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Hausdorff and conformal measures for expanding piecewise monotonic maps of the interval II

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Abstract. We construct examples of expanding piecewise monotonic maps on the interval which have a closed topologically transitive invariant subset A with Darboux property, Hausdorff dimension $d \in (0,1)$ and zero d-dimensional Hausdorff measure. This shows that the results about Hausdorff and conformal measures proved in the first part of this paper are in some sense best possible.

1. Introduction. For the investigation of the dynamics of a map T on an interval I Lebesgue measure on I is often considered as a natural measure. It is also the Hausdorff measure of dimension 1. If one wants to investigate T on an invariant Cantor set $A \subset I$ of Hausdorff dimension $d < 1$, one can try to find measures with similar properties. On the one hand, one can consider d-conformal measures which are defined below (Lebesgue measure is a 1-conformal measure), and on the other hand, one has the d-dimensional Hausdorff measure ν restricted to A (for the definition and properties of Hausdorff measures and Hausdorff dimension see [1]). For certain T-invariant subsets A of expanding piecewise monotonic maps T equality of the unique d-conformal measure on A and of $\nu/\nu(A)$ is shown in [5], provided that $\nu(A) > 0$. Therefore the question arises whether $\nu(A) = 0$ is possible for an expanding piecewise monotonic map T. In this paper we give examples which show that the answer is yes. The reason why these examples have this unexpected behavior is that they are far from having the Markov property.

We begin with a description of the results of [5]. A map T from the interval I to itself is called *piecewise monotonic* if there is a finite partition $\mathcal{Z}$ of I into intervals such that $T|Z$ is continuous and monotone for all $Z \in \mathcal{Z}$. For simplicity we only consider piecewise monotonic maps which are continuous on I. Furthermore, we assume that T' exists and is Hölder

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continuous in the interior of each $Z \in \mathcal{Z}$, and that T is expanding, which means that $\inf |(T^k)'| > 1$ for some $k \geq 1$, where the infimum is taken over the set on which $(T^k)'$ exists. Let A be a closed T-invariant topologically transitive subset of I which has the *Darboux property*. This means that

$$(1.1)\qquad T(Y \cap A) = T(Y) \cap A \quad \text{if } Y \subset Z \text{ for some } Z \in \mathcal{Z}.$$

A probability measure m concentrated on A which satisfies

$$(1.2)\qquad m(T(Y)) = \int_Y |T'|^d \, dm \quad \text{if } Y \subset A \cap Z \text{ for some } Z \in \mathcal{Z}$$

is called *d-conformal*.

Under the above assumptions on T and A, it is shown in [5] that there is a unique d-conformal measure m on A with support A and without atoms if d is greater than zero and equals the Hausdorff dimension $\mathrm{HD}(A)$ of A. Furthermore, if ν is the d-dimensional Hausdorff measure restricted to A, the existence of $c \in [0, \infty)$ is shown such that $\nu = cm$. If additionally T satisfies the Misiurewicz condition, then c cannot be zero. The question is, therefore, whether $c = 0$ or equivalently $\nu \equiv 0$ is possible in general. The result of this paper is that $\nu \not\equiv 0$ can only be proved under additional assumptions like the Misiurewicz condition, which is a weaker version of the Markov property. Actually, the set A considered here is the closure of the set A considered in [5]. This makes no difference, since this closure only adds a countable set to A and since the measures we investigate have no atoms. As only continuous maps are considered in this paper, it is not necessary to use the more complicated definitions of [5].

The Darboux property is quite natural. Let F be a T-invariant subset of I. Set $A = I \setminus \bigcup_{i=0}^{\infty} T^{-i}(F)$. Then A is T-invariant and even satisfies $T^{-1}(A) = A$. This implies $T(Y \cap A) = T(Y \cap T^{-1}(A)) = T(Y) \cap A$, proving (1.1). Hence A also has the Darboux property.

In this paper we consider unimodal maps. We say that $T : \mathbb{R} \to \mathbb{R}$ is *unimodal* if T is continuous, $T|(-\infty, 0]$ is strictly increasing, and $T|[0, \infty)$ is strictly decreasing. Interesting behaviour occurs if

$$(1.3)\qquad T^2(0) < 0 < T(0) \quad \text{and} \quad T^2(0) \leq T^3(0).$$

In this case the interval $I := [T^2(0), T(0)]$ is T-invariant and $T|I$ is piecewise monotonic with $\mathcal{Z} = \{[T^2(0), 0), [0, T(0)]\}$.

We consider the following family of unimodal maps:

$$(1.4)\qquad T(x) = \begin{cases} \beta x + \beta t + 1 & \text{for } x \in (-\infty, 0], \\ -\beta x + \beta t + 1 & \text{for } x \in [0, t], \\ -x + t + 1 & \text{for } x \in [t, 1], \\ -\beta x + \beta + t & \text{for } x \in [1, \infty), \end{cases}$$

where $\beta \geq 2$ and $t \in [0,1]$. If the parameters β and t in (1.4) are chosen such that (1.3) holds and hence $I := [T^2(0), T(0)]$ is T-invariant, set

$$(1.5)\qquad A = I \setminus \bigcup_{i=0}^{\infty} T^{-i}(F) \quad \text{with } F = (t, 1).$$

Then A is T-invariant and closed, since F is T-invariant. Above we have shown that A has the Darboux property. Furthermore, $|T'(x)| = \beta$ for $x \in \mathbb{R} \setminus F$ and $|T'(x)| = 1$ for $x \in F$. Observe that A and $T|A$ do not change if one redefines T on F. For example, one can redefine T on F to have $T(F) \subset F$, $T|F$ piecewise monotonic, and $|T'| \equiv \beta$ on all of $\mathbb{R}$. Then T is no more unimodal, but it becomes an expanding piecewise monotonic map on I with piecewise Hölder continuous derivative, so that the results of [5] apply.

We shall show that for each $d \in (0, 1)$ there are $\beta > 2$ and $t \in [0, 1]$ such that T defined by (1.4) satisfies (1.3) and such that A defined by (1.5) is topologically transitive, has Hausdorff dimension d and satisfies $\nu(A) = 0$, where ν denotes the d-dimensional Hausdorff measure. The main tools for the construction of these maps are kneading sequences and Markov diagrams of unimodal maps. In Section 2 we show how one can construct unimodal maps with prescribed kneading sequences. In Section 3 the Markov diagram is introduced. It is a directed graph which reflects the dynamics of the transformation T and which is used to estimate the length of certain intervals. Finally, in Section 4 particular members of the family (1.4) of unimodal maps are constructed and the result stated above is shown.

2. Unimodal maps and their kneading sequences. For a unimodal map which satisfies $T^i(0) \neq 0$ for $i \geq 1$, we define the *kneading sequence* $e_1 e_2 \ldots \in \{0,1\}^{\mathbb{N}}$ as follows:

$$(2.1)\qquad e_i = \begin{cases} 0 & \text{if } T^i(0) < 0, \\ 1 & \text{if } T^i(0) > 0. \end{cases}$$

Furthermore, we say that $(T_t)_{t \in [0,1]}$ is a *continuous family of unimodal maps* if each $T_t : \mathbb{R} \to \mathbb{R}$ is unimodal, $T_t(0) > 0$ for all $t \in [0,1]$, $(t, x) \mapsto T_t(x)$ is continuous as a map from $[0,1] \times \mathbb{R}$ to $\mathbb{R}$, $T_0^2(0) \geq 0$ and $T_1^3(0) \leq T_1^2(0) < 0$. The goal of this section is to find a member of a continuous family of unimodal maps with a prescribed kneading sequence.

To this end we consider maps $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ which satisfy

$$(2.2)\qquad Q(k) \leq k - 1 \quad \text{for } k \in \mathbb{N}.$$

We say that such a map Q *generates* a sequence $(r_i)_{i \geq 1}$ of integers if

$$(2.3)\qquad r_i = R_{Q(i)} + 1 \quad \text{for } i \geq 1 \quad \text{where} \quad R_0 = 0 \text{ and } R_j = \sum_{l=1}^{j} r_l.$$

Because of (2.2) the sequence $(r_i)_{i\ge1}$ is uniquely determined by (2.3). Note that $r_1 = 1$, since (2.2) implies $Q(1) = 0$.

