3}$$

In fact, (2) follows easily from the fact that if $j \le k$ then

$$\lambda_j = \lambda_k - \frac{1}{2}\beta \sum_{i=j}^{k-1} \sigma_i \le \lambda_k - \frac{1}{2}(k-j)\beta \,.$$

For (3) we note that if $j \ge k$,

$$\begin{aligned} \|u_j\|_F &\le C_1 \varrho(a_j)^{-1} \le C_1 C_0 2^{-\beta(a_j - b_k)} \varrho(b_k)^{-1} \\ &\le C_1^2 C_0 2^{-\beta(a_j - b_k)} \|u_k\|_F \le C_1^2 C_0 2^{-2(\lambda_j - \lambda_k)} \|u_k\|_F \end{aligned}$$

so that

$$2^{\lambda_j} \|u_j\|_F \le C_1^2 C_0 2^{-\frac{1}{2}\beta(j-k)} 2^{\lambda_k} \|u_k\|_F$$

from which (3) will follow.

In particular, let us define $z = \sum_{k\in A} 2^{\lambda_k} x_k$. The above calculations show that $z \in E + F$. Since (E,F) is a Calderón couple there is a constant $M_0 = M_0(z)$ so that if $u \in E + F$ and $K(t,u) \le K(t,z)$ for all t then there exists $T \in \mathcal{A}(E,F)$ with $\|T\|_{(E,F)} \le M_0$ and $Tz = u$.

Now suppose $(y_k)_{k\in A}$ is any sequence with $\|y_k\|_E = 1$ and $\operatorname{supp} y_k \subset J_k$. We set $v = \sum 2^{\lambda_k} y_k \in E + F$. We turn to comparing $K(t,v)$ with $K(t,z)$. Let us note first that for every $k \in A$ we have

$$\|y_k\|_F \le C_1 \varrho(d_k)^{-1} \le C_1^2 \|x_k\|_F .$$

If t satisfies $t \le \varrho(a_k)$ for all k then we must have $A = \mathbb{Z}_+$ and we make the estimate

$$K(t,v) \le t\|v\|_F \le C_3 t 2^{\lambda_0} \|y_0\|_F \le C_3 C_1^2 t 2^{\lambda_0} \|x_0\|_F$$

so that by Lemma 3.3,

$$K(t,v) \le C_3 C_2 C_1^2 K(t,z) .$$

Similarly, if $t \ge \varrho(b_k)$ for all $k \in A$ then we can have $A = \mathbb{Z}_-$ and we make a similar estimate

$$K(t,v) \le \|v\|_E \le C_3 2^{\lambda-1} \le C_3 \|z_{[a_{-1},b_{-1}]}\|_E \le C_2 C_3 K(t,z) .$$

In the other cases we first consider the case when $\varrho(n) \le t \le \varrho(n+1)$ for some n in an interval $[a_k, d_k)$. Then $K(t,x_k) \ge tC_1^{-1}\varrho(d_k)^{-1}$ by Lemma 3.2. Hence

$$\begin{aligned} K(t, v_{[a_k,\infty)}) &\le C_3 t \|v_{[a_k,a_{k+1}]}\|_F = C_3 t 2^{\lambda_k} \|y_k\|_F \le C_3 C_1 t 2^{\lambda_k} \varrho(d_k)^{-1} \\ &\le C_3 C_1^2 2^{\lambda_k} K(t,x_k) \le C_3 C_1^2 K(t,z) . \end{aligned}$$

If k is the initial element of A we are done. Otherwise,

$$K(t, v_{(-\infty,a_k)}) \le C_3 2^{\lambda_{k-1}} \le C_3 \|z_{(-\infty,a_k)}\|_E \le C_3 K(t,z) .$$

Combining in this case we have $K(t,v) \le CK(t,z)$ for some constant C depending only on C_1, C_2 and C_3.

For the final case, we can suppose there exists n not in any interval $[a_k, d_k)$ and such that $\varrho(n) \le t \le \varrho(n+1)$; it may also be assumed that there exists $k \in A$ with $k+1 \in A$ and $d_k \le n \le a_{k+1}$. Then by Lemma 3.3,

$$\|z_{(-\infty,n]}\|_E + t\|z_{(n,\infty)}\|_F \le C_2 K(t,z) .$$

Now

$$\|v_{(-\infty,b_k]}\|_E \le C_3 2^{\lambda_k} \le C_3 \|z_{(-\infty,n]}\|_E .$$

Also

$$\|v_{(b_k,\infty)}\|_F \le C_3 2^{\lambda_{k+1}} \|y_{k+1}\|_F \le C_1^2 C_3 \|z_{(n,\infty)}\|_F .$$

Thus combining all the cases there exists C_4 independent of (y_k) so that $K(t,v) \le C_4 K(t,z)$. Hence there is an operator $T \in \mathcal{A}(E,F)$ with $Tz = v$ and $\|T\|_{(E,F)} \le C_4 M_0$.

Now for fixed $k \in A$ assume first that k is not the initial element of A. Then

$$\|z_{(-\infty,a_k)}\|_E \le C_3 2^{\lambda_{k-1}} \le C_3 2^{-\frac{1}{2}\beta\sigma_{k-1}} 2^{\lambda_k} .$$

Thus we have

$$\|T(z_{(-\infty,a_k)})\|_E \le C_4 C_3 M_0 2^{-\frac{1}{2}\beta\sigma_{k-1}} 2^{\lambda_k} .$$

If k is not the final element,

$$\begin{aligned} \|z_{(b_k,\infty)}\|_F &\le C_3 2^{\lambda_{k+1}} \|y_{k+1}\|_F \\ &\le C_1 C_3 2^{\lambda_{k+1}} \varrho(a_{k+1})^{-1} \le C_0 C_1 C_3 2^{\lambda_{k+1} - \beta\sigma_k} \varrho(b_k)^{-1} . \end{aligned}$$

Thus if $f = T(z_{(b_k,\infty)})_{(d_k,b_k]}$ then

$$\begin{aligned} \|f\|_E &\le C_1 \varrho(b_k) \|f\|_F \\ &\le C_1 C_4 M_0 \varrho(b_k) \|z_{(b_k,\infty)}\|_F \le C_0 C_1^2 C_3 C_4 M_0 2^{-\frac{1}{2}\beta\sigma_k} 2^{\lambda_k} . \end{aligned}$$

It follows that if k is not an initial or final element of A,

$$\|y_k - P_{J_k} T x_k\|_E \le C_5 M_0 2^{-\frac{1}{2}\beta\tau_k}$$

where C_5 is a constant depending only on E and F and $\tau_k = \min(\sigma_{k-1}, \sigma_k)$. Now if we set $S = \sum_{k\in A} P_{J_k} T P_{I_k}$ then $\|S\|_{(E,F)} \le C_4 M_0$. Further, if we let A_1 be the set of $k \in A$ so that k is not an initial or final element and $C_5 M_0^2 2^{-\frac{1}{2}\beta\tau_k} < 1/2$ then $A_0 = A \setminus A_1$ is finite and $\|Sx_k - y_k\| < 1/2$ for $k \in A_1$. Thus the claim is established and the proof is complete. ∎

LEMMA 4.3. *Suppose (E,F) satisfies* (RSP). *Then there is a constant C so that if $0 \le x, y \in E+F$ and $\|y_{(-\infty,a]}\|_E \le \|x_{(-\infty,a]}\|_E$ for all $a \in \mathbb{J}$ then there exists a positive $T \in \mathcal{A}(E,F)$ with $\|T\|_{(E,F)} \le C$ and $Tx = y$.*

Proof. By applying the argument of Lemma 2.6 we deduce from (RSP) the existence of a constant C_0 so that if A is an interval in $\mathbb{Z}$, $(J_k)_{k\in A}$ is a collection of finite intervals in $\mathbb{J}$ with $J_k < J_{k+1}$ whenever $k, k+1 \in A$ and $(x_k)_{k\in A}, (y_k)_{k\in A}$ are positive and satisfy $\operatorname{supp} x_k, \operatorname{supp} y_k \subset J_k$ and $\|y_{k+1}\|_E \le \|x_k\|_E$ for $k, k+1 \in A$ then there is a positive matrix operator T with $\|T\|_{(E,F)} \le C_0$ and $Tx_k = y_{k+1}$ whenever $k, k+1 \in A$.

Let us prove the lemma when x, y have disjoint supports. We first define a function $\sigma : \mathbb{Z} \to \mathbb{Z} \cup \{\pm\infty\}$ by setting $\sigma(k) = -\infty$ if $k < \mathbb{J}$, $\sigma(k) = \infty$ if $k > \mathbb{J}$ and otherwise $\sigma(k)$ is the greatest $j \in \mathbb{Z} \cup \{-\infty\}$ so that $\|x_{(-\infty,k]}\|_E \ge 4^j$.

Let $I_0 = \{k \in \mathbb{J} : \sigma(k) > \sigma(k-1)\}$. We then let I be the subset of I_0 of all k so that if $n \in I_0$ with $n < k$ then $\|x_{(-\infty,n]}\|_E \le \frac{1}{2}\|x_{(-\infty,k]}\|_E$.

We can now index I as $(a_n)_{n\in A}$ where A is an interval in $\mathbb{Z}$ which can be assumed to have 0 as its initial element if I is bounded below.

We now define B to be $\mathbb{Z}_-$ when $\inf_{k\in\mathbb{Z}} \sigma(k) > -\infty$, and empty otherwise. We only need to introduce B in the case $\lim_{a\to-\infty} \|x_{(-\infty,a]}\|_E > 0$. If B is nonempty then I is bounded below and there exists a greatest λ so that $\|x_{(-\infty,k]}\|_E \ge 4^\lambda$ for every k (in this case $\mathbb{J}$ cannot be bounded below). We must have $\|x_{(-\infty,k]}\|_E < 4^{\lambda+1}$ whenever $k < a_0$. It follows that we may

pick a_{-1} so that $\|x_{(a_{-1},a_0]}\|_E \geq 4^{\sigma(a_0)-1}$ and then inductively a_{-n} so that $\|x_{(a_{-n},a_{-(n-1)}]}\|_E \geq 4^{\lambda-1}$.

In this way we define $(a_n)_{n\in A\cup B}$. We now let $x_n = x_{(a_{n-1},a_n]}$ and $y_n = y_{(a_{n-1},a_n]}$ if $n \in A \cup B$ is not the initial element of $A \cup B$; if $n = 0$ is the initial element we let $x_0 = x_{(-\infty,a_0]}$ and $y_0 = y_{(-\infty,a_0]}$. If n is the final element of $A \cup B$ we set $y_{n+1} = y_{(a_n,\infty]}$. We may now verify that $\sum_{n\in A\cup B} x_n \leq x$. We also claim that $\sum_{n\in A\cup B} y_{n+1} = y$. If $A\cup B = \mathbb{Z}$ this is clear. If $A\cup B = (-\infty, n]$ for some n it follows from our definition of y_{n+1}. If $A\cup B$ is bounded below (by 0) then B is empty and hence $\sigma(a_0 - 1) = -\infty$. Thus $y_0 = 0$ and we obtain our claim easily.

We first prove that if $n, n+1 \in A$ then $\sigma(a_{n+1} - 1) \leq \sigma(a_n) + 1$. If not, there exists a first k_1 so that $\sigma(k_1) \geq \sigma(a_n) + 1$ and a first k_2 so that $\sigma(k_2) \geq \sigma(a_n) + 2$ and $a_n < k_1 \leq k_2 \leq a_{n+1} - 1$. Then k_1, k_2 are in $I_0 \setminus I$. Thus $\|x_{(-\infty,a_n]}\|_E > \frac{1}{2}\|x_{(-\infty,k_1]}\|_E$. The equality $k_1 = k_2$ would entail $\|x_{(-\infty,k_1]}\|_E \geq 4^{\sigma(a_n)+2}$ and thus $\|x_{(-\infty,a_n]}\|_E > 4^{\sigma(a_n)+1}$, which contradicts the definition of $\sigma(a_n)$. Thus $k_1 < k_2$ and we conclude also that $\|x_{(-\infty,k_1]}\|_E > \frac{1}{2}\|x_{(-\infty,k_2]}\|_E$ so that $\|x_{(-\infty,a_n]}\|_E > \frac{1}{4}\|x_{(-\infty,k_2]}\|_E$, which implies the absurd conclusion $\sigma(k_2) \leq \sigma(a_n) + 1$. Thus, as claimed, $\sigma(a_{n+1} - 1) \leq \sigma(a_n) + 1$.