For $e \in \{0,1\}$ set $e' = 1$ if $e = 0$ and $e' = 0$ if $e = 1$. We say that a sequence $(r_i)_{i\ge1}$ of integers *generates* $e_1e_2\ldots \in \{0,1\}^{\mathbb{N}}$ if $e_1 = 1$ and

$$(2.4)\qquad e_{R_i+j+1} = e_j \quad \text{for } 1 \le j < r_{i+1} \quad \text{and} \quad e_{R_{i+1}+1} = e'_{r_{i+1}}$$

for $i \ge 0$, where R_i is as in (2.3). Again (2.4) determines a unique 0-1-sequence with first element 1.

Finally, we say that a map Q satisfying (2.2) *generates* a 0-1-sequence $e_1e_2\ldots$ if Q generates $(r_i)_{i\ge1}$ and $(r_i)_{i\ge1}$ generates $e_1e_2\ldots$, in which case $e_2 = 0$ by (2.4), since $r_1 = 1$.

In [6] a map $Q : \mathbb{N} \to \mathbb{N}\cup\{0\}$ satisfying (2.2) is called a *kneading map* if

$$(Q(k+l))_{l\ge1} \ge (Q(Q^2(k)+l))_{l\ge1} \quad \text{for all } k \ge 1 \text{ with } Q(k) \ge 1,$$

where $\ge$ denotes the lexicographic order in $(\mathbb{N}\cup\{0\})^{\mathbb{N}}$ and Q^2 denotes the second iterate of Q. It is shown in [6] that a 0-1-sequence occurs as the kneading sequence of a member of certain families of unimodal C^1-maps if it is generated by a kneading map.

In this paper we consider unimodal maps which are not everywhere differentiable. Hence we need additional assumptions on Q to get similar results. We use the following conditions:

$$(2.5)\qquad Q(1) = 0 \quad \text{and} \quad Q(k+1) < k \quad \text{for } k \ge 1,$$

$$(2.6)\qquad Q(k+1) > Q(Q^2(k)+1) \quad \text{for all } k \ge 1 \text{ with } Q(k) \ge 1.$$

Note that (2.5) implies (2.2). In order to show that 0-1-sequences generated by maps Q satisfying (2.5) and (2.6) occur as kneading sequences of unimodal maps, we need some lemmas. For a 0-1-sequence $e_1e_2\ldots$ with $e_1 = 1$ set

$$(2.7)\qquad n = \min\{k \ge 2 : e_k = 1\}$$

and define $a : \mathbb{N} \to \mathbb{N}$ by $a(1) = 1$ and

$$(2.8)\qquad a(k+1) = \begin{cases} a(k) & \text{if } e_{k+1} = e_{k+1-a(k)} \\ k+1 & \text{if } e_{k+1} = e'_{k+1-a(k)} \end{cases} \qquad \text{for } k \ge 1,$$

and $b : \{k \in \mathbb{N} : k \ge n\} \to \{k \in \mathbb{N} : k \ge n\}$ by $b(n) = n$ and

$$(2.9)\qquad b(k+1) = \begin{cases} b(k) & \text{if } e_{k+1} = e_{k+1-b(k)} \\ k+1 & \text{if } e_{k+1} = e'_{k+1-b(k)} \end{cases} \qquad \text{for } k \ge n.$$

We allow that $n = \infty$, which means that $e_k = 0$ for $k \ge 2$. In this case no b is defined and some statements in the following lemmas are vacuous or have to be interpreted in the obvious way.

LEMMA 1. *Let $e_1e_2\ldots$ be a 0-1-sequence generated by a map $Q : \mathbb{N} \to \mathbb{N}\cup\{0\}$ which satisfies* (2.5) *and* (2.6). *Let n and the maps a and b be as in* (2.7)–(2.9). *Then $n \ge 3$ and*

(i) *$a(k) = k$ for $k < n$, $a(n) = n-1$, $Q(k) = 0$ for $k \le n-2$, $Q(k) \ge 1$ for $k \ge n-1$,*

(ii) *$k+1-a(k) < a(k)$ for $k \ge 2$ and $k+1-b(k) < b(k)$ for $k \ge n$,*

(iii) *$a(k) \ne b(k)$ for $k \ge n$,*

(iv) *$e_{k+1-a(k)} = e_{k+1-b(k)} \Rightarrow e_{k+1} = e_{k+1-a(k)}$ if $k \ge n$.*

Proof. Since $Q(1) = 0$ by (2.5), we get $r_1 = 1$ by (2.3) and $e_2 = 0$ by (2.4). Hence $n \ge 3$. Since $e_1 = e_n = 1$ and $e_k = 0$ for $1 < k < n$ by (2.7), we get $a(k) = k$ for $k < n$ and $a(n) = n-1$ by (2.8). Together with (2.4) this also implies that

$$(2.10)\qquad r_k = 1 \text{ for } k \le n-2, \quad r_{n-1} \ge 2, \quad R_{n-2} = n-2, \quad R_{n-1} \ge n.$$

Furthermore, (2.10) and (2.3) imply $Q(k) = 0$ for $k \le n-2$ and $Q(n-1) \ge 1$. If $Q(k) \ge 1$, then (2.6) implies $Q(k+1) \ge 1$, which finishes the proof of (i).

In order to show the other assertions, we write S_k instead of $R_k + 1$ and set $p_k = \max\{i : S_i \le k\}$. We show

$$(2.11)\qquad a(k) = S_{p_k} \quad \text{for } k \ge 1.$$

For $k = 1$ we have $p_k = 0$ and $a(k) = 1$, which shows (2.11) for $k = 1$. We proceed by induction and suppose that (2.11) is shown for $k = l$. By definition of p_l we have $S_{p_l} \le l < S_{p_l+1}$. If $l+1 < S_{p_l+1}$, then $p_{l+1} = p_l$ and $e_{l+1} = e_{l+1-S_{p_l}} = e_{l+1-a(l)}$ by (2.4) and the induction hypothesis. Hence (2.8) implies $a(l+1) = a(l)$, and $a(l+1) = S_{p_{l+1}}$ follows from the induction hypothesis. If $l+1 = S_{p_l+1}$, then $p_{l+1} = p_l + 1$ and $e_{l+1} = e'_{l+1-S_{p_l}} = e'_{l+1-a(l)}$ by (2.4) and the induction hypothesis. Hence (2.8) implies $a(l+1) = l+1$, which equals $S_{p_{l+1}}$. This finishes the proof of (2.11).

If now $k \ge 2$, then $p_k \ge 1$ as $r_1 = 1$, and $k+1-a(k) \le r_{p_k+1} = S_{Q(p_k+1)} < S_{p_k} = a(k)$ follows from (2.11), (2.3) and (2.5). This is the first part of (ii).

Next we do the same for the $b(k)$'s. By (2.7), (2.10) and (2.4) we get $e_n = 1$ and $e_k = 0$ for $n+1 = S_{n-2}+2 \le k < S_{n-1}$, as $r_{n-1} \le S_{n-2} = n-1$ by (2.3) and (2.5). Since $e_1 = 1$, we see by (2.9) that

$$(2.12)\qquad b(k) = k \quad \text{for } n \le k < S_{n-1}.$$

If $k \ge S_{n-1}$ then $p_k \ge n-1$ and we have $Q(p_k) \ge 1$ by (i). Set $u_k = Q(Q^2(p_k)+1)$ and $v_k = Q(p_k+1)$. Note that $u_k < v_k$ by (2.6). For $k \ge S_{n-1}$ we show that

$$(2.13)\qquad b(k) = \begin{cases} S_{p_k} - r_{Q(p_k)} & \text{if } S_{p_k} \le k < S_{p_k} + S_{u_k}, \\ S_{p_k} + S_q & \text{if } S_{p_k} + S_q \le k < S_{p_k} + S_{q+1},\ u_k \le q < v_k. \end{cases}$$