The same argument shows that if A is bounded above then if $k > a_n$ we must have $\sigma(k) \leq \sigma(a_n) + 1$.

Now if $n, n+1 \in A$ we can argue that since x, y have disjoint supports, y_{n+1} is supported on (a_n, a_{n+1}) and thus $\|y_{n+1}\|_E \leq \|x_{(-\infty,a_{n+1}-1]}\|_E \leq 4^{\sigma(a_n)+2}$. Similarly, let n be the last element of A. Then for all $k > a_n$, $\|y_{(-\infty,k]}\|_E \leq \|x_{(-\infty,k]}\|_E \leq 4^{\sigma(k)+1} \leq 4^{\sigma(a_n)+2}$. Thus $\|y_{n+1}\|_E \leq 4^{\sigma(a_n)+2}$.

On the other hand, if n is not the initial element of A,

$$\|x_n\|_E \geq \|x_{(-\infty,a_n]}\|_E - \|x_{(-\infty,a_{n-1}]}\|_E \geq \frac{1}{2}\cdot 4^{\sigma(a_n)}.$$

If $n = 0$ is the initial element, we either have, if $B = \emptyset$, $x_0 = x_{(-\infty,a_0]}$ so that $\|x_0\| \geq 4^{\sigma(a_0)}$, or if $B \neq \emptyset$ then $\|x_1\| \geq 4^{\sigma(a_0)-1}$. In all such cases, if $n \in A$ we have $\|y_{n+1}\|_E \leq 4^3\|x_n\|_E$.

Next suppose $n, n+1 \in B$. Then $\|y_{n+1}\|_E \leq \|x_{(-\infty,a_{n+1})}\|_E \leq 4^{\lambda+1}$ while $\|x_n\| \geq 4^{\lambda-1}$. Thus $\|y_{n+1}\|_E \leq 4^2\|x_n\|_E$.

Finally, consider the case $n = -1 \in B$ and $n+1 = 0 \in A$. Then since a_0 is in the support of x we have $\|y_{n+1}\|_E \leq \|x_{(-\infty,a_0-1)}\|_E < 4^{\lambda+1}$. However, $\|x_n\|_E \geq 4^{\lambda-1}$ so that $\|y_{n+1}\|_E \leq 4^2\|x_n\|_E$.

Combining all cases, we conclude that there is a positive operator T with $\|T\|_{(E,F)} \leq 4^3 C_0$ so that $Tx_n = y_{n+1}$. Now it is clear that $\sum_{n\in A\cup B} x_n \leq x$ while $\sum_{n\in A\cup B} y_{n+1} = y$. Thus if $S = \sum_{n\in A\cup B} P_{\operatorname{supp} y_{n+1}} T P_{\operatorname{supp} x_n}$ then $\|S\|_{(E,F)} \leq 64C_0$ and $Sx = y$. Thus the lemma is established in the case when x and y have disjoint supports.

For the general case we let $I = \{n \in \mathbb{J} : y_n > 2x_n\}$. Let $J = \mathbb{J}\setminus I$. Then set $u = x_J$ and $v = y_I$. For any $k \in \mathbb{J}$ we have $\|x_{I\cap(-\infty,k]}\|_E \leq \frac{1}{2}\|y_{(-\infty,k]}\|_E \leq \frac{1}{2}\|x_{(-\infty,k]}\|_E$. Thus $\|u_{(-\infty,k]}\|_E \geq \frac{1}{2}\|x_{(-\infty,k]}\|_E$. Hence there is a positive operator S in $\mathcal{A}(E,F)$ with $Su = v$ and $\|S\|_{(E,F)} \leq 128C_0$. On the other hand, $y_J \leq 2x$ and so there is a multiplication operator $V \in (E,F)$ with $\|V\|_{(E,F)} \leq 2$ and $Vx = y_J$. Finally, the operator $T = SP_J + V$ establishes the lemma. ∎

LEMMA 4.4. *Suppose (E,F) is exponentially separated and satisfies* (SP). *Then there exists a constant C so that if $0 \leq x, y \in E + F$ and $K(t,y) \leq K(t,x)$ for all $t \geq 0$ then there exists a positive matrix $T \in \mathcal{A}(E,F)$ with $\|T\|_{(E,F)} \leq C$ and $Tx = y$.*

Proof. It follows from Lemma 4.1 that there is a constant C_2 so that for all $a \in \mathbb{J}$ we have, whenever $K(t,y) \leq K(t,x)$ for all $t \geq 0$,

$$\max(\|y_{(-\infty,a]}\|_E, \varrho(a)\|y_{[a,\infty)}\|_F) \leq 2C_2 \max(\|x_{(-\infty,a]}\|_E, \varrho(a)\|x_{(a,\infty)}\|_F).$$

Thus for every a either

$$\|y_{(-\infty,a]}\|_E \leq 2C_2\|x_{(-\infty,a]}\|_E \tag{4}$$

or

$$\|y_{[a,\infty)}\|_F \leq 2C_2\|x_{[a,\infty)}\|_F. \tag{5}$$

Let J_1 be the set of a so that (4) holds and let $J_2 = \mathbb{J}\setminus J_1$. Since (E,F) has (RSP) we can apply Lemma 4.3 to deduce the existence of a positive matrix T_1 with $\|T_1\|_{(E,F)} \leq C_3$ where C_3 depends only on (E,F) and $T_1 x = y_{J_1}$. Similarly, since (E,F) has (LSP) we can find a positive matrix T_2 with $\|T_2\|_{(E,F)} \leq C_3$ so that $T_2 x = y_{J_2}$. Then $(T_1 + T_2)(x) = y$. ∎

THEOREM 4.5. *Let (E,F) be a pair of Köthe sequence spaces. Suppose either*

(a) $\kappa_-(E)\kappa_+(F) < 1$, *or*

(b) *(E,F) is exponentially separated, F is r-concave for some $r < \infty$ and there exists p with $1 \leq p < \infty$ so that E has a lower p-estimate and F has an upper p-estimate.*

Then (E,F) is a (uniform) Calderón couple if and only if E has (RSP) *and F has* (LSP).

Proof. This is an immediate deduction from Proposition 3.6, Theorem 4.2 and Lemma 4.4. ∎

Remark. Note that in fact Lemma 4.4 implies that under these circumstances if $K(t,y) \leq K(t,x)$ for all t and $x, y \geq 0$ then there is a positive operator T with $\|T\|_{(E,F)} < \infty$ and $Tx = y$.

The following theorem is similar to results of Cwikel and Nilsson [18].

THEOREM 4.6. *Let E, F be symmetric sequence spaces on $\mathbb{Z}_+$ and suppose $(E, F(w))$ is a Calderón pair for a weight sequence $w = (w_n)$. Then either $F(w) = F$ (i.e. $0 < \inf w_n \le \sup w_n < \infty$) or $E = \ell_p$, $F = \ell_q$ for some $1 \le p, q \le \infty$.*

Proof. If (w_n) is unbounded we can pass to a subsequence satisfying $w_{n_k} > 2w_{n_{k-1}}$. Then the pair $(E, F(w_{n_k}))$ is a Calderón pair and we can apply Theorem 4.2 to deduce that F has (RSP) and E has (LSP). An application of Proposition 2.3 gives the result. If (w_n^{-1}) is unbounded we can argue similarly. ∎

5. Calderón couples of r.i. spaces. Let Ω denote one of the sets $[0,\infty)$, $[0,1]$ and $\mathbb{N}$. Let $\mathbb{J}$ be the set $\mathbb{Z}$, $\mathbb{Z}_-$, or $\mathbb{Z}_+$ respectively. If X is an r.i. space on Ω (or a symmetric sequence space if $\Omega = \mathbb{N}$) we will associate with X a Köthe sequence space E_X on $\mathbb{J}$. To do this let e_n, $n \in \mathbb{J}$, be defined by $e_n = \chi_{[2^n, 2^{n+1})}$. We then define for $x \in \omega(\mathbb{J})$,

$$\|x\|_{E_X} = \Big\| \sum_{k\in\mathbb{J}} x(k) e_k \Big\|_X .$$

(Here we use e_k with a dual meaning as both the canonical basis element of $\omega(\mathbb{J})$ and as an element of $X(\Omega)$.) We observe that E_X regarded as a subspace of X is 1-complemented by the natural averaging operator. Notice also that $E_{X^*} = E_X^*(2^n)$ is a weighted version of E_X^*. We also note that on E_X we can compute $\|\tau_n\|_{E_X} \le \|D_{2^n}\|_X$ where D_s is the natural dilation operator. Furthermore, it is easy to see that for $f \in X$ we have $D_{2^n} f^* \le \tau_{n+1} P f^*$ where P is the natural averaging projection of X onto E_X; thus $\|D_{2^n}\|_X \le \|\tau_{n+1}\|_{E_X}$. Thus $\kappa_+(E_X) = 2^{1/p_X}$ and $\kappa_-(E_X) = 2^{-1/q_X}$ where p_X and q_X are the Boyd indices of X.

We now show how to build examples of r.i. spaces from sequence spaces. To keep the notation straight we prove our results for the case of function spaces $\Omega = [0,1]$ or $\Omega = [0,\infty)$. However, simple modifications give the analogous results for sequence spaces.

PROPOSITION 5.1. *Let E be a Köthe sequence space on $\mathbb{J}$. Then:*

(1) If $\kappa_+(E) < 2$ there is an r.i. space $X = X(\Omega)$ so that $\|f\|_X$ is equivalent to $\|\sum_{n\in\mathbb{J}} f^*(2^n) e_n\|_E$.

(2) If $\kappa_-(E) < 1 \le \kappa_+(E) < 2$, and X is an r.i. space so that $\|f\|_X$ is equivalent to $\|\sum_{n\in\mathbb{J}} f^*(2^n) e_n\|_E$ then $E_X = E$ (up to equivalence of norm).

Proof. (1) We define X to be the set of measurable functions on Ω such that

$$\|f\|_X = \Big\| \sum_{n\in\mathbb{J}} f^*(2^n) e_n \Big\|_E < \infty .$$

We show that the functional $\|f\|_X$ is equivalent to a norm by computing $\|f^{**}\|_X$ where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$. Then

$$f^{**}(2^n) \le 2^{-n} \sum_{k<n} 2^k f^*(2^k) .$$

Thus

$$\|f^{**}\|_X \le \sum_{j=1}^{\infty} 2^{-j} \Big\| \tau_j \Big(\sum_{n\in\mathbb{J}} f^*(2^n) e_n \Big) \Big\|_E .$$

Thus since $\kappa_+(E) < 2$ there is a constant C_1 so that $\|f^{**}\|_X \le C_1 \|f\|_X$. Since $f \to \|f^{**}\|_X$ is plainly an r.i. norm and the set $\{f : \|f^{**}\|_X \le 1\}$ is closed in measure it is clear that X is an r.i. space.

(2) Let $\|\ \|_X$ denote the quasinorm induced by E. We remark that it follows from (1) that there exists a constant C_2 so that for $f \in X$ we have $\|f^{**}\|_X \le C_2 \|f\|_X$. Now, considering the E_X-quasinorm induced on $\omega(\mathbb{J})$ it is clear that if x is a nonincreasing sequence then $\|x\|_{E_X} = \|x\|_E$. In general, we note that if $f \in E_X$ then for some $C_3 = \sum_{j\ge 0} \|\tau_{-j}\|_E$, since $\kappa_-(E) < 1$,

$$\|\max_{j\ge 0} |\tau_{-j} f|\|_E \le C_3 \|f\|_E$$

so that

$$\|f\|_{E_X} \le C_1 \|f\|_E .$$

For the converse direction we observe that if $f \in X$ it is trivial that $\|D_{2^j} f\|_X \le \|\tau_j\|_E \|f\|_X$. Then

$$\|f\|_E \le \|\max_{j\ge 0} |\tau_{-j} f|\|_E = \|\max_{j\ge 0} |\tau_{-j} f|\|_X \le \Big\| \sum_{j\ge 0} D_{2^{-j}} |f| \Big\|_X$$

$$\le \sum_{j\ge 0} \|D_{2^{-j}} f^{**}\|_X \le C_3 \|f^{**}\|_X \le C_2 C_3 \|f\|_X .$$

Thus E_X is (up to equivalence of norm) identical with E. ∎

Remark. It follows from the above proposition that there is a natural one-one correspondence between r.i. spaces X with Boyd indices satisfying $1 < p_X \le q_X < \infty$ and sequence spaces E on $\mathbb{J}$ with $\kappa_-(E) < 1 \le \kappa_+(E) < 2$ determined by $E = E_X$. Under this correspondence, if $1 < p < \infty$ an r.i. space X with $q_X < \infty$ is a Lorentz space (of order p) if and only if E_X is a weighted ℓ_p-space. For if

$$\|f\|_X = \Big(\int_0^{\infty} (f^*(t) w(t))^p \frac{dt}{t} \Big)^{1/p}$$

where w is an increasing function satisfying $1 < \inf w(2t)/w(t) \leq \sup w(2t)/w(t) < \infty$ then the above proposition shows that $E_X = \ell_p(w_n)$, where $w_n = w(2^n)$. Conversely, if E_X is an ℓ_p-space then $E_X = \ell_p(w_n)$ where the assumption that $q_X < \infty$ enables us to assume $\inf w_{n+1}/w_n > 1$. If we define $w(t) = w_n$ whenever $2^{n-1} < t \leq 2^n$ then it is easy to see that X is a Lorentz space.

We now prove the elementary

PROPOSITION 5.2. *Let (X,Y) be a pair of r.i. spaces on Ω. Then (X,Y) is a Calderón couple if and only if (E_X, E_Y) is a Calderón couple.*

Proof. By using the averaging projection it is clear that if (X,Y) is a Calderón couple then so is (E_X, E_Y). In fact, it is trivial to see that for $f \in E_X + E_Y$ we have $K(t,f;X,Y) = K(t,f;E_X,E_Y)$. Thus if $K(t,g;E_X,E_Y) \leq K(t,f;E_X,E_Y)$ for all t there exists $T \in \mathcal{A}(X,Y)$ so that $Tf = g$. If P is the averaging projection then $PT \in \mathcal{A}(E_X,E_Y)$ and $PTf = g$.

Conversely, suppose (E_X,E_Y) is a Calderón couple. Suppose $f,g \in X+Y$ and $K(t,g;X,Y) \leq K(t,f;X,Y)$ for all $t \geq 0$. We then observe that if $G = \sum_{n\in\mathbb{J}} g^*(2^n)e_n$ and $F = \sum_{n\in\mathbb{J}} f^*(2^n)e_n$ then $g^* \leq G \leq D_2 g^*$ and $f^* \leq F \leq D_2 f^*$, and

$$K(t,G;X,Y) \leq K(t,D_2g^*;X,Y) \leq 2K(t,f;X,Y) \leq 2K(t,F;X,Y)\,.$$

Since F,G are in $E_X + E_Y$ we can deduce the existence of $T \in \mathcal{A}(E_X,E_Y)$ with $TF = G$. Now since $F \leq D_2 f^*$ and $g^* \leq G$ it is clear that there exists $S \in \mathcal{A}(X,Y)$ with $Sf = g$. ∎

Remark. It now follows that every pair of Lorentz spaces whose Boyd indices are finite is a Calderón couple, since every pair of weighted ℓ_p-spaces is a Calderón couple (cf. [36], [13]); this result is due to Cwikel [14] and Merucci [30] for certain special cases.

We introduce the following definitions. We say X is *stretchable* if E_X has (RSP), and we say that X is *compressible* if E_X has (LSP). If X is both stretchable and compressible, we say that it is *elastic*. It is immediate from Proposition 2.1 that X is stretchable if and only if X^* is compressible and *vice versa*; thus elasticity is a self-dual property. We remark that we have no example of a stretchable (or compressible) space which is not already elastic. In fact, we shall see that for Orlicz spaces these concepts do indeed coincide.

THEOREM 5.3. *Let (X,Y) be a pair of r.i. spaces on Ω whose Boyd indices satisfy $p_Y > q_X$. Then (X,Y) is a Calderón couple if and only if X is stretchable and Y is compressible.*

Proof. Since $\kappa_-(E_X) = 2^{-1/q_X}$ and $\kappa_+(E_Y) = 2^{1/p_Y}$ we have $\kappa_-(E_X)\kappa_+(E_Y) < 1$ and so the theorem is immediate from Theorem 4.5 and Proposition 5.2. ∎

If one space is L_∞ we can do rather better.

THEOREM 5.4. *Let X be an r.i. space on $\Omega = [0,1]$ or $\Omega = [0,\infty)$. Then (X,L_∞) is a (uniform) Calderón couple if and only if X is stretchable. Similarly, if X is a symmetric sequence space then (ℓ_∞, X) is a (uniform) Calderón couple if and only if X is stretchable.*

Before proving Theorem 5.4 we state a result which has a very similar proof. We remark that Theorem 5.5 only improves on Theorem 5.3 under the assumption that $p_Y = p = q_X$ since the case $p_Y < q_X$ is already covered.

THEOREM 5.5. *Suppose (X,Y) is a couple of r.i. spaces on Ω so that for some $1 \leq p < \infty$ X is p-concave and Y is p-convex and suppose also that Y is r-concave for some $r < \infty$. Then (X,Y) is a (uniform) Calderón couple if and only if X is stretchable and Y is compressible.*

Proof of Theorems 5.4 and 5.5. Theorem 5.4 corresponds to the case $p = \infty$, and $Y = L_\infty$. We can and do assume that the p-convexity constant of Y and the p-concavity constant of X are both equal to one. Under this hypothesis it is easy to see that, when $p < \infty$, $2^{-k/p}\|e_k\|_X$ is increasing and $2^{-k/p}\|e_k\|_Y$ is decreasing. Thus for $p \leq \infty$, $\varrho(k) = \|e_k\|_X/\|e_k\|_Y$ is an increasing function and $\varrho(k+1) \leq 2\varrho(k)$ whenever $k, k+1 \in \mathbb{J}$. Then for $k \in \mathbb{J}$ we let $I_k = \{n \in \mathbb{J} : 2^k < \varrho(n) \leq 2^{k+1}\}$.

Before continuing let us make a remark which we use several times in the proof. Assuming $p < \infty$ suppose f, g are two finitely supported functions in E_X which satisfy $\|f\|_p = \|g\|_p$ and

$$\int_0^t (f^*(s))^p\,ds \leq \int_0^t (g^*(s))^p\,ds$$

for every $t \geq 0$. Then we have the inequalities $\|f\|_Y \leq \|g\|_Y$ and $\|f\|_X \leq \|g\|_X$. In fact, it follows from a well-known lemma of Hardy, Littlewood and Pólya, [19], [25], p. 124, that $|f^*|^p$ is in the convex hull of the set of all rearrangements of $|g^*|^p$; this can be proved by partitioning the supports of f^*, g^* into finitely many sets of equal measure. The assertion is then a direct consequence of the definitions of p-convexity and p-concavity.

We make some initial remarks which will be needed in both directions of the proof. Each set I_k is an interval (possibly infinite) or is empty. The set of k so that I_k is nonempty is an interval A. Let $E(I_k)$ be the linear span of $(e_n : n \in I_k)$ when $k \in A$. We state the following lemma.

LEMMA 5.6. *If $f, g \in E(I_k)$ then, under the hypotheses of Theorem* 5.5,

$$\|f\|_X\|g\|_p \le 2\|f\|_p\|g\|_X\,, \qquad \|f\|_Y\|g\|_p \le 2\|f\|_p\|g\|_X$$

where $\|\ \|_p$ denotes the usual L_p-norm, so that $\|\sum \alpha_k e_k\|_p = (\sum 2^k|\alpha_k|^p)^{1/p}$. Under the hypotheses of Theorem 5.4, *we have*

$$\|f\|_X\|g\|_\infty \le 4\|f\|_\infty\|g\|_X\,.$$

Proof. In fact, suppose $f, g \in [e_n : a \le n \le b]$ where $a, b \in I_k$, and that neither is zero. We may observe that for all $t \ge 0$ we have

$$2^{-a}\int_0^t (e_a^*(s))^p\,ds \ge \|f\|_p^{-p}\int_0^t (f^*(s))^p\,ds \ge 2^{-b}\int_0^t (e_b^*(s))^p\,ds$$

with similar inequalities for g. It thus follows from the remarks above that

$$2^{-a/p}\|e_a\|_Y \ge \|f\|_p^{-1}\|f\|_Y \ge 2^{-b/p}\|e_b\|_Y\,.$$

Similarly,

$$2^{-a/p}\|e_a\|_X \le \|f\|_p^{-1}\|f\|_X \le 2^{-b/p}\|e_b\|_X\,.$$

There are similar inequalities for g. Since $2^k < \varrho(a) \le \varrho(b) \le 2^{k+1}$,

$$2^{-b/p}\|e_b\|_X \le 2^{k+1-b/p}\|e_b\|_Y \le 2^{k+1-a/p}\|e_a\|_Y \le 2\cdot 2^{-a/p}\|e_a\|_X\,.$$

Combining these we see that

$$\|f\|_p^{-1}\|f\|_X \le 2\|g\|_p^{-1}\|g\|_X$$

and

$$\|f\|_p^{-1}\|f\|_Y \le 2\|g\|_p^{-1}\|g\|_Y$$

whence the claimed inequalities follow. For the last part, we observe that

$$\|f\|_\infty\|e_a\|_X \le \|f\|_X \le \|f\|_\infty\|e_a + \ldots + e_b\|_X \le 2\|f\|_\infty\|e_b\|_X$$

and proceed similarly. ∎

We draw immediately the conclusion that if A is finite (so that ϱ is bounded) then both X and Y coincide with $L_p(\mu)$ and there is nothing to prove. In other cases at most one I_k is infinite. We write $I_k = [a_k, b_k]$ if I_k is finite and $I_k = [a_k, \infty)$ or $I_k = (-\infty, b_k]$ if I_k is infinite. Let A_0 be the set of k so that $k-1, k+1 \in A$. We define a set J by taking one point d_k from each I_k for $k \in A$. We introduce the sequence spaces F_X and F_Y modelled on A by setting $\|x\|_{F_X} = \|\sum_{k\in A} 2^{-d_k/p}x(k)e_{d_k}\|_X$ and $\|x\|_{F_Y} = \|\sum_{k\in A} 2^{-d_k/p}x(k)e_{d_k}\|_Y$. In the case $p = \infty$ we define $\|x\|_{F_X} = \|\sum_{k\in A} x(k)e_{d_k}\|_X$.