This covers all $k \geq S_{n-1}$, since $S_{p_k} + S_{v_k} = S_{p_k+1}$. For $k = S_{n-1} - 1$ we have $p_k = n-2$. Hence $k = S_{p_k} + S_{v_k} - 1$ by (2.3). By (2.5) and (2.10) we get $r_{v_k} = r_{Q(n-1)} = 1$ and therefore $k = S_{p_k} + S_{v_k - 1}$. Hence (2.13) gives the correct result $b(k) = k$ already for $k = S_{n-1} - 1$. We proceed by induction and suppose that (2.13) is shown for $k = l$, where $l \geq S_{n-1} - 1$. In order to show (2.13) for $k = l+1$, suppose first that $S_{p_l} \leq l < S_{p_l} + S_{u_l}$. This implies $p_{l+1} = p_l$, since $S_{u_l} < r_{p_l+1}$ by (2.6). If $l+1 < S_{p_l} + S_{u_l}$, applying the induction hypothesis and (2.4) twice we get

$$e_{l+1-b(l)} = e_{l+1-S_{p_l}+r_{Q(p_l)}} = e_{l+1-S_{p_l}+S_{Q^2(p_l)}} = e_{l+1-S_{p_l}} = e_{l+1}$$

since $l+1-S_{p_l} < r_{Q^2(p_l)+1} = S_{u_l} < S_{v_l} = r_{p_l+1}$ by (2.3) and (2.6). By (2.9) we get $b(l+1) = b(l)$. As $p_{l+1} = p_l$ the induction hypothesis implies now (2.13) for $k = l+1$. If $l+1 = S_{p_l} + S_{u_l}$, applying the induction hypothesis and (2.4) twice we get

$$e_{l+1-b(l)} = e_{l+1-S_{p_l}+r_{Q(p_l)}} = e_{l+1-S_{p_l}+S_{Q^2(p_l)}} = e'_{l+1-S_{p_l}} = e'_{l+1}$$

since $l+1-S_{p_l} = r_{Q^2(p_l)+1} = S_{u_l} < S_{v_l} = r_{p_l+1}$ by (2.3) and (2.6). By (2.9) we get $b(l+1) = l+1$. As $p_{l+1} = p_l$ this equals $S_{p_{l+1}} + S_{u_l}$, and we have (2.13) for $k = l+1$.

Now suppose that $S_{p_l} + S_q \leq l < S_{p_l} + S_{q+1}$ and $u_l \leq q < v_l$. If $l+1 < S_{p_l} + S_{q+1}$ we have $p_{l+1} = p_l$, as $S_{q+1} \leq r_{p_l+1}$ by (2.3). Applying the induction hypothesis and (2.4) twice we get

$$e_{l+1-b(l)} = e_{l+1-S_{p_l}-S_q} = e_{l+1-S_{p_l}} = e_{l+1}$$

since $l+1-S_{p_l}-S_q < r_{q+1}$ and $l+1-S_{p_l} < S_{q+1} \leq r_{p_l+1}$. By (2.9) we get $b(l+1) = b(l)$. As $p_{l+1} = p_l$, the induction hypothesis now implies (2.13) for $k = l+1$. If $l+1 = S_{p_l} + S_{q+1}$ and $q+1 < v_l$ we have $p_{l+1} = p_l$, as $S_{q+1} < r_{p_l+1}$ by (2.3). Applying the induction hypothesis and (2.4) twice we get

$$e_{l+1-b(l)} = e_{l+1-S_{p_l}-S_q} = e'_{l+1-S_{p_l}} = e'_{l+1}$$

since $l+1-S_{p_l}-S_q = r_{q+1}$ and $l+1-S_{p_l} = S_{q+1} < r_{p_l+1}$. By (2.9) we get $b(l+1) = l+1$. As $p_{l+1} = p_l$, this equals $S_{p_{l+1}} + S_{q+1}$, and (2.13) for $k = l+1$ follows. Finally, if $l+1 = S_{p_l} + S_{q+1}$ and $q+1 = v_l$, we have $l+1 = S_{p_l+1}$ by (2.3) and $p_{l+1} = p_l + 1$. Applying the induction hypothesis and (2.4) twice we get

$$e_{l+1-b(l)} = e_{l+1-S_{p_l}-S_q} = e'_{l+1-S_{p_l}} = e'_{r_{p_l+1}} = e_{S_{p_l+1}} = e_{l+1}$$

since $l+1-S_{p_l}-S_q = r_{q+1}$. By (2.9) we get $b(l+1) = b(l)$. As $S_{v_l} = r_{p_l+1}$ we have $b(l) = S_{p_l+1} - r_{v_l}$ by the induction hypothesis. As $p_{l+1} = p_l + 1$ we get $b(l+1) = S_{p_{l+1}} - r_{Q(p_{l+1})}$, which is (2.13) for $k = l+1$. This finishes the proof of (2.13).

Now we can show the second part of (ii). For $n \leq k < S_{n-1}$ it is trivial, since $b(k) = k$ by (2.12). Hence suppose that $k \geq S_{n-1}$, so that $Q(p_k) \geq 1$ and (2.13) applies. By (2.5) we have $u_k < Q(p_k)$ and $v_k < p_k$. If $S_{p_k} \leq k < S_{p_k} + S_{u_k}$ then $k+1-b(k) \leq S_{u_k} + r_{Q(p_k)}$, which does not exceed r_{p_k}, as $u_k \leq Q(p_k) - 1$, and $b(k) = S_{p_k} - S_{Q^2(p_k)}$, which is larger than r_{p_k} by (2.5). If $S_{p_k} + S_q \leq k < S_{p_k} + S_{q+1}$ and $u_k \leq q < v_k$, then $k+1-b(k) \leq r_{q+1}$ and $b(k) > S_{p_k} > S_{q+1}$, as $p_k > v_k \geq q+1$. This shows (ii) in all cases.

We have $a(k) = S_{n-2} \neq b(k)$ for $n = S_{n-2} + 1 \leq k < S_{n-1}$ by (2.10)–(2.12). If $k \geq S_{n-1}$, then $Q(p_k) \geq 1$ by (i) and $a(k) = S_{p_k} \neq b(k)$ by (2.11) and (2.13). This proves (iii).

Finally, suppose that $e_{k+1-a(k)} = e_{k+1-b(k)} \neq e_{k+1}$ for some $k \geq n$. By (2.8) and (2.9) we get $a(k+1) = k+1 = b(k+1)$, which contradicts (iii). Hence (iv) is also shown. ■

For a continuous family $(T_t)_{t\in[0,1]}$ of unimodal maps we define

(2.14) $$P_k : [0,1] \to \mathbb{R} \quad \text{by} \quad P_k(t) = T_t^k(0) \text{ for } k \geq 0\,.$$

The maps P_k are continuous and we have

(2.15) $$P_0 \equiv 0 \quad \text{and} \quad P_k(t) = T_t(P_{k-1}(t)) \quad \text{for } k \geq 1 \text{ and } t \in [0,1]\,.$$

This recursion formula implies immediately that

(2.16) $$P_m(t) = 0 \;\Rightarrow\; P_{m+k}(t) = P_k(t) \text{ for } k \geq 1\,.$$

Since $T_t(x) \leq T_t(0)$ for all $x \in \mathbb{R}$ by the definition of unimodal maps, (2.15) implies that

(2.17) $$P_k(t) \leq P_1(t) \quad \text{for } k \geq 1 \text{ and } t \in [0,1]\,.$$

The assumptions $T_t(0) > 0$ for $t \in [0,1]$, $T_0^2(0) \geq 0$ and $T_1^3(0) \leq T_1^2(0) < 0$ imply

(2.18) $$P_1(t) > 0 \text{ for } t \in [0,1], \quad P_2(0) \geq 0 \quad \text{and} \quad P_k(1) < 0 \text{ for } k \geq 2$$

since $T|(-\infty,0]$ is increasing and hence $T_1^3(0) \leq T_1^2(0) < 0$ implies $T_1^k(0) \leq T_1^2(0) < 0$ for $k \geq 2$. After these preparations we can show

LEMMA 2. *Let $e_1e_2\ldots$ be the 0-1-sequence generated by a map $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ satisfying (2.5) and (2.6). Let n and the maps a and b be as in (2.7)–(2.9). Let $(T_t)_{t\in[0,1]}$ be a continuous family of unimodal maps and let the maps P_k for $k \geq 0$ be as in (2.14). For $k \geq 1$ there are nonempty intervals $(c_k, d_k) \subset (0,1)$ with the following properties.*

(i) *We have $c_1 = 0$ and $P_2(c_2) = 0$. If $k \geq 3$ and $a(k) = a(k-1)$, then $c_k = c_{k-1}$. If $k \geq 3$ and $a(k) = k$, which is equivalent to $a(k) \neq a(k-1)$ by (2.8), then $c_k > c_{k-1}$ and $P_k(c_k) = 0$.*

(ii) *If* $k < n$, *then* $d_k = 1$. *If* $k \geq n$ *and* $b(k) = b(k-1)$, *then* $d_k = d_{k-1}$. *If* $k \geq n$ *and* $b(k) = k$, *which is equivalent to* $b(k) \neq b(k-1)$ *by* (2.9), *then* $d_k < d_{k-1}$ *and* $P_k(d_k) = 0$.