LEMMA 5.7. *Under the hypotheses of Theorem* 5.5, *suppose $E_Y(J)$ has* (LSP). *Then there is a constant C_0 so that if $f \in E_Y$ then $\|f\|_Y$ is C_0-equivalent to $\|(\|f_{I_k}\|_p)\|_{F_Y}$.*

Proof. It suffices to prove such an equivalence if $f \in E_Y$ satisfies $f_{I_k} = 0$ for $k \notin A_0$, since there are at most two values of $k \notin A_0$ and Lemma 5.6 shows that the Y-norm on each such $E(I_k)$ is equivalent to the L_p-norm. Next observe that for such f if $g = \sum_{k\in A_0} 2^{-d_{k+1}/p}\|f_{I_k}\|_p e_{d_{k+1}}$ then for all $t \ge 0$, $\int_0^t (g^*(s))^p\,ds \le \int_0^t (f^*(s))^p\,ds$. Thus we have immediately by the p-convexity and rearrangement-invariance of Y, $\|g\|_Y \le \|f\|_Y$. Similarly, if $h = \sum_{k\in A_0} 2^{-d_{k-1}/p}\|f_{I_k}\|_p e_{d_{k-1}}$ then $\|h\|_Y \ge \|f\|_Y$. Next let $\tilde f = \sum_{k\in A_0} 2^{-d_k/p}\|f_{I_k}\|_p e_{d_k}$. We complete the proof by showing that for some C, $\|h\|_Y \le C\|\tilde f\|_Y$ and $\|\tilde f\|_Y \le C\|g\|_Y$. Once this is done it will be clear that $\|f\|_Y$ is actually equivalent to $\|\tilde f\|_Y$ as claimed.

The proofs of these statements are essentially the same, so we concentrate on the first. Note that

$$\begin{aligned}2^{-d_{k-1}/p}\|e_{d_{k-1}}\|_Y &\le 2^{-(k-1)-d_{k-1}/p}\|e_{d_{k-1}}\|_X\\ &\le 2^{-(k-1)-d_k/p}\|e_{d_k}\|_X \le 2^{1-d_k/p}\|e_{d_k}\|_Y\end{aligned}$$

and so if C is the (LSP) constant of $E_Y(J)$ we have $\|h\|_Y \le 2C\|\tilde f\|_Y$. Similarly, $\|\tilde f\|_Y \le 2C\|g\|_Y$. ∎

In a very similar way, exploiting the p-concavity of X one has

LEMMA 5.8. *Suppose $E_X(J)$ has* (RSP). *Then there is a constant* (*which we also name C_0*) *so that if $f \in E_X$ then $\|f\|_X$ is C_0-equivalent to $\|(\|f_{I_k}\|_p)\|_{F_X}$.*

Sketch of proof. First consider the case of Theorem 5.5. We assume $f \in E_X$ is finitely supported. Proceed as in Lemma 5.7, defining $g, h, \tilde f$ as before. In this case we have $\|g\|_X \ge \|f\|_X \ge \|h\|_X$. The remainder of the argument mirrors that of Lemma 5.7.

Let us also sketch the argument when $p = \infty$ (i.e. for Theorem 5.4). Analogously to Lemma 5.7 we note that $\|g\|_X \ge \|f\|_X \ge \|h\|_X$ where $g = \sum_{k\in A_0}\|f_{I_k}\|_\infty e_{d_{k+1}}$ and $h = \sum_{k\in A_0}\|f_{I_k}\|_\infty e_{d_k}$. The remainder of the argument is the same. ∎

Now let us turn to the proofs of Theorems 5.4 and 5.5. Suppose first that the couple (X, Y) is a Calderón couple. Then the couple $(E_X(J), E_Y(J))$ must also be a Calderón couple since there is a common averaging projection from (X, Y) onto (E_X, E_Y). Now it is clear that $(E_X(J), E_Y(J))$ is exponentially separated (when J is indexed as a sequence). We can thus apply Theorem 4.2 to deduce that $E_Y(J)$ has (LSP) and $E_X(J)$ has (LSP). We conclude this direction of the proof by showing that if $E_Y(J)$ (and hence F_Y) has (LSP) then E_Y has (LSP) and so Y is compressible. A very similar argument shows that X is stretchable.

To prove this we suppose that $\{f_j, g_j\}_{j\in B}$ is an interlaced pair of positive sequences in E_Y with $\|f_j\|_Y \le \|g_j\|_Y = 1$. For given j let $l(j)$ be the largest k so that $P_{I_k} f_j \neq 0$. (Note here that if such a largest k does not exist then j is the maximal element of B and $g_j = 0$; hence this case can be ignored.) We then split $f_j = f_j' + f_j''$ where $f_j'' = P_{I_{l(j)}} f_j$. Similarly we let $g_j = g_j' + g_j''$ where $g_j' = P_{I_{l(j)}} g_j$. Let $B_0 = \{j : \|f_j'\|_Y \ge 1/2\}$, and let $B_1 = B \setminus B_0$.

For $j \in B_0$ we set $v_j = (\|P_{I_k} f_j'\|_p)_{k\in A} \in F_Y$; for $j \in B_1$ we set $v_j = (\|P_{I_k} f_j''\|_p)_{k\in A}$. For all $j \in B$ we set $w_j' = (\|P_{I_k} g_j'\|_p)$ and $w_j'' = (\|P_{I_k} g_j''\|_p)$.

Let $(\alpha_j)_{j\in B}$ be positive and finitely nonzero. First observe that for j in B_0 we must have $\operatorname{supp} v_j < \operatorname{supp}(w_j' + w_j'')$. Further, $\|v_j\|_{F_Y} \ge (2C_0)^{-1}$ while $\|w_j + w_j''\|_{F_Y} \le C_0$. Thus, since F_Y has (LSP) applying Lemma 2.6 we get the existence of a constant C_1 depending only on (E, F) so that

$$\Big\| \sum_{j\in B_0} \alpha_j (w_j' + w_j'') \Big\|_{F_Y} \le C_1 \Big\| \sum_{j\in B_0} \alpha_j v_j \Big\|_{F_Y}.$$

Notice also that $(w_j' + w_j'')_{j\in B_0}$ have disjoint supports so that we can conclude that

$$\Big\| \sum_{j\in B_0} \alpha_j g_j \Big\|_Y \le C_0 \Big\| \sum_{j\in B_0} \alpha_j (w_j' + w_j'') \Big\|_{F_Y}.$$

Similarly,

$$\Big\| \sum_{j\in B_0} \alpha_j v_j \Big\|_{F_Y} \le C_0 \Big\| \sum_{j\in B_0} \alpha_j f_j \Big\|_Y.$$

Combining we have

$$\Big\| \sum_{j\in B_0} \alpha_j g_j \Big\|_Y \le C_0^2 C_1 \Big\| \sum_{j\in B_0} \alpha_j f_j \Big\|_Y.$$

We now obtain a similar estimate on B_1. In fact, if we set $B_2 = \{j \in B_1 : w_j'' \neq 0\}$ then we can argue as above to show that

$$\Big\| \sum_{j\in B_2} \alpha_j w_j'' \Big\|_{F_Y} \le C_1 \Big\| \sum_{j\in B_2} \alpha_j v_j \Big\|_{F_Y}$$

and hence obtain an estimate

$$\Big\| \sum_{j\in B_1} \alpha_j g_j'' \Big\|_Y \le C_0^2 C_1 \Big\| \sum_{j\in B_1} \alpha_j f_j \Big\|_Y.$$

Finally, we observe that for $j \in B_1$, $\|P_{l(j)} g_j\|_p \le 4\|P_{l(j)} f_j\|_p$ by Lemma 5.6. Thus for any k,

$$\Big\| P_{I_k} \sum_{j\in B_1} \alpha_j g_j' \Big\|_p = \Big(\sum_{l(j)=k} |\alpha_j|^p \|P_{I_k} g_j'\|_p^p \Big)^{1/p}$$

$$\le 4 \Big(\sum_{l(j)=k} |\alpha_j|^p \|P_{I_k} f_j''\|_p^p \Big)^{1/p} = 4 \Big\| P_{I_k} \sum_{j\in B_1} \alpha_j f_j'' \Big\|_p.$$

Thus

$$\Big\| \sum_{j\in B_1} \alpha_j g_j' \Big\|_Y \le 4C_0^2 \Big\| \sum_{j\in B_1} \alpha_j f_j'' \Big\|_Y.$$

Combining these estimates gives

$$\Big\| \sum_{j\in B} \alpha_j g_j \Big\|_Y \le C \Big\| \sum_{j\in B} \alpha_j f_j \Big\|_Y$$

for a suitable constant C. This completes the proof that Y is compressible and, as explained above, a similar argument shows that X is stretchable.

We now consider the other direction in Theorems 5.4 and 5.5. We suppose X is stretchable and Y is compressible. It follows that E_X has (RSP) and E_Y has (LSP) and we can apply both Lemmas 5.7 and 5.8. We can immediately deduce

LEMMA 5.9. *There exists C so that if $0 \le f, g \in E_X + E_Y$ and $\|f_{I_k}\|_p \ge \|g_{I_k}\|_p$ for all $k \in A$ then there exists $0 \le T \in \mathcal{A}(E_X, E_Y)$ with $\|T\|_{(E_X,E_Y)} \le C$ and $Tf = g$.*

Now suppose $f, g \ge 0$ in $E_X + E_Y$ and that $K(t, g) \le K(t, f)$ for all $t \ge 0$. We define $f' = \sum_{k\in A} 2^{-d_k/p} \|f_{I_k}\|_p e_{d_k}$ and $g' = \sum_{k\in A} 2^{-d_k/p} \|g_{I_k}\|_p e_{d_k}$. Then Lemma 5.9 yields the conclusion that $K(t, g') \le CK(t, g) \le K(t, f) \le C^2 K(t, f')$. Now $(E_X(J), E_Y(J))$ is exponentially separated.

Now for Theorem 5.5 we quote Theorem 4.5 to conclude that $(E_X(J), E_Y(J))$ is a Calderón couple and hence there exists $S \in \mathcal{A}(E_X(J), E_Y(J))$ with $\|S\|_{(E_X(J),E_Y(J))} \le C_2$, where C_2 depends only on (E, F), and $Sf' = g'$. It follows easily from Lemma 5.9 that (E_X, E_Y) and hence (X, Y) is a uniform Calderón couple.

In the case of Theorem 5.4 we note that it suffices to consider the case when f and g are decreasing functions; then f' and g' are also decreasing. Then $K(t, g') \le C^2 K(t, f')$ for all t implies that

$$\|g' \chi_{[0,t]}\|_X \le C^2 \|f' \chi_{[0,t]}\|_X.$$

We further note that $(E_X(J), \ell_\infty(J))$ has (RSP) by Lemma 3.4 and then apply Lemma 4.3 to obtain a positive $S \in \mathcal{A}(E_X(J), \ell_\infty(J))$ with $\|S\|_{E_X(J),\ell_\infty(J)} \le C_2$ and $Sf' = g'$. This leads to the desired conclusion. ∎

COROLLARY 5.10. *Let X be an r.i. space on $[0, 1]$ or $[0, \infty)$. Suppose X is r-concave for some $r < \infty$. In order that both (L_1, X) and (L_∞, X) be Calderón couples it is necessary and sufficient that X be elastic.*

EXAMPLES. We begin with the obvious remark that the spaces L_p for $1 \le p \le \infty$ are elastic and so our results include the classical results cited in the introduction. On the space $[0, \infty)$ one can basically separate behavior at ∞ from behavior at 0 so that spaces of the form $L_p + L_q$ and $L_p \cap L_q$ are also elastic. Note, however, that we cannot apply Theorems 5.3 or

5.5 unless we have appropriate assumptions on either the Boyd indices or convexity/concavity assumptions; thus pairs of such spaces are not always Calderón couples.

Let us now specialize to $[0,1]$. In certain special cases we can easily see that an r.i. space is elastic. For example, suppose X is the Lorentz space on $[0,1]$, for which $q_X < \infty$. Then it is immediately clear that X is elastic since E_X is a weighted ℓ_p-space. Rather more obscure elastic spaces can be built using a weighted Tsirelson space for E_X.

On the other hand, it is possible to give easy examples where E_X fails (RSP) or (LSP). Indeed, if one takes any symmetric sequence space E on $\mathbb{J}$ which is not an ℓ_p-space and considers $E(w^n)$ where $1 < w < 2$ then there is an r.i. space X for which $E_X = E(w^n)$. By Proposition 2.3, E_X fails (RSP) and (LSP). In this case we note that since $\kappa_+(E_X) = w$ and $\kappa_-(E_X) = w^{-1}$, we have $p_X = q_X = (\log_2 w)^{-1}$. If, say, $E = \ell_F(\mathbb{Z}_-)$ for some Orlicz function F satisfying the Δ_2-condition then X is an "Orlicz–Lorentz space" given by

$$\|f\|_X \sim \int_0^1 F(f^*(t)t^{-1/p})\,\frac{dt}{t}$$

where $p = p_X = q_X$. Note that for such a space the pair (L_∞, X) fails to be a Calderón couple. This answers a well-known question (cf. [8], [28]).

In the next section we will investigate Orlicz spaces in more detail. We will also give examples of Orlicz spaces L_F for which (L_∞, L_F) is not a Calderón couple.

We will conclude this section by considering a situation suggested by the example of Ovchinnikov [34] (cf. [29]).

THEOREM 5.11. *Suppose $1 < p < \infty$ and that X is an r.i. space on $[0,\infty)$ whose Boyd indices satisfy either $q_X < p$ or $p < p_X \le q_X < \infty$. Then $(X \cap L_p, X + L_p)$ is a Calderón couple if and only if X is a Lorentz space of order p.*

Proof. If X is a Lorentz space of order p, then both $X + L_p$ and $X \cap L_p$ are also Lorentz spaces of order p, and so form a Calderón couple. Conversely, suppose $(X \cap L_p, X + L_p)$ is a Calderón couple; then so is $(E_{X\cap L_p}, E_{X+L_p})$. Let us consider the case $q_X < p$; the other case is similar. Then $E_0 = E_{X\cap L_p} = E_X(\mathbb{Z}_-) \oplus E_{L_p}(\mathbb{Z}_+)$ and $E_1 = E_{X+L_p} = E_{L_p}(\mathbb{Z}_-) \oplus E_X(\mathbb{Z}_+)$. Note that for all n we have $\|e_n\|_{X\cap L_p} \ge \|e_n\|_{X+L_p}$; further, if we rearrange the sequence $(e_n)_{n\in\mathbb{Z}}$ so that $\|e_n\|_{X\cap L_p}/\|e_n\|_{X+L_p}$ increases, it is not difficult to see that (E_0, E_1) is exponentially separated. Thus E_0 has (RSP) and E_1 has (LSP) for this ordering. It also follows easily from our assumptions on the Boyd indices that there exists k so that the gap in the new ordering for E_0 between two consecutive elements of $\mathbb{Z}_+$ is at most k. Indeed, the ratio $\|e_n\|_{X\cap L_p}/\|e_n\|_{X+L_p}$ behaves like $2^{-n/p}\|e_n\|_X$ for $n < 0$ and like $2^{n/p}\|e_n\|_X^{-1}$ for $n \ge 0$ and we have an estimate for $k > 0$, $C^{-1}2^{k/r} \le \|e_{n+k}\|_X/\|e_n\|_X \le C2^k$ for suitable C and r with $q_X < r < p$. Thus E_0 must be a weighted ℓ_p-space by the argument of Proposition 2.3. Hence $E_X(\mathbb{Z}_-)$ is a weighted ℓ_p-space. Similarly, $E_X(\mathbb{Z}_+)$ is a weighted ℓ_p-space and so X is a Lorentz space of order p. ∎

6. Orlicz spaces. Let F be an Orlicz function, i.e. a strictly increasing convex function $F : [0,\infty) \to [0,\infty)$ satisfying $F(0) = 0$. We will also assume that F satisfies the Δ_2-condition with constant Δ, i.e. $F(2x) \le \Delta F(x)$ for every $x > 0$. We will use the notation $F_t(x) = F(tx)/F(x)$.

We recall first that F is said to be *regularly varying at* ∞ (resp. *at* 0), in the sense of Karamata, if the limit $\lim_{t\to\infty} F_t(x)$ (resp. $\lim_{t\to 0} F_t(x)$) exists for all x (in fact, it suffices that the limit exists when $x \le 1$). In this case there exists p, $1 \le p < \infty$, so that $\lim_{t\to\infty} F_t(x) = x^p$ (resp. $\lim_{t\to 0} F_t(x) = x^p$); F is then said to be regularly varying with order p. See [6] for details.

LEMMA 6.1. *The following conditions are equivalent:*

(1) *F is equivalent to an Orlicz function G which is regularly varying with order p at ∞ (resp. 0).*

(2) *There exists a constant C so that if $x_0 \le 1$ there exists $0 < t_0 < \infty$ so that if $t \ge t_0$ (resp. $t \le t_0$) and $x_0 \le x \le 1$,*

$$C^{-1}x^p \le F_t(x) \le Cx^p\,.$$

(3) *There exists a constant C so that if $x \le 1$, $\limsup_{t\to\infty} F_t(x) \le C \liminf_{t\to\infty} F_t(x)$ (resp. $\limsup_{t\to 0} F_t(x) \le C \liminf_{t\to 0} F_t(x)$).*

Proof. The implication (1)⇒(3) is immediate and (3)⇒(2) is a simple compactness argument. We indicate the details of (2)⇒(1). Let $f(x) = \log F(e^x)$ for $x \in \mathbb{R}$. The function $f(x) - x$ is then increasing. Then it is easy to translate (2) as:

(2)′ There exists c so that if $y_0 \ge 0$ there exists x_0 so that if $0 \le y \le y_0$ then

$$|f(x) - f(x-y) - py| \le c\,,$$

whenever $x \ge x_0$.

Now we can pick a function $u = u(x)$ for $x \in \mathbb{R}$ so that $u(x) = 0$ for $x \le 0$, u is differentiable, increasing, $u'(x) \le 1$, $\lim_{x\to\infty} u(x) = \infty$, and $|f(x) - f(x-y) - py| \le c$ for $0 \le y \le u(x)$. Now define $g(x) = f(x)$ if $u(x) = 0$ and

$$g(x) = \frac{1}{u}\int_{x-u}^{x} (f(s) + p(x-s))\,ds$$

if $u > 0$. It is easy to show that $f - g$ is bounded. Further, if $u > 0$,

$$g'(x) = -\frac{u'}{u}(g(x) - f(x-u) - pu) + \frac{1}{u}(f(x) - f(x-u) - pu) + p\,.$$

Since

$$\frac{u'(x)}{u(x)} \le \frac{1}{u}$$

it is easy to see that $\lim g'(x) = p$ and so if $G_0(x) = \exp(g(\log x))$ then G_0 is regularly varying and equivalent to F.

It remains to construct a convex G with the same properties. First note that since $f(x) - x$ is increasing we have if $u > 0$,

$$g(x) - f(x) \le \frac{1}{u}\int_{x-u}^{x}(p-1)(x-s)\,ds \le \frac{p-1}{2}u\,.$$

Hence

$$\begin{aligned} g'(x) &\ge p + \frac{1-u'}{u}(f(x) - f(x-u) - pu) - \frac{p-1}{2}u' \\ &\ge p - (p-1)(1-u') - (p-1)u' \ge 1\,. \end{aligned}$$

It now follows that $G_0(x)/x$ is increasing. The proof is completed by setting $G(x) = \int_0^x (G_0(x)/x)dx$ and it is then easy to verify that G has the desired properties. ∎

If F is an Orlicz function, $0 < x \le 1$, and $C > 1$ we can define $\Psi_p^\infty(x, C)$ (resp. $\Psi_p^0(x, C)$) to be the supremum (possibly ∞) of all N so that there exist $a_1 < a_2 < \ldots < a_N$ with $a_k/a_{k-1} \ge 2$ for $k \le N-1$ and $a_1 \ge 1$ (resp. $a_N \le 1$) so that for all k either $F_{a_k}(x) \ge Cx^p$ or $x^p \ge CF_{a_k}(x)$. It is easy to show the following:

PROPOSITION 6.2. *F is equivalent to a regularly varying function of order p at ∞ (resp. at 0) if and only if for some C and all $0 < x \le 1$ we have $\Psi_p^\infty(x, C) < \infty$ (resp. $\Psi_p^0(x, C) < \infty$).*

We omit the proof which is immediate. However, we can now state the result of Montgomery-Smith [33] which characterizes Orlicz spaces which are Lorentz spaces (see Lorentz [26]).

THEOREM 6.3. *In order that $L_F[0,1]$ coincides with a Lorentz space of order p it is necessary and sufficient that there exist C_0, C_1 and $r > 0$ so that for every x with $0 < x \le 1$ we have $\Psi_p(x, C_0) \le C_1 x^{-r}$.*

This is a somewhat disguised restatement of Montgomery-Smith's result. However, we will not pause in our exposition to derive this result as a proof is implicit in our approach to elastic Orlicz spaces. Further, we state the result in order to motivate the following definition.

For $C > 1$ and $0 < x \le 1$ let us define $\Phi_+^\infty(x, C)$ (resp. $\Phi_+^0(x, C)$) to be the supremum of all n so that there exist $a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n$ with $a_1 \ge 1$ (resp. $b_n \le 1$) so that $F_{b_k}(x) \ge CF_{a_k}(x)$, for $1 \le k \le n$. For $C > 1$ and $0 < x \le 1$ let us define $\Phi_-^\infty(x, C)$ (resp. $\Phi_-^0(x, C)$) to be the supremum of all n so that there exist $a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_n < b_n$ with $a_1 \ge 1$ (resp. $b_n \le 1$) so that $F_{a_k}(x) \ge CF_{b_k}(x)$, for $1 \le k \le n$. We say that F is *elastic at* ∞ (resp. *at* 0) if there exist $C_0, C_1 > 1$ and $r > 0$ so that for $0 < x \le 1$ we have $\Phi_+^\infty(x, C_0) + \Phi_-^\infty(x, C_0) \le C_1 x^{-r}$ (resp. $\Phi_+^0(x, C_0) + \Phi_-^0(x, C_0) \le C_1 x^{-r}$). From now on, we will consider only the case at ∞ although similar results can always be proved at 0.

LEMMA 6.4. *F is elastic at ∞ if and only if there exist constants $C_0, C_1 > 1$ and $r > 0$ so that if $0 < x \le 1$, $\Phi_+^\infty(x, C_0) \le C_1 x^{-r}$ (resp. $\Phi_-^\infty(x, C_0) \le C_1 x^{-r}$).*