(iii) *If* $k \geq 1$ *and* $e_k = 1$ *then* $P_k(t) > 0$ *for* $t \in (c_k, d_k)$. *If* $k \geq 1$ *and* $e_k = 0$ *then* $P_k(t) < 0$ *for* $t \in (c_k, d_k)$.

(iv) *For* $k \geq n$ *and* $t \in (c_k, d_k)$ *we have either* $P_{k+1-a(k)}(t) \leq P_{k+1}(t) \leq P_{k+1-b(k)}(t)$ *or* $P_{k+1-b(k)}(t) \leq P_{k+1}(t) \leq P_{k+1-a(k)}(t)$.

Proof. We use induction on k. Set $c_1 = 0$ and $d_1 = 1$. As $P_1(t) > 0$ for $t \in (c_1, d_1)$ by (2.18), as $e_1 = 1$, and as $n \geq 3$, all assertions hold for $k = 1$. Therefore suppose that $l \geq 2$ and that all assertions are shown for $k < l$.

We first consider the case $l \leq n$. The induction hypothesis implies $P_{l-1}(c_{l-1}) = 0$ if $l \geq 3$, since $a(j) = j$ for $j < n$ by Lemma 1(i). Hence $P_l(c_{l-1}) = P_1(c_{l-1}) > 0$ by (2.16) and (2.18) if $l \geq 3$. Furthermore, $P_2(c_1) \geq 0$ and $P_l(d_{l-1}) < 0$ for $l \geq 2$ by (2.18) and induction hypothesis. If now $l < n$, set $d_l = 1$ and let c_l be the largest zero of P_l in $[c_{l-1}, 1]$, which exists as $P_l(c_{l-1}) \geq 0$. As $P_l(d_l) = P_l(1) < 0$, we have $c_l < d_l$ and $P_l(t) < 0$ for $t \in (c_l, d_l)$. If $l \geq 3$ we have $P_l(c_{l-1}) > 0$ and hence $c_{l-1} < c_l$. All assertions are shown for $k = l$ if $l < n$, since $e_l = 0$ and $a(l) = l$ in this case. If $l = n \geq 3$, set $c_l = c_{l-1}$ and let d_l be the smallest zero of P_l in (c_{l-1}, d_{l-1}), which exists since $P_l(c_{l-1}) > 0$ and $P_l(d_{l-1}) < 0$. We have $P_l(t) > 0$ for $t \in (c_l, d_l)$, proving (iii) for $k = l$, as $e_n = 1$. Furthermore, $c_{l-1} < d_l < d_{l-1}$, which shows (ii) for $k = l$, since $b(n) = n$. As $a(n) = n - 1 = a(n-1)$ by Lemma 1(i) we also have (i) for $k = l$. Finally, using (2.17), for $t \in (c_l, d_l)$ we get $P_0(t) = 0 \leq P_l(t) \leq P_1(t)$. Together with (2.15) this implies $P_2(t) \leq P_{l+1}(t) \leq P_1(t)$, as $T|[0, \infty)$ is decreasing. We have shown (iv) for $k = l = n$, since $a(n) = n - 1$ and $b(n) = n$, finishing the induction step for $l \leq n$.

Now we consider the case $l > n$. As $a(2) = 2$ we have $P_{a(2)}(c_2) = 0$. If $P_{a(k-1)}(c_{k-1}) = 0$ is shown and if $k \leq l - 1$, we get $P_{a(k)}(c_k) = 0$ by (i), which says that either $P_{a(k)}(c_k) = 0$ or $P_{a(k)}(c_k) = P_{a(k-1)}(c_{k-1})$. Hence $P_{a(l-1)}(c_{l-1}) = 0$. Similarly we get $P_{b(l-1)}(d_{l-1}) = 0$ using $b(n) = n$ and (ii). Setting $i = l - a(l-1)$ and $j = l - b(l-1)$, from (2.16) we get

$$P_l(c_{l-1}) = P_i(c_{l-1}) \quad \text{and} \quad P_l(d_{l-1}) = P_j(d_{l-1}). \tag{2.19}$$

Furthermore, Lemma 1(ii) implies $i < a(l-1)$ and $j < b(l-1)$. This gives

$$c_i < c_{l-1} \quad \text{and} \quad d_{l-1} < d_j, \tag{2.20}$$

because otherwise the induction hypothesis would imply $a(i) = a(l-1) > i$ or $b(j) = b(l-1) > j$, which contradicts (2.8) or (2.9).

Suppose first that $e_i = e_j$. Set $s = 1$ if $e_i = e_j = 1$, and $s = -1$ if $e_i = e_j = 0$. By (iii) for $k = i$ and $k = j$ we have $\operatorname{sign} P_i(t) = \operatorname{sign} P_j(t) = s$ for $t \in (c_{l-1}, d_{l-1})$. By (iv) for $k = l-1$ we get $\operatorname{sign} P_l(t) = s$ for $t \in (c_{l-1}, d_{l-1})$. Set $c_l = c_{l-1}$ and $d_l = d_{l-1}$. Since $e_l = e_i = e_j$ by Lemma 1(iv) and since this implies $a(l) = a(l-1)$ and $b(l) = b(l-1)$ by (2.8) and (2.9), we get (i), (ii) and (iii) for $k = l$. Finally, P_l, P_i and P_j have the same sign on $(c_{l-1}, d_{l-1}) = (c_l, d_l)$ and T_t is monotone on $(-\infty, 0]$ and on $[0, \infty)$. Hence we get (iv) for $k = l$ from (iv) for $k = l-1$ and from (2.15), as $a(l) = a(l-1)$ and $b(l) = b(l-1)$.

Suppose now that $e_i \neq e_j$. In this case (2.20) and (iii) for $k = i$ and $k = j$ imply that $\operatorname{sign} P_i(c_{l-1}) \neq \operatorname{sign} P_j(d_{l-1})$. Therefore $\operatorname{sign} P_l(c_{l-1}) \neq \operatorname{sign} P_l(d_{l-1})$ by (2.19). We have either $e_l = e_i$ or $e_l = e_j$. In the first case set $c_l = c_{l-1}$ and let d_l be the smallest zero of P_l in (c_{l-1}, d_{l-1}). Then $c_l < d_l < d_{l-1}$. As $e_l = e_i$ and as $\operatorname{sign} P_l(c_l) = \operatorname{sign} P_i(c_l)$ by (2.19), we get (iii) for $k = l$ from (iii) for $k = i$. Since $e_l = e_i = e'_j$ we get $a(l) = a(l-1)$ and $b(l) = l$. Hence (i) and (ii) hold for $k = l$. By (iv) for $k = l - 1$ and (2.19) we get either $P_0(t) = 0 \leq P_l(t) \leq P_i(t)$ or $P_0(t) = 0 \geq P_l(t) \geq P_i(t)$ for $t \in (c_l, d_l)$. Since T_t is monotone on $(-\infty, 0]$ and on $[0, \infty)$, together with (2.15) this implies (iv) for $k = l$, as $a(l) = a(l-1)$ and $b(l) = l$. This finishes the case $e_l = e_i \neq e_j$. The proof in the case $e_l = e_j \neq e_i$ is similar, setting $d_l = d_{l-1}$ and choosing c_l to be the largest zero of P_l in (c_{l-1}, d_{l-1}). ■

Now we can show

THEOREM 1. *Let* $(T_t)_{t \in [0,1]}$ *be a continuous family of unimodal maps. Suppose that* $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ *satisfies* (2.5) *and* (2.6). *Then there is* $q \in [0, 1]$ *such that* $T_q^k(0) \neq 0$ *for* $k \geq 1$, *the* 0-1-*sequence* $e_1 e_2 \ldots$ *generated by* Q *is the kneading sequence of* T_q, $T_q^2(0) < 0 < T_q(0)$ *and* $T_q^2(0) \leq T_q^3(0)$.