Proof. Assume $\Phi_+^\infty(x, C_0) \le C_1 x^{-r}$. Suppose $1 \le a_1 < b_1 \le \ldots \le a_n < b_n$ with $F_{a_k}(x) \ge eC_0 F_{b_k}(x)$ for $1 \le k \le n$. Consider an interval $[b_k, a_{k+1}]$ where $1 \le k \le n-1$. Let $\nu = \nu_k$ be the integer part of $(\log C_0)^{-1}(\log F_{a_{k+1}}(x) - \log F_{b_k}(x))$. Then we can find $b_k = c_0 < c_1 < \ldots < c_\nu \le a_{k+1}$ so that $\log F_{c_k}(x) - \log F_{c_{k-1}}(x) = \log C_0$. It follows that

$$\sum_{k=1}^{n-1} \nu_k \le \Phi_+(x, C_0)$$

and hence that

$$\sum_{k=1}^{n-1}(\log F_{a_{k+1}}(x) - \log F_{b_k}(x)) \le (\log C_0)(\Phi_+(x, C_0) + n - 1)$$

and thus

$$\log F_{b_n}(x) - \log F_{a_1}(x) \le (\log C_0)(C_1 x^{-r} - 1) - n\,.$$

Now

$$\log F_{b_n}(x) - \log F_{a_1}(x) \ge \log F_{b_n}(x) \ge -C_2|\log x| - C_3$$

for suitable C_2, C_3 by the Δ_2-condition. Hence

$$n \le (\log C_0)(C_1 x^{-r} - 1) + C_2|\log x| + C_3$$

and so

$$\Phi_-^\infty(x, eC_0) \le C_4 x^{-2r}$$

for a suitable C_4. The other case is similar. ∎

PROPOSITION 6.5. *The following conditions on F are equivalent:*

(1) *F is elastic at ∞.*

(2) *There exist constants* $C_0, C_1 > 1$ *so that if* $1 \le a_1 < b_1 \le \ldots \le a_n < b_n$ *and* $0 \le x \le 1$ *then:*

$$\sum_{k=1}^{n}(F_{b_k}(x) - C_0F_{a_k}(x)) \le C_1\,.$$

(3) *There exist constants* $C_0, C_1 > 1$ *so that if* $1 \le a_1 < b_1 \le \ldots \le a_n < b_n$ *and* $0 \le x \le 1$ *then*

$$\sum_{k=1}^{n}(F_{a_k}(x) - C_0F_{b_k}(x)) \le C_1\,.$$

(4) *There exists a bounded monotone increasing function* $w : [1,\infty) \to \mathbb{R}$ *and a constant* C_0 *so that if* $1 \le s \le t$ *and* $0 \le x \le 1$ *then*

$$F_t(x) \le CF_s(x) + w(t) - w(s)\,.$$

(5) *There exists a bounded monotone increasing function* $w : [1,\infty) \to \mathbb{R}$ *and a constant* C_0 *so that if* $1 \le s \le t$ *and* $0 \le x \le 1$ *then*

$$F_s(x) \le CF_t(x) + w(t) - w(s)\,.$$

Proof. (1)⇒(2). We assume that for suitable constants $C_2, C_3 > 1$ and $r > 0$, we have

$$\Phi_+^\infty(x, C_2) \le C_3x^{-r}\,.$$

We will assume that $C_2 > \Delta$ from which it follows easily that $F_b(x)/F_a(x) \ge C_2$ implies $b > 2a$. First suppose m is an integer with $m > r$. We will estimate $\Phi_+^\infty(x, C_2^m)$. Suppose $1 \le a_1 < b_1 \le \ldots \le a_n < b_n$ and $F_{b_k}(x) > C_2^mF_{a_k}(x)$. Let s be the smallest integer greater than $|\log_2 x| + 1$. Then $a_s \ge x^{-1}$ and $a_{ks} > x^{-1}b_{(k-1)s}$ for $2 \le k \le [n/s]$. Let $\xi = x^{1/m}$.

Now, for each $1 \le k \le [n/s]$ there exists σ_k with $0 \le \sigma_k \le m-1$ so that $F_{\xi^{\sigma_k}b_{sk}}(\xi) \ge C_2F_{\xi^{\sigma_k}a_{sk}}(\xi)$ and the intervals $[\xi^{\sigma_k}a_{sk}, \xi^{\sigma_k}b_{sk}]$ are disjoint in $[1,\infty)$. Hence we have an estimate

$$\Phi_+^\infty(\xi, C_2) \ge [n/s]$$

and this means that

$$[n/s] \le C_3\xi^{-r}\,.$$

Thus

$$n \le C_3(s+1)x^{-r/m} \le C_4 + C_5|\log x|x^{-r/m}$$

for suitable constants C_4, C_5. This leads to an estimate

$$\Phi_+^\infty(x, C_2^m) \le C_6x^{-\alpha}$$

where $0 < \alpha < 1$.

Now suppose $C_0 = \Delta C_2^m$. Suppose $1 \le a_1 < b_1 \le \ldots \le a_n < b_n$ and that $0 \le x_k \le 1$ for $1 \le k \le n$. For $j \in \mathbb{N}$ let I_j be the set of k such that

$2^{-j} < x_k \le 2 \cdot 2^{-j}$. Then

$$\begin{aligned}\sum_{k\in I_j}(F_{b_k}(x_k) - C_0F_{a_k}(x_k)) &\le \sum_{k\in I_j}(\Delta F_{b_k}(2^{-j}) - C_0F_{a_k}(2^{-j}))\\ &\le \Delta\Phi_+^\infty(2^{-j}, C_2^m)\max_k F_{b_k}(2^{-j})\\ &\le C_6\Delta 2^{-(1-\alpha)j}\,.\end{aligned}$$

Thus

$$\sum_{k=1}^{n}(F_{b_k}(x) - C_0F_{a_k}(x)) \le C_6\Delta\sum_{j=1}^{\infty}2^{-(1-\alpha)j}\,.$$

This establishes (2).

(2)⇒(4). We define $w(t)$, for $1 \le t < \infty$, to be the supremum of $\sum_{k=1}^{n}(F_{b_k}(x_k) - C_0F_{a_k}(x_k))$ over all n and all $1 \le a_1 < b_1 \le \ldots \le a_n < b_n \le t$ and all $0 \le x_k \le 1$ for $1 \le k \le n$. Clearly $w(t)$ is increasing and bounded above by C_1. Condition (4) is immediate from the definition.

(4)⇒(1). Suppose $0 < x \le 1$ and that $1 \le a_1 < b_1 \le \ldots \le a_n < b_n$ are such that $F_{b_k}(x) > 2C_0F_{a_k}(x)$. Then we have

$$C_0\sum_{k=1}^{n}F_{b_k}(x) \le \sum_{k=1}^{n}(w(b_k) - w(a_k)) \le C_1$$

where $C_1 = \lim_{x\to\infty}w(x) - w(1)$. Now $F_t(x) \ge C_2x^r$ for all t, for a suitable C_2, by the Δ_2-condition. Thus

$$\Phi_+^\infty(x, 2C_0) \le C_1(C_0C_2)^{-1}x^{-r}\,.$$

The implication now follows from Lemma 6.4.

The remaining implications are similar. ∎

LEMMA 6.6. *If* F *is elastic at* ∞ *then* F *is equivalent to an Orlicz function which is regularly varying at* ∞.

Proof. It follows immediately from (4) above that

$$\limsup_{t\to\infty}F_t(x) \le C_0\liminf_{t\to\infty}F_t(x)$$

for $0 < x \le 1$. Apply Lemma 6.1. ∎

We now come to our main theorem on elastic Orlicz functions.

THEOREM 6.7. *Let* F *be an Orlicz function satisfying the* Δ_2*-condition. Then the following are equivalent:*

(1) F *is elastic at* ∞.
(2) $L_F[0,1]$ *is stretchable.*

(3) $L_F[0,1]$ *is compressible.*

(4) $L_F[0,1]$ *is elastic.*

Proof. We will only show (1)⇒(2) and (2)⇒(1). The other implications will then be clear. We will write $E = E_F$ for E_X where $X = L_F[0,1]$. Then E_F is the modular sequence space of $\mathbb{Z}_-$ defined by $\|x\|_{E_F} = 1$ if and only if $\sum_{n\in\mathbb{Z}_-} F(x(n))2^n = 1$. Let us define λ_n for $n \in \mathbb{Z}_-$ by $F(\lambda_n) = 2^{-n}$. Then $(\lambda_n)_{n\in\mathbb{Z}_-}$ is strictly decreasing and $\lambda_{n-1} \le 2\lambda_n$ for $n < 0$.

(1)⇒(2). We must show that E has (RSP). It suffices to show the existence of a constant C so that if $a_1 < b_1 < c_1 \le a_2 < b_2 < c_2 \le \ldots \le a_n < b_n < c_n \le 0$ and $\operatorname{supp} x_k \subset [a_k, b_k)$, $\operatorname{supp} y_k \subset [b_k, c_k)$ and $\|y_k\|_E \le \|x_k\|_E = 1$ then

$$\Big\|\sum_{k=1}^n \alpha_k y_k\Big\|_E \le C\Big\|\sum_{k=1}^n \alpha_k x_k\Big\|_E .$$

To do this let us suppose n, a_k, b_k, c_k, x_k are fixed and let Γ be the least constant C for which this inequality holds. We show a uniform bound on Γ. We can suppose the existence of constants C_0, C_1 and an increasing function $w : [1,\infty) \to \mathbb{R}$ with $\lim_{x\to\infty} w(x) = w(1) + C_1$ so that if $1 \le s \le t$,

$$F_s(x) \le C_0 F_t(x) + w(t) - w(s)$$

for $0 \le x \le 1$.

Define $x_k' = \sum_{|x_k(j)|\ge \frac12\lambda_{b_k}} x_k(j)e_j$ and $y_k' = \sum_{|y_k(j)|\ge\frac12\lambda_{c_k}} y_k(j)e_j$. Then

$$\sum 2^j F(2|x_k(j) - x_k'(j)|) \le 2^{-b_k}\sum_{j<b_k} 2^j \le 1 .$$

Thus $\|x_k - x_k'\|_E \le 1/2$ and similarly $\|y_k - y_k'\|_E \le 1/2$. We let $x_k''(j) = \min(2|x_k'(j)|, \lambda_{a_k})$ and $y_k''(j) = \min(2|y_k'(j)|, \lambda_{b_k})$. Then $\|x_k''\|_E, \|y_k''\|_E \le 2$.

Now for any $\alpha_1, \ldots, \alpha_n$ such that $\|\sum_{k=1}^n \alpha_k x_k\|_E = 1$ we set $z = \sum_{k=1}^n \alpha_k y_k'$ and $v = \sum_{k=1}^n \alpha_k x_k'$. We also let $u = \sum_{k=1}^n \alpha_k \lambda_{b_k} e_{b_k}$. Then for fixed k,

$$\begin{aligned}\sum_{j\in[b_k,c_k)} 2^j F(|\alpha_k y_k'(j)|) &\le \sum 2^j F(|\alpha_k y_k''(j)|)\\ &\le C_0 2^{b_k} F(|\alpha_k|\lambda_{b_k}) \sum_{y_k''(j)\ne 0} 2^j F(y_k''(j))\\ &\quad + \sum_{y_k''(j)\ne 0} 2^j F(y_k''(j))(w(\lambda_{b_k}) - w(y_k''(j)))\\ &\le C_0\Delta 2^{b_k} F(|\alpha_k|\lambda_{b_k}) + \Delta(w(\lambda_{b_k}) - w(\lambda_{c_k})) .\end{aligned}$$

On summing, we get

$$\sum_j 2^j F(|z(j)|) \le C_0\Delta\sum_j 2^j F(|u(j)|) + \Delta C_1 .$$

Now, in the other direction, for fixed k,

$$2^{b_k} F(|\alpha_k|\lambda_{b_k}) \le C_0 F(|\alpha_k|x_k''(j))F(x_k''(j))^{-1} + w(\lambda_{a_k}) - w(\lambda_{b_k})$$

whenever $x_k''(j) \ne 0$.

Thus

$$\begin{aligned}2^{b_k} F(|\alpha_k|\lambda_{b_k})\sum_j 2^j F(x_k''(j)) &\le C_0\sum_j 2^j F(|\alpha_k x_k''(j)|)\\ &\quad + \sum_j 2^j F(x_k''(j))(w(\lambda_{a_k}) - w(\lambda_{b_k})) .\end{aligned}$$

Now we observe that $1/2 \le \|x_k'\|_E \le 1$ so that $1/2 \le \|x_k''\|_E \le 2$. Hence $1/2 \le \sum_j 2^j F(x_k''(j)) \le \Delta$. Thus we have

$$2^{b_k} F(|\alpha_k|\lambda_{b_k}) \le 2C_0\Delta\sum_j 2^j F(|\alpha_k x_k'(j)|) + 2\Delta(w(\lambda_{a_k}) - w(\lambda_{b_k})) .$$

Summing as before,

$$\sum_j 2^j F(|u(j)|) \le 2C_0\Delta\sum_j 2^j F(|v(j)|) + 2C_1\Delta .$$

We thus have an estimate

$$\sum_j 2^j F(|z(j)|) \le C_2\sum_j 2^j F(|v(j)|) + C_3$$

for constants C_2, C_3 depending only on F. This in turn implies an estimate $\|z\|_E \le C_4\|v\|_E$ for some constant C_4.

Now we conclude by noting that

$$\begin{aligned}\Big\|\sum_{k=1}^n \alpha_k y_k\Big\|_E &\le \|z\|_E + \Big\|\sum_{j=1}^k \alpha_k(y_k - y_k')\Big\|_E\\ &\le C_4\|v\|_E + \Gamma\max_j\|y_j - y_j'\|_E\Big\|\sum_{k=1}^n \alpha_k x_k\Big\|_E \le C_4 + \frac{\Gamma}{2} .\end{aligned}$$

Thus $\Gamma \le C_4 + \Gamma/2$ and so $\Gamma \le 2C_4$ and E has (RSP).

(2)⇒(1). Suppose E has (RSP). This implies that for some C_0, if $a_1 < b_1 < a_2 < \ldots < a_n < b_n$ are negative integers, and $0 \le x_k$ with $\sum 2^{a_k} F(\lambda_{a_k} x_k) \le 1$, then $\sum 2^{b_k} F(\lambda_{b_k} x_k) \le C_0$. We also note from the Δ_2-condition that we can suppose $C_1 F_t(x) \ge x^r$ for some C_1 and r and all $t > 0$, $0 \le x \le 1$.

For any constant $C > 2C_0\Delta^3$ and $0 < x \leq 1$ suppose now that $1 \leq c_1 < d_1 \leq c_2 < \ldots < d_{n-1} \leq c_n < d_n$ and $F_{c_k}(x) \geq CF_{d_k}(x)$. Then we must have $d_k > 2^3 c_k$. Now choose $b_{n-k+1} \in \mathbb{Z}_-$ to be the largest integer so that $\lambda_{b_{n-k+1}} > c_k$ and let a_{n-k+1} be the smallest integer so that $\lambda_{a_{n-k+1}} < d_k$. It is clear that $a_1 < b_1 < a_2 < \ldots < a_n < b_n$. Further, $\lambda_{b_{n-k+1}} \leq 2c_k$ and $\lambda_{a_{n-k+1}} \geq d_k/2$. It follows that for every k with $1 \leq k \leq n$ we have

$$2^{b_k} F(\lambda_{b_k} x) \geq C\Delta^{-2} 2^{a_k} F(\lambda_{a_k} x)\,.$$

Now suppose $n > C_1 x^{-r}$. Then we can select a subset J of $\{1, \ldots, n\}$ so that $1/2 \leq \sum_{k\in J} 2^{a_k} F(\lambda_{a_k} x) \leq 1$. Then we can conclude that

$$\frac{C\Delta^{-2}}{2} \leq \sum_{k\in J} 2^{b_k} F(\lambda_{b_k} x) \leq C_0\,.$$

Since $C > 2C_0\Delta^2$ we reach a contradiction and conclude that $n \leq C_1 x^{-r}$. Thus

$$\Phi_-^\infty(x, C) \leq C_1 x^{-r}$$

and F is elastic by Lemma 6.4. ∎

Of course there are corresponding results for sequence spaces and Orlicz spaces on $[0, \infty)$. We will omit the proofs.

THEOREM 6.8. *Suppose F is an Orlicz function satisfying the Δ_2-condition. Then:*

(1) *In order that ℓ_F be elastic (resp. compressible, resp. stretchable) it is necessary and sufficient that F be elastic at 0.*

(2) *In order that $L_F[0, \infty)$ be elastic (resp. compressible, resp. stretchable) it is necessary and sufficient that F be elastic at both 0 and ∞.*

Remark. It is perhaps worth pointing out that the theorem of Montgomery-Smith (Theorem 6.3) cited above can be proved in much the same manner as Theorem 6.7; the problem in this case is to show that E is a weighted ℓ_p-space. In fact, our proof of Theorem 6.7 is derived from the arguments used by Montgomery-Smith [33].

Returning to the case of $[0, 1]$ we note the following simple deduction.

PROPOSITION 6.9. *If the Orlicz space $L_F[0, 1]$ is elastic then its Boyd indices $p_F = p_{L_F}$ and $q_F = q_{L_F}$ coincide.*

Proof. In fact, we can suppose F is regularly varying by Lemma 6.6 and so the conclusion is immediate. ∎

Remark. The analogous result holds for sequence spaces, but not for $L_F[0, \infty)$ where one must consider behavior at both 0 and ∞. Thus $L_p \cap L_q$ is elastic for any p, q. Let us also mention at this point that Proposition 6.9 allows us very easily to give examples of Orlicz function spaces $L_F[0, 1]$ so that (L_∞, L_F) is not a Calderón couple by simply ensuring that $p_F \neq q_F$.

EXAMPLES. We now give two examples to separate the concepts implicit in our discussion above. We first construct a regularly varying Orlicz function which is not elastic. To do this first suppose (ξ_n) is a positive sequence, bounded by one and tending monotonically to zero. We define $\phi(x) = 2$ if $x \leq 1$ and then $\phi(x) = 2 + (-1)^n \xi_n$ if $2^{n-1} < x \leq 2^n$. Define $f(x) = \int_0^x \phi(t)\,dt$ and $F(x) = \exp(f(\log x))$. Then $F(x)/x$ is increasing and hence $F_1(x) = \int_0^x (F(t)/t)\,dt$ is an Orlicz function equivalent to F. Further, F and F_1 are regularly varying of order 2. It remains to show that F_1 or equivalently F is not elastic at ∞. Suppose $C > 1$ and that $0 < x \leq 1$. If $2^{n-1} > \log x^{-1}$ then

$$\left|\log \frac{F(e^{2^n} x)}{F(e^{2^n})} - \log \frac{F(e^{2^{n+1}} x)}{F(e^{2^{n+1}})}\right| \geq (\xi_n + \xi_{n+1}) \log x^{-1}\,.$$

If we assume that ξ_n goes to zero slowly enough, say $\xi_n \sim (\log\log n)^{-1}$, this will exceed $\log C$, $O(\exp(x^{-r}))$ times for some $r > 0$ and so F_1 cannot be elastic.

Our second construction is of an elastic Orlicz space which is not a Lorentz space. It is of course clear that, conversely, every Lorentz space is elastic. We note first that if $F(x) = \exp(f(\log x))$ where f is convex then F is elastic at ∞, by applying Proposition 6.5(4) (the same conclusion holds when f is concave). We thus consider a function $\phi(t) = 2 + \psi(t)$ where $\psi(t)$ is bounded by one and decreases monotonically to 0. Let $f(x) = \int_0^t \phi(t)\,dt$ as above. As usual, it may be necessary to convexify F by constructing F_1; however, this is equivalent to F. Now we show that for $L_F[0, 1]$ to be a Lorentz space it is necessary that ψ tends to zero at a certain rate. In fact, if $\Psi_2^\infty(x, C_0) \leq C_1 x^{-r}$ it follows that $\psi(2^{C_1 x^{-r}+1}) < \log C_0 / \log x^{-1}$ and hence that $\psi(u) = O((\log\log u)^{-1})$. Thus if we choose ψ converging to zero slowly enough then $L_F[0, 1]$ is an elastic non-Lorentz space.

We now turn to the general problem of determining when a pair of Orlicz spaces $L_F[0, 1]$ and $L_G[0, 1]$ forms a Calderón couple. Of course if the Boyd indices satisfy $q_F < p_G$ this can only happen if both F and G are elastic at ∞ in which case $p_F = q_F$ and $p_G = q_G$. Brudnyĭ [8] has conjectured that if L_F and L_G are distinct then if (L_F, L_G) forms a Calderón couple then we must have $p_F = q_F$ and $p_G = q_G$. The next theorem shows that if either $p_F \neq q_F$ or $p_G \neq q_G$ then F and G must in some sense be similar functions. However, following the theorem we will give a counterexample to Brudnyĭ's conjecture.

THEOREM 6.10. *Suppose F and G are Orlicz functions satisfying the Δ_2-condition and such that $(L_F[0,1], L_G[0,1])$ forms a Calderón pair. Then either F and G are both elastic or $p_F = p_G$ and $q_F = q_G$.*

Proof. Let us assume that $q_F > q_G$. The other case is similar. It will be convenient to pick q_0, q_1 so that $q_G < q_0 < q_F < q_1$ and to suppose (by passing to equivalent functions) that $F(x)/x^{q_1}$ and $G(x)/x^{q_0}$ are decreasing.

Let $\mathcal{F}$ be the closure of the set of functions $\{F_t : t \geq 1\}$ in $C[0,1]$. This set is relatively compact. For each $M > 1$ let $\mathcal{F}_M$ be the closure of the set of functions $\{F_t : t \geq 1,\ F(t)/G(t) \geq M\}$ and let $\mathcal{F}_\infty = \bigcap_M \mathcal{F}_M$. Similarly, if $a < 1$ we let $\mathcal{F}_a$ be the closure of the set of functions $\{F_t : t \geq 1,\ F(t)/G(t) \leq a\}$ and set $\mathcal{F}_0 = \bigcap_a \mathcal{F}_a$.

Now suppose $t \geq 1$ and A_t is a measurable subset of $[0,1]$ such that $\mu(A_t) = F(t)^{-1}$. Then $\|\chi_{A_t}\|_{L_F} = t^{-1}$ while $\|\chi_{A_t}\|_{L_G} = s^{-1}$ where $G(s) = F(t)$. If $F(t) > G(t)$ we conclude that $s > t$ and further from the Δ_2-condition for G we have $\|\chi_{A_t}\|_{L_G} \geq \phi(F(t)/G(t))\|\chi_{A_t}\|_{L_F}$ where ϕ is a function satisfying $\lim_{u\to\infty} \phi(u) = \infty$.

Suppose $\mathcal{F}_\infty$ is nonempty and $H_1, H_2 \in \mathcal{F}_\infty$. Then we can find a sequence $(t_n)_{n\geq 1}$ such that $t_1 \geq 2$, $t_n > 2t_{n-1}$ and $F(t_n)/G(t_n) \to \infty$,

$$\frac{\|\chi_{A_{t_n}}\|_{L_G}}{\|\chi_{A_{t_n}}\|_{L_F}} \geq 2\frac{\|\chi_{A_{t_{n-1}}}\|_{L_G}}{\|\chi_{A_{t_{n-1}}}\|_{L_F}}$$

for $n \geq 1$ and such that $F_{t_{2n}} \to H_1$ while $F_{t_{2n-1}} \to H_2$. Since $\mu A_{t_n} \leq 2^{-n}$ we can suppose these sets are disjoint. If we restrict to the sub-σ-algebra $\mathcal{A}$ of the Borel sets generated by (A_{t_n}) then $(L_F(\mathcal{A}), L_G(\mathcal{A}))$ forms a Calderón couple. Regarded as a couple of sequence spaces it is exponentially separated and hence the Orlicz modular space $\ell_{F_{t_n}}$ has (LSP) by Theorem 4.2. By passing to a subsequence of the unit vectors it follows that both the Orlicz sequence spaces ℓ_{H_1} and ℓ_{H_2} have (LSP) and further that the space obtained by interlacing their bases has (LSP). Hence from Proposition 2.4, $H_1(x)$ and $H_2(x)$ are both equivalent to some (common) x^{p_0}. We thus conclude that there exists p_0 so that any $H \in \mathcal{F}_\infty$ is equivalent to x^{p_0}.

By similar reasoning, if $\mathcal{F}_0$ is nonempty there exists p_1 so that every $H \in \mathcal{F}_0$ is equivalent to x^{p_1}.

Now suppose $q_0 < r_1 < r_2 < q_F$. We pick an integer m large enough so that $(m-2)r_1 + 2q_1 < mr_2$. Then for any $\xi < 1$ the function $F_0(x) = \max\{x^{r_2}t^{-r_2}F(t) : \xi^m x \leq t \leq x\}$ is equivalent to F and therefore $F_0(x)/x^{r_2}$ cannot be decreasing everywhere. Thus for any x_0 there exists $x \geq x_0$ such that for some $\delta > 0$ we have $F_0(u)/u^{r_2} < F_0(x)/x^{r_2}$ if $x - \delta < u < x$. It follows that $F_0(x) = F(x)$ and hence $F(x) \geq x^{r_2}t^{-r_2}F(t)$ if $\xi^m x \leq t \leq x$.