Proof. For $k \geq 1$ let (c_k, d_k) be the intervals found in Lemma 2. Set $q = \lim_{k \to \infty} c_k$, which exists, since $(c_k)_{k \geq 1}$ is increasing and bounded by 1. If $n < \infty$ we have $\operatorname{card}\{k : a(k) \neq a(k-1)\} = \infty$ and $\operatorname{card}\{k : b(k) \neq b(k-1)\} = \infty$ by Lemma 1(ii). Hence Lemma 2 implies that $q \in (c_k, d_k)$ for all $k \geq 1$. Now Lemma 2(iii) and (2.14) imply that $T_q^k(0) \neq 0$ for $k \geq 1$ and that $e_1 e_2 \ldots$ is the kneading sequence of T_q. If $n = \infty$, then $a(k) = k$ for $k \geq 1$ by Lemma 1(i), and $e_1 e_2 \ldots = 100\ldots$ by (2.7). By Lemma 2 we get $q \in (c_k, d_k] = (c_k, 1]$ for $k \geq 1$, and Lemma 2(iii), (2.14) and (2.18) imply that $T_q(0) > 0$ and that $T_q^k(0) < 0$ for $k \geq 1$. Hence $e_1 e_2 \ldots$ is the kneading sequence of T_q also in the case $n = \infty$.

It remains to show the last assertion. Since $n \geq 3$ by Lemma 1 we have $e_1 = 1$ and $e_2 = 0$. Hence $T_q^2(0) < 0 < T_q(0)$ by (2.1). We have $P_k(c_k) = 0$ for infinitely many $k \geq 2$ by Lemmas 1(ii) and 2(i). For these k we get $P_{k+1}(c_k) > 0$ by (2.16) and (2.18) and hence $T_{c_k}^{k+1}(0) > 0$ by (2.14). By Lemma 2 we have $T_{c_k}^2(0) = P_2(c_k) \leq 0$ for $k \geq 2$ as $e_2 = 0$. We see by induction that $T_{c_k}^3(0) \leq T_{c_k}^2(0)$ implies $T_{c_k}^l(0) \leq T_{c_k}^2(0)$ for $l \geq 3$, since $T|(-\infty, 0]$ is increasing. This contradicts $T_{c_k}^{k+1}(0) > 0$, and hence $T_{c_k}^3(0) >$

$T^2_{c_k}(0)$ holds for infinitely many k. Since $t \mapsto T^j_t(0)$ is continuous for $j = 2$ and $j = 3$, letting $k \to \infty$ we conclude that $T^3_q(0) \geq T^2_q(0)$, which finishes the proof. ■

Remark. Theorem 1 is not the best possible result. One can show that the assertion of Theorem 1 remains true if (2.5) is replaced by

$$Q(1) = 0 \quad \text{and} \quad (Q(k+l))_{l\geq 1} < (k, k, \ldots) \quad \text{for } k \geq 1\,,$$

and if (2.6) is replaced by

$$(Q(k+l))_{l\geq 1} > (Q(Q^2(k)+l))_{l\geq 1} \quad \text{for all } k \geq 1 \text{ with } Q(k) \geq 1\,,$$

where $>$ denotes the lexicographic order in $(\mathbb{N}\cup\{0\})^{\mathbb{N}}$. We have only proved the simpler result, since it is sufficient for the examples we construct and since it has a simpler proof.

3. The Markov diagram and its properties. Our main tool for the investigation of unimodal maps is the Markov diagram, which we now introduce. Let $T : \mathbb{R} \to \mathbb{R}$ be a unimodal map satisfying (1.3), so that $I := [T^2(0), T(0)]$ is T-invariant. Set $I_0 = (T^2(0), 0)$, $I_1 = (0, T(0))$ and $\mathcal{Y} = \{I_0, I_1\}$, so that $T|Y$ is monotone for $Y \in \mathcal{Y}$. Let C be an open interval contained in an element of $\mathcal{Y}$. We say that D is a *successor* of C if $D \neq \emptyset$ and if $D = T(C) \cap Y$ for some $Y \in \mathcal{Y}$. Since $T|Y$ is strictly monotone for $Y \in \mathcal{Y}$, D is again an open subinterval of some element of $\mathcal{Y}$, so that we can iterate the formation of successors. We write $C \to D$ if D is a successor of C. Let $\mathcal{D}$ be the minimal set which contains $\mathcal{Y}$ and which is closed under taking successors. The finite or countable directed graph $(\mathcal{D}, \to)$ is called the *Markov diagram* of $T|I$.

Now suppose that $T^k(0) \neq 0$ for $k \geq 1$ and that the kneading sequence $e_1 e_2 \ldots$ of T is determined by a sequence $(r_i)_{i\geq 1}$ of integers with $r_1 = 1$ as in (2.4). We describe $(\mathcal{D}, \to)$ in terms of the r_i's. To this end set $c_i = T^i(0)$ for $i \geq 0$. Let $\langle a, b\rangle$ denote the open interval (a, b) if $a < b$, and (b, a) if $a > b$. For $i \geq 1$ set $V_i = \langle c_i, c_{i-R_j-1}\rangle$ where j is such that $R_j < i \leq R_{j+1}$. Since $r_1 = 1$, by (2.1) and (2.4) we get

$$(3.1) \qquad V_1 = I_1, \quad V_2 = I_0 \quad \text{and} \quad V_i \subset I_{e_i} \quad \text{for } i \geq 1\,.$$

Furthermore, for $i \geq 1$ we show that

$$(3.2) \quad T(V_i) \cap I_{e_{i+1}} = V_{i+1}\,, \quad T(V_i) \cap I_{e'_{i+1}} = \begin{cases} \emptyset & \text{if } i \notin \{R_k : k \geq 1\}\,, \\ V_{r_k} & \text{if } i = R_k,\ k \geq 1\,. \end{cases}$$

By (3.1) we have $T(V_i) = \langle c_{i+1}, c_{i-R_j}\rangle$, where $R_j < i \leq R_{j+1}$. If $i < R_{j+1}$ then both c_{i+1} and c_{i-R_j} are in $I_{e_{i+1}}$ by (2.1) and (2.4). This implies that $T(V_i) \cap I_{e_{i+1}} = T(V_i) = V_{i+1}$ and $T(V_i) \cap I_{e'_{i+1}} = \emptyset$. If $i = R_{j+1}$ then $c_{i+1} \in I_{e_{i+1}}$ and $c_{i-R_j} = c_{r_{j+1}} \in I_{e'_{i+1}}$ by (2.1) and (2.4). Hence $T(V_i) \cap I_{e_{i+1}} = \langle c_{i+1}, c_0\rangle = V_{i+1}$ and $T(V_i) \cap I_{e'_{i+1}} = \langle c_{r_{j+1}}, c_0\rangle = V_{r_{j+1}}$. This last equality follows from the definition of $V_{r_{j+1}}$, since $r_{j+1} = R_{Q(j+1)} + 1$ by (2.3). Hence (3.2) is shown.

Now (3.1) and (3.2) imply that $\mathcal{D} = \{V_i : i \geq 1\}$ and that the arrows in $(\mathcal{D}, \to)$ are

$$(3.3) \qquad V_i \to V_{i+1} \quad \text{for } i \geq 1 \quad \text{and} \quad V_{R_j} \to V_{r_j} \quad \text{for } j \geq 1\,.$$

For later use we deduce some further properties of the Markov diagram.

LEMMA 3. *If D is the only successor of C then $T(C) = D$. If D and E are different successors of C, then $T(C) = D \cup E \cup \{0\}$ and $D \cap E = \emptyset$.*

Proof. Since $(T^2(0), T(0)) = I_0 \cup I_1 \cup \{0\}$ and $I_0 \cap I_1 = \emptyset$, this follows from the definition of successor. ■

LEMMA 4. *If $D_0 D_1 \ldots D_n$ is a path in $(\mathcal{D}, \to)$, then $T^j(\bigcap_{i=0}^n T^{-i}(D_i)) \subset D_j$ for $0 \leq j \leq n$ and $T^n(\bigcap_{i=0}^n T^{-i}(D_i)) = D_n$.*

Proof. We get $T^j(\bigcap_{i=0}^n T^{-i}(D_i)) = \bigcap_{i=0}^{n-j} T^{-i}(D_{i+j})$ for $0 \leq j \leq n$ by induction on j, since $T(D_j) \supset D_{j+1}$ by Lemma 3. This implies the desired result. ■

LEMMA 5. *Fix $D \in \mathcal{D}$ and $n \geq 1$. Let $\mathcal{P}$ be the set of all paths in $(\mathcal{D}, \to)$ of length n starting with D. Then $D \setminus \bigcup_{i=0}^{n-1} T^{-i}(\{0\})$ is the disjoint union of the sets $\bigcap_{i=0}^{n-1} T^{-i}(D_i)$ for all $D_0 D_1 \ldots D_{n-1} \in \mathcal{P}$.*

Proof. This follows easily by induction on n using Lemma 3. ■

Now set $\mathcal{Y}_n = \{\bigcap_{i=0}^{n-1} T^{-i}(Y_i) \neq \emptyset : Y_i \in \mathcal{Y}\}$, which is the set of maximal open intervals on which T^n is monotone. Set $K = \{0, T(0), T^2(0)\}$, which are the endpoints of the intervals in $\mathcal{Y}$. The intervals in $\mathcal{Y}_n$ are pairwise disjoint and their union is $I \setminus \bigcup_{i=0}^{n-1} T^{-i}(K)$. For $x \in I \setminus \bigcup_{i=0}^{n-1} T^{-i}(K)$ let $Y_n(x)$ be the unique element of $\mathcal{Y}_n$ which contains x. Furthermore, for $x \in I \setminus \bigcup_{i=0}^{\infty} T^{-i}(K)$ set $D_0(x) = Y_1(x)$ and $D_i(x) = T(D_{i-1}(x)) \cap Y_1(T^i(x))$ for $i \geq 1$. We then have

LEMMA 6. *If $x \in I \setminus \bigcup_{i=0}^{\infty} T^{-i}(K)$ and $D_i(x)$ for $i \geq 0$ is as above, then $D_0(x) D_1(x) \ldots$ is an infinite path in $(\mathcal{D}, \to)$ with $T^n(x) \in D_n(x)$ and $T^n(Y_{n+1}(x)) = D_n(x)$ for $n \geq 0$.*

Proof. We get $T^n(x) \in D_n(x)$ for $n \geq 0$ by induction. In particular, $D_n(x) \neq \emptyset$ for $n \geq 0$ and $D_0(x) D_1(x) \ldots$ is a path in $(\mathcal{D}, \to)$. It remains to show $T^n(Y_{n+1}(x)) = D_n(x)$ for $n \geq 0$. For $n = 0$ this is the definition of $D_0(x)$. Suppose that $l \geq 1$ and that this is shown for $n = l - 1$. We have $Y_{l+1}(x) = Y_l(x) \cap T^{-l}(Y_1(T^l(x)))$, because the right hand side is an element of $\mathcal{Y}_{l+1}$ and contains x. Hence $T^l(Y_{l+1}(x)) = T^l(Y_l(x)) \cap Y_1(T^l(x)) = T(D_{l-1}(x)) \cap Y_1(T^l(x)) = D_l(x)$ and the lemma is proved by induction. ■

Lemma 7. *Suppose that T is a unimodal map with $T^i(0) \neq 0$ for $i \geq 1$. For each $k \geq 1$ there is an open interval U which contains 0 such that each path $D_0D_1 \ldots D_k$ in $(\mathcal{D}, \to)$ of length $k+1$ satisfying $U \cap \bigcap_{i=0}^k T^{-i}(D_i) \neq \emptyset$ contains a $V_j \in \mathcal{D}$ with $j \geq k$.*

Proof. Since $c_i := T^i(0) \neq 0$ for $i \geq 1$, the definition of the intervals V_i and (3.1) imply that 0 is in the closure of V_i if and only if $i = R_m + 1$ for some $m \geq 0$. Let Y be the interval in $\mathcal{Y}_k$ which has c_1 as right endpoint. Then $T^{-1}(\overline{Y})$ contains 0 in its interior. Choose $U \subset T^{-1}(\overline{Y})$ such that $V_j \cap U = \emptyset$ for $j \in \{1, \ldots, k\} \setminus \{R_m + 1 : m \geq 0\}$. Since $\bigcap_{i=0}^{k-1} T^{-i}(I_{e_{i+1}})$ is in $\mathcal{Y}_k$ and contains c_1 in its closure, it equals Y. We get $T^i(U) \subset \overline{I}_{e_i}$ for $1 \leq i \leq k$ by Lemma 4. If now $D_0D_1 \ldots D_k$ is a path in $(\mathcal{D}, \to)$ with $U \cap \bigcap_{i=0}^k T^{-i}(D_i) \neq \emptyset$, then $U \cap D_0 \neq \emptyset$ and hence $D_0 = V_j$ with $j > k$ or with $j = R_m + 1$ for some $m \geq 0$.

In the first case the proof is finished. In the second case note that $D_i \subset I_{e_i}$ for $1 \leq i \leq k$, since D_i is contained in one of the two disjoint intervals I_0 and I_1 by (3.1), since $T^i(U) \subset \overline{I}_{e_i}$ for $1 \leq i \leq k$, and since $T^i(U) \cap D_i \neq \emptyset$ by Lemma 4 and the choice of $D_0D_1 \ldots D_k$. As V_{i+1} is the only successor of V_i for $R_m < i < R_{m+1}$ by (3.3), $D_0 = V_{R_m+1}$ implies $D_i = V_{R_m+i+1}$ for $1 \leq i < r_{m+1}$. Furthermore, (3.1) and (3.3) imply $D_i = V_i$ for $r_{m+1} \leq i \leq k$, since $D_i \subset I_{e_i}$. We have either $D_k = V_{R_m+k+1}$ or $D_k = V_k$. In both cases we get $D_k \in \{V_i : i \geq k\}$, finishing the proof. ∎

Lemma 8. *Let T be a unimodal map with $T^i(0) \neq 0$ for $i \geq 1$ and suppose that $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ is a map which satisfies (2.2) and which generates the kneading sequence $e_1e_2 \ldots$ of T. Suppose that there is an $l \geq 2$ with $Q(i) = 0$ for $i \leq l$ and with $Q(i) \geq 2$ for $i > l$. Then*

(i) *$e_{R_k} = 0$ for $k \geq 2$,*
(ii) *$V_i \cap J = \emptyset$ for $i \geq 2$, where J is any subset of I_1 with $T(J) \subset J$,*
(iii) *$e_1e_2 \ldots$ does not contain three consecutive ones,*
(iv) *if $D_0D_1D_2$ is a path in $(\mathcal{D}, \to)$ with $D_i \subset I_1$ for $0 \leq i \leq 2$, then $D_i = V_1$ for some $i \in \{0, 1, 2\}$.*

Proof. We show (i) by induction. If $2 \leq j \leq l$ then $Q(j) = Q(j-1) = 0$ and hence $r_j = r_{j-1} = 1$ by (2.3). Therefore (2.4) implies $e_{R_j} = e_{R_{j-1}+1} = e'_1 = 0$. Now suppose that $j > l$ and that (i) is shown for all $k \in \{2, \ldots, j-1\}$. By (2.4) and (2.3) we get $e_{R_j} = e_{r_j - 1} = e_{R_{Q(j)}}$. This equals 0, since $Q(j) \in \{2, \ldots, j-1\}$ by (2.2) and the assumptions. Hence (i) is shown.

In order to show (ii) suppose that $V_i \cap J \neq \emptyset$ for some $i \geq 2$. Let k be such that $R_{k-1} < i \leq R_k$. Then $k \geq 2$. Since V_{j+1} is the only successor of V_j for $R_{k-1} < j < R_k$ by (3.3), we get $T^{R_k - i}(V_i) = V_{R_k}$ by Lemma 3. As $T(J) \subset J$, we get $V_{R_k} \cap J \neq \emptyset$. But (3.1) and (i) imply that $V_{R_k} \subset I_0$, which contradicts $J \cap I_0 = \emptyset$, and (ii) is shown.