Next define $F_1(y) = \max\{y^{r_1}t^{-r_1}F(t) : \xi y \leq t \leq y\}$. Notice that $F_1(y) \leq \xi^{q_1-r_1}F(\xi y)$. We will argue that $F_1(x)/x^{r_1}$ cannot be decreasing on $(\xi^{m-1}x, \xi x)$. If it is then $F_1(\xi x) \leq F_1(\xi^{m-1}x)$ and hence

$$\begin{aligned} F(x) &\leq \xi^{-q_1}F(\xi x) \leq \xi^{-q_1}F_1(\xi x) \\ &\leq \xi^{-(m-2)r_1-q_1}F_1(\xi^{m-1}x) \leq \xi^{-(m-2)r_1-2q_1}F(\xi^m x)\,, \end{aligned}$$

and thus $(m-2)r_1 + 2q_1 > mr_2$, contrary to assumption.

We now argue as above and conclude similarly that there exists u with $\xi^{m-1}x \leq u \leq \xi x$ such that $F(u) \geq u^{r_1}t^{-r_1}F(t)$ for $\xi u \leq t \leq u$.

Now notice that

$$F(u)/G(u) \leq u^{r_2-q_0}x^{q_0-r_2}F(x)/G(x)$$

and so

$$F(u)/G(u) \leq \xi^{r_2-q_0}F(x)/G(x)\,.$$

It follows that given any $\xi < 1$ any x_0 we can pick $x \geq x_0$ so that $F(x) \geq x^{r_1}t^{-r_1}F(t)$ for $\xi x \leq t \leq x$ and either $F(x)/G(x) \geq \xi^{-(r_2-q_0)/2}$ or $F(x)/G(x) \leq \xi^{(q_0-r_2)/2}$. Thus we can find a sequence $t_n \to \infty$ such that $F_{t_n}(x) \geq x^{r_1}$ for $n^{-1} \leq x \leq 1$ and either $F(t_n)/G(t_n) \to \infty$ or $F(t_n)/G(t_n) \to 0$.

Consider the former case. Then there exists $H \in \mathcal{F}_\infty$ with $H(x) \geq x^{r_1}$. Hence $p_0 \geq r_1$. In the latter case $p_1 \geq r_1$. Since $r_1 < q_F$ is arbitrary we conclude that either $p_0 = q_F$ or $p_1 = q_F$.

Consider the case $p_0 = q_F$; in particular, $F(t)/G(t)$ is unbounded for $t \geq 1$. We will argue that $F(t)/G(t)$ tends to infinity. For any $\xi < 1$ consider the function $h(x) = \min\{F(t)/G(t) : \xi x \leq t \leq x\}$. If h does not converge to ∞ then given any M and x_0 there exist $x > x_0$ and $\delta > 0$ so that $h(x) = M$ and $h(x) < h(u)$ for $x - \delta < u < x$ and this implies that $F(t)/G(t) \leq F(x)/G(x)$ for $\xi x \leq t \leq x$. Thus we can construct $t_n \to \infty$ so that $F(t_n)/G(t_n) \to \infty$ and $F_{t_n}(x) \leq G_{t_n}(x) \leq x^{q_0}$ for $n^{-1} \leq x \leq 1$. Thus $\mathcal{F}_\infty$ contains a function H with $H(x) \leq x^{q_0}$. This contradicts the fact that $q_0 < p_F$. Thus $F(t)/G(t) \to \infty$ and it follows easily that since $\mathcal{F}_\infty = \mathcal{F}$ we have $p_F = q_F$. We can invoke Theorem 5.3 to conclude that both F and G must be elastic.

The case $p_1 = q_F$ is similar. In this case $G(t)/F(t)$ is unbounded and we use the same argument as above to show that $G(t)/F(t) \to \infty$.

We omit the case $p_F < p_G$; the reasoning is much the same. ∎

EXAMPLE. It remains to construct an example of a Calderón couple $(L_F[0,1], L_G[0,1])$ with F and G nonequivalent and $p_F = p_G < q_F = q_G$. Such an example is a counterexample to the previously mentioned conjecture of Brudnyĭ [8]. Our construction depends on the following lemma:

LEMMA 6.11. *Let (Y_0, Y_1) be a Calderón couple and let X be a Banach space. Then the pair $(X \oplus Y_0, X \oplus Y_1)$ also forms a Calderón couple.*

Proof. We suppose the direct sums are ℓ_1-sums. Suppose that $(x_0, y_0), (x_1, y_1) \in X \oplus (Y_0 + Y_1)$ satisfies

$$K(t, (x_0, y_0), X \oplus Y_0, X \oplus Y_1) \le K(t, (x_1, y_1), X \oplus Y_0, X \oplus Y_1)\,.$$

Then we observe that

$$\|x_0\|_X \le \|x_1\|_X + K(1, y_1, Y_0, Y_1)\,.$$

Thus there is an operator $S : X \oplus (Y_0 + Y_1) \to X$ with $\|S_0\| \le 1$ and $S(x_1, y_1) = x_0$. On the other hand,

$$K(t, y_0, Y_0, Y_1) \le \min(1, t)\|x_1\| + K(t, y_1, Y_0, Y_1)\,.$$

Now (Y_0, Y_1) is Gagliardo complete ([13], Lemma 3) so by K-divisibility ([4], [7], [15]) we can write $y_0 = u + v$ where

$$K(t, u, Y_0, Y_1) \le \gamma \min(1, t)\|x_1\|$$

and

$$K(t, v, Y_0, Y_1) \le \gamma K(t, y, Y_0, Y_1)$$

and γ is an absolute constant. The former inequality implies that $\max(\|u\|_{Y_0}, \|u\|_{Y_1}) \le \gamma\|x_1\|_X$ and hence that there exists $S_1 : X \to Y_0 \cap Y_1$ with $\|S_1\| \le \gamma$ and $S_1 x_1 = u$. The latter inequality yields the existence of $S_2 : Y_0 + Y_1 \to Y_0 + Y_1$ with $S_2 \in \mathcal{A}(Y_0, Y_1)$ and $S_2 y_1 = v$. Let $S(x, y) = (S_0 x, S_1 x + S_2 y)$. Then S is bounded on each $X \oplus Y_i$ and maps (x_1, y_1) to (x_0, y_0). ∎

We now construct the example. We suppose $q > p > 1$; we set $r = \frac{1}{2}(p + q)$, $\alpha = p - 1$ and $\beta = q - r$. We next define $a_1 = 1$ and then inductively $(b_n)_{n\ge1}, (c_n)_{n\ge1}, (d_n)_{n\ge1}$ and $(a_n)_{n\ge2}$ by letting $b_n = 2^n a_n$, $c_n = 4b_n$, $d_n = c_n + 2n$ and $a_{n+1} = 4d_n$.

We then can construct an unbounded nonnegative Lipschitz function $\phi : \mathbb{R} \to \mathbb{R}$ so that $\operatorname{supp}\phi \subset \bigcup_{n\ge1}[a_n, b_n]$ and $|\phi'(x)| \le \alpha x^{-1}$ a.e. (or equivalently $|\phi(x) - \phi(y)| \le |\log x - \log y|$ for $x, y \ge 1$). We then also define a nonnegative Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ with $\operatorname{supp}\psi \subset \bigcup_{n\ge1}[c_n, d_n]$ by defining $\psi(x) = \beta(x - c_n)$ for $c_n \le x \le c_n + n$ and $\psi(x) = \beta(d_n - x)$ for $c_n + n \le x \le d_n$. Finally, we put $F(x) = x^r \exp(\psi(\log x))$ and $G(x) = x^r \exp(\psi(\log x) + \phi(\log x))$.

Now observe that F and G both satisfy the Δ_2-condition and both $F(x)/x$ and $G(x)/x$ are increasing functions so that F and G are equivalent to convex Orlicz functions. We prefer to work directly with F and G. We consider the pair (E_F, E_G). For $n < 0$ let λ_n be the unique solution of $F(\lambda_n) = 2^{-n}$ and let ν_n be the unique solution of $G(\nu_n) = 2^{-n}$. We split $\mathbb{Z}_-$ into two disjoint sets J_0, J_1 by setting $J_0 = \{n : \log\lambda_n \in \bigcup_k [c_k/2, 2d_k]\}$ and $J_1 = \mathbb{Z}_- \setminus J_0$.

We claim that on $\omega(J_0)$ the norms $\|\ \|_{L_F}$ and $\|\ \|_{L_G}$ are equivalent. In fact, since $F \le G$ we need only bound $\sum_{n\in J_0} 2^n G(\xi_n)$ subject to $\sum_{n\in J_0} 2^n F(\xi_n) = 1$. To do this observe that if $n \in J_0$ and $0 \le \xi_n \le \lambda_n$ then $F(\xi_n) = G(\xi_n)$ unless $\log\xi_n < \log\lambda_n/2$. Thus

$$\sum_{n\in J_0} G(\xi_n) \le 1 + \sum_{n\in J_0} 2^n G(\sqrt{\lambda_n}) \le 1 + \sum_{n\in J_0} \lambda_n^{-1/2}$$

and this establishes the required estimate since λ_n increases geometrically.

On $\omega(J_1)$ we claim both $\|\ \|_F$ and $\|\ \|_G$ are equivalent to weighted ℓ_r-norms and hence form a Calderón couple by the result of Sparr [36]. Let us do this for the case of $\|\ \|_G$ which we claim is equivalent to $(\sum_{n\in J_1} |\xi_n/\nu_n|^r)^{1/r}$. It suffices to (a) bound $\sum_{n\in J_1} 2^n G(\xi_n)$ subject to $\sum_{n\in J_1} \xi_n^r \nu_n^{-r} = 1$ and (b) conversely, bound $\sum_{n\in J_1} \xi_n^r \nu_n^{-r}$ subject to $\sum_{n\in J_1} 2^n G(\xi_n) = 1$.

For (a) note that if $0 \le \xi_n \le \nu_n$ then

$$\left|\log G(\xi_n) - \log G(\nu_n) - r\log\frac{\xi_n}{\nu_n}\right| \le \alpha \log\frac{\log\nu_n}{\log\xi_n}$$

as long as $\log\xi_n > \log\nu_n/2$. Hence

$$G(\xi_n) \le 2^\alpha 2^{-n} \nu_n^{-r} \xi_n^r + G(\sqrt{\nu_n})$$

for $n \in J_1$. Thus

$$\sum_{n\in J_1} 2^n G(\xi_n) \le 2^\alpha + \sum_{n\in J_1} 2^n G(\sqrt{\nu_n}) \le 2^\alpha + \sum_{n\in J_1} \nu_n^{-1/2}$$

and this gives the required estimate. (b) is similar. The argument that $\|\ \|_F$ is equivalent to $(\sum_{n\in J_1} |\xi_n|^r \lambda_n^{-r})^{1/r}$ is slightly simpler and we omit it.

This completes the construction of the example. It is clear from Lemma 6.11 that (E_F, E_G) and hence $(L_F[0,1], L_G[0,1])$ is a Calderón couple with $p_F = p_G = p$ and $q_F = q_G = q$ but that F and G are nonequivalent.

We remark in closing that it is possible to find Orlicz function spaces $L_F[0,1]$ so that if (L_F, L_G) forms a Calderón pair then $L_F = L_G$. (We assume the Δ_2-condition for both F and G.) We sketch the details. The argument of Theorem 6.10 can be used to establish that if F and G are not equivalent at ∞ then there exists p with $1 \le p < \infty$ so that x^p is equivalent, for $0 \le x \le 1$, to a function of the form $\lim F_{t_n}(x)$ where $t_n \to \infty$. Now, there are many examples of functions F which fail this property; for example one can take the minimal Orlicz function:

$$F(x) = x^2 \exp\left(\alpha \sum_{n=0}^{\infty} (1 - \cos(2\pi(\log t)/2^n))\right)$$

(see [20]).

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