We show by induction on k that $e_1e_2 \ldots e_{R_k}$ does not contain three consecutive ones for $k \geq 1$. Since $R_1 = 1$ and $R_2 = 2$, this is trivial for $k \leq 2$. If $j \geq 3$ and if it is shown for $k < j$, it also follows for $k = j$ using (2.4) for $i = j - 1$ and observing that $e_{R_{j-1}+2} = e_{R_j+1} = e'_1 = 0$ if $Q(j) = 0$, that $e_{R_{j-1}+3} = e_2 = 0$ if $Q(j) \geq 2$, and that $e_{R_{j-1}} = 0$ by (i). Hence (iii) is shown.

If $D_0D_1D_2$ is a path in $(\mathcal{D}, \to)$ with $D_i \subset I_1$ for $0 \leq i \leq 2$, it is not possible that $D_i = V_{k+i}$ for $0 \leq i \leq 2$ and some $k \geq 1$, because otherwise (3.1) implies $e_k = e_{k+1} = e_{k+2} = 1$, contradicting (iii). The only elements V_k of $\mathcal{D}$ with another successor besides V_{k+1} are those with $k = R_j$ for some j by (3.3). Hence there are an $i \in \{0, 1, 2\}$ and a $j \geq 1$ with $D_i = V_{R_j}$. But then $e_{R_j} = 1$ by (3.1) and (i) implies that $j = 1$. Hence $D_i = V_1$, showing (iv). ∎

4. Construction of the examples. In this section we construct unimodal maps which have the desired properties. To this end we use the results of Section 2 and define maps $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ satisfying (2.5) and (2.6). Fix an integer $u \geq 4$ and a sequence $(v_i)_{i \geq 1}$ of integers such that $2v_i \leq v_{i+1} \leq v_i^2$ for $i \geq 1$. Set

$$w_i = u + \sum_{j=1}^{i-1} (2v_j - 1) + v_i \qquad \text{for } i \geq 1$$

and define

$$\begin{aligned} &Q(j) = 0 && \text{for } 1 \leq j \leq u\,, \\ &Q(w_i - j) = w_i - j - 2 && \text{for } i \geq 1 \text{ and } 0 \leq j < v_i\,, \\ &Q(w_i + j) = w_i - j - 2 && \text{for } i \geq 1 \text{ and } 0 \leq j < v_i\,. \end{aligned} \tag{4.1}$$

It follows easily that Q defined by (4.1) satisfies (2.5). In order to show (2.6) let $k \geq 1$ be such that $Q(k) \geq 1$, which is equivalent to $k > u$. If $Q^2(k) = 0$ we have $Q(Q^2(k)+1) = Q(1) = 0 < Q(k+1)$, since $k > u$. If $Q^2(k) \geq 1$ and hence $Q(k) > u \geq 4$, applying (2.5) twice we get $Q(Q^2(k) + 1) < Q^2(k) < Q(k) - 1$. This implies (2.6), since $Q(k) - 1 \leq Q(k+1)$ for $k \geq 1$ follows easily from (4.1).

Therefore the map Q defined by (4.1) satisfies (2.5) and (2.6). Furthermore, it satisfies the assumptions of Lemma 8. Theorem 1 gives the existence of unimodal maps with kneading sequences generated by Q. We investigate the Markov diagram $(\mathcal{D}, \to)$ of such maps.

We use the description of $(\mathcal{D}, \to)$ given in Section 3 in terms of the sequence $(r_i)_{i \geq 1}$ defined by (2.3). We have $\mathcal{D} = \{V_i : i \geq 1\}$ with the arrows of (3.3). We say that a path $D_0D_1 \ldots D_{n-1}$ of length n is *uniquely extendable* to a path of length $m > n$ if there are $D_i \in \mathcal{D}$ for $n \leq i < m$ such that D_i is the only successor of D_{i-1} in $(\mathcal{D}, \to)$. In this case it follows easily from

Lemma 3 that

$$(4.2)\qquad \bigcap_{i=0}^{n-1} T^{-i}(D_i) = \bigcap_{i=0}^{m-1} T^{-i}(D_i)\,,$$

which we need later. Fix $s \ge 1$ and set

$$l = \sum_{j=w_s-v_s}^{w_s} r_j + \sum_{k=1}^{s-1} r_{w_k-v_k}\,.$$

Let $\mathcal{P}$ be the set of all paths in $(\mathcal{D},\to)$ starting with $V_{R_{w_s-1}+1}$ which have length l. Now we can show

LEMMA 9. *Suppose that $s \ge 2$. Then the paths in $\mathcal{P}$ do not contain V_1 and their number is $(s-1)v_s + 1 + \frac{1}{2}v_s(v_s+1)$. The number of paths in $\mathcal{P}$ ending at V_u is at least v_s. There are at least $\frac{1}{2}v_s(v_s+1)$ paths in $\mathcal{P}$ which are uniquely extendable to a path of length $l + v_{s-1}$.*

Proof. We give the proof by examination of all paths in $(\mathcal{D},\to)$ starting at $V_{R_{w_s-1}+1}$. To this end let $P(j)$ be the path $V_{R_{j-1}+1}V_{R_{j-1}+2}\ldots V_{R_j}$ in $(\mathcal{D},\to)$ for $j \ge 1$. It has length r_j. It follows from (3.3) that the paths in $(\mathcal{D},\to)$ starting at $V_{R_{w_s-1}+1}$ are exactly the paths $P(j_1)P(j_2)P(j_3)\ldots$, where $j_1 = w_s$ and where $j_{i+1} = j_i + 1$ or $j_{i+1} = Q(j_i) + 1$ for $i \ge 1$. This defines a directed graph in which each vertex has exactly two successors; part of it is shown below:

$$\begin{array}{ccccccccccccc}
w_s & \to & w_s+1 & \to & \ldots & \to & w_s+v_s-1 & \to & w_s+v_s & & & & \\
\downarrow & & \downarrow & & & & \downarrow & & & & & & \\
w_s-1 & \to & w_s-2 & \to & \ldots & \to & w_s-v_s & \to & w_{s-1}-v_{s-1} & \to \ldots \to & w_2-v_2 & \to & w_1-v_1 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
w_s & & w_s-1 & & & & w_s-v_s+1 & & w_{s-1}-v_{s-1}+1 & & w_2-v_2+1 & &
\end{array}$$

Examination of all paths in this graph starting at w_s corresponds to examination of all paths in $(\mathcal{D},\to)$ starting at $V_{R_{w_s-1}+1}$.

First we consider the paths in the above figure which end at $w_1 - v_1$. They correspond to the paths

$$P(w_s)P(w_s+1)\ldots P(w_s+i)P(w_s-i-1)\ldots P(w_s-v_s)\ldots P(w_1-v_1)$$

in $(\mathcal{D},\to)$, where $0 \le i < v_s$. They all end at $P(w_1-v_1) = P(u) = V_u$ and have length

$$\sum_{j=w_s}^{w_s+i} r_j + \sum_{j=w_s-v_s}^{w_s-i-1} r_j + \sum_{k=1}^{s-1} r_{w_k-v_k}\,,$$

which equals l for each i by (4.1) and (2.3). Hence we have found v_s paths in $\mathcal{P}$ of length l ending at V_u.

Next consider the path in the above figure which ends at $w_s + v_s$. It corresponds to the path

$$P(w_s)P(w_s+1)\ldots P(w_s+v_s) = V_{R_{w_s-1}+1}V_{R_{w_s-1}+2}\ldots V_{R_{w_s+v_s}}$$

in $(\mathcal{D},\to)$. Its length is

$$\sum_{k=w_s}^{w_s+v_s} r_k = \sum_{k=w_s-v_s+1}^{w_s} r_k + r_{w_s+v_s}$$

and its length without the last part $P(w_s+v_s)$ is $\sum_{k=w_s-v_s+1}^{w_s} r_k$. As $r_{w_s+v_s} = R_{w_s+v_s-2} + 1$ by (4.1) and (2.3) we get

$$\sum_{k=w_s-v_s+1}^{w_s} r_k < l < \sum_{k=w_s}^{w_s+v_s} r_k\,.$$

Therefore the initial segment of length l of this path ends at some element of its last part $P(w_s+v_s)$ and belongs to $\mathcal{P}$.

Next consider the paths in the above figure which end at one of the last $s-2$ elements in the last row. They correspond to the paths

$$P(w_s)\ldots P(w_s+i)P(w_s-i-1)\ldots P(w_s-v_s)\ldots P(w_j-v_j)P(w_j-v_j+1)$$

in $(\mathcal{D},\to)$, where $0 \le i < v_s$ and $2 \le j \le s-1$. Using (4.1) and (2.3) one sees that the length of these paths does not depend on i and equals $l_j + r_{w_j-v_j+1}$, where $l_j = \sum_{k=w_s-v_s}^{w_s} r_k + \sum_{k=j}^{s-1} r_{w_k-v_k}$. The length without the last part $P(w_j-v_j+1)$ is l_j. We have $l_j < l < l_j + r_{w_j-v_j+1}$ since $\sum_{k=1}^{j-1} r_{w_k-v_k} \le R_{w_{j-1}-v_{j-1}}$, since $w_{j-1}-v_{j-1} \le w_j - v_j - 1$, and since $R_{w_j-v_j-1} + 1 = r_{w_j-v_j+1}$ by (4.1) and (2.3). Therefore the initial segment of length l of each of these $(s-2)v_s$ paths ends at some element of its last part $P(w_j-v_j+1)$ and belongs to $\mathcal{P}$.

Finally, consider the paths in the above figure which end at one of the first v_s elements in the last row. They correspond to the paths

$$P(w_s)P(w_s+1)\ldots P(w_s+i)P(w_s-i-1)\ldots P(w_s-j)P(w_s-j+1)$$

in $(\mathcal{D},\to)$, where $0 \le i < j \le v_s$. For fixed i and j, the length of such a path is

$$\sum_{k=w_s}^{w_s+i} r_k + \sum_{k=w_s-j}^{w_s-i-1} r_k + r_{w_s-j+1}\,.$$

Using (4.1) and (2.3) one sees that this equals

$$\sum_{k=w_s-j}^{w_s} r_k + R_{w_s-j-1} + 1 = R_{w_s} + 1$$

and hence depends neither on i nor on j. The length of these paths without the last part $P(w_s-j+1)$ is $\sum_{k=w_s-j}^{w_s} r_k$, which is less than l. Furthermore, (4.1) implies $r_{w_{s-1}} = R_{w_{s-1}-2}+1 \geq w_{s-1}-1 \geq v_{s-1}$, and $l \leq R_{w_s}-r_{w_{s-1}} < R_{w_s}+1-v_{s-1}$ follows. This shows that the initial segment of length l of each of these $\frac{1}{2}v_s(v_s+1)$ paths ends at some element of its last part $P(w_s-j+1)$ and belongs to $\mathcal{P}$. Each of them is uniquely extendable to a path of length $l+v_{s-1}$, since each element of $P(w_s-j+1)$ except the last one has a unique successor in $(\mathcal{D},\to)$.

This finishes the examination of the paths in $(\mathcal{D},\to)$ corresponding to paths in the above figure. All paths in $(\mathcal{D},\to)$ of infinite length starting at $V_{R_{w_s-1}+1}$ are continuations of the examined paths. Hence we have found all paths of $\mathcal{P}$. All assertions of the lemma follow immediately except the first one. But $P(j)$ contains V_1 only if $j=1$. Since $P(1)$ does not occur in the examined paths, the first assertion also follows. ■

Now we can estimate certain elements of $(\mathcal{D},\to)$. The length of an interval J is denoted by $|J|$.

LEMMA 10. *Let T be defined by* (1.4), *suppose that* (1.3) *holds, so that $I := [T^2(0), T(0)]$ is T-invariant, and suppose that the kneading sequence of T is generated by Q defined in* (4.1). *Let A be defined by* (1.5). *Suppose that $d\in(0,1)$ and that m is an atomless d-conformal measure on A. Then* $\inf_{k\geq 1}|V_k|^d/m(V_k)=0$.

Proof. Let F be as in (1.5). Since $|T'|\equiv\beta$ outside F by (1.4), we see from (1.2) that

$$m(T(U)) = \beta^d m(U) \quad \text{if } U\subset Y\setminus F \text{ for some } Y\in\mathcal{Y} \tag{4.3}$$

since $m(T(U)) = m(T(U)\cap A) = m(T(U\cap A))$ by (1.1). For $s\geq 2$ let l and $\mathcal{P}$ be as before Lemma 9. Set $\mathcal{R} = \{\bigcap_{i=0}^{l-1}T^{-i}(D_i) : D_0D_1\dots D_{l-1}\in\mathcal{P}\}$. By Lemma 8 we get $V_i\cap F=\emptyset$ for $i\geq 2$ and hence Lemma 9 implies that the elements of the paths in $\mathcal{P}$ have empty intersection with F. Now Lemma 4 implies that

$$\text{if } W\in\mathcal{R} \text{ and } 0\leq i<l \text{ then } T^i(W)\subset Y\setminus F \text{ for some } Y\in\mathcal{Y}. \tag{4.4}$$

Set $k_s = R_{w_s-1}+1$. Then $m(V_{k_s}) = \sum_{W\in\mathcal{R}} m(W)$ and $|V_{k_s}| = \sum_{W\in\mathcal{R}}|W|$ by Lemma 5, since m has no atoms, and $m(W) = m(T^{l-1}(W))\beta^{-d(l-1)}$ for $W\in\mathcal{R}$ by (4.3) and (4.4). Since v_s different paths in $\mathcal{P}$ end with V_u by Lemma 9 and since elements of $\mathcal{R}$ corresponding to different paths in $\mathcal{P}$ are disjoint by Lemma 5, we get $m(V_{k_s})\geq v_s m(V_u)\beta^{-d(l-1)}$ by Lemma 4. By (4.4) and the mean value theorem we find that $|W| = |T^{l-1}(W)|\beta^{-(l-1)}$ for $W\in\mathcal{R}$ and using (4.2) that $|W| = |T^{l+v_{s-1}-1}(W)|\beta^{-(l+v_{s-1}-1)}$ for those $W\in\mathcal{R}$ which correspond to the $\frac{1}{2}v_s(v_s+1)$ paths in $\mathcal{P}$ which are shown in Lemma 9 to be uniquely extendable to paths of length $l+v_{s-1}$. As $|T^i(W)|\leq|I| =: c$ for $i\geq 0$ and $W\in\mathcal{R}$ and as $v_s\geq 1$, we get $|V_{k_s}|\leq sv_sc\beta^{-(l-1)} + v_s^2c\beta^{-(l+v_{s-1}-1)}$. The estimates of $m(V_{k_s})$ and $|V_{k_s}|$ now imply

$$\frac{|V_{k_s}|^d}{m(V_{k_s})} \leq \frac{c^d}{m(V_u)}(s+v_s\beta^{-v_{s-1}})^d v_s^{-(1-d)}. \tag{4.5}$$

By (4.1) and (2.3) we have $r_i=1$ for $1\leq i\leq u$ and $R_u=u$. Hence (3.3), Lemma 3 and (3.1) imply $T(V_u)\supset V_1$ and $T^2(V_u)=I$. By (1.2) we get $m(V_u)>0$. Since $2v_{s-1}\leq v_s\leq v_{s-1}^2$ for $s\geq 2$, since $\beta>1$, and since $d\in(0,1)$, the right hand side of (4.5) tends to zero for $s\to\infty$. This proves the lemma. ■

THEOREM 2. *For each $d\in(0,1)$ there are $\beta>2$ and $t\in[0,1]$ such that the map T defined by* (1.4) *satisfies $T^2(0)<0<T(0)$ and $T^2(0)\leq T^3(0)$, and such that A defined by* (1.5) *is topologically transitive, has Hausdorff dimension d and satisfies $\nu(A)=0$, where ν denotes the d-dimensional Hausdorff measure.*

Proof. Let Q be defined by (4.1), where $u\geq 4$, and where $(v_i)_{i\geq 1}$ is such that $2v_i\leq v_{i+1}\leq v_i^2$ for $i\geq 1$. Let $(\mathcal{D},\to)$ be the directed graph determined by Q using (2.3) and (3.3). Here it is not yet clear what the vertices V_i in this graph are, but the arrows are fixed. Choose u so large that the spectral radius ϱ of the 0-1-matrix associated with $(\mathcal{D},\to)$ and considered as an l^∞-operator is greater than 2^d. This is possible since $d<1$. Set $\beta=\varrho^{1/d}>2$. By Theorem 1 there is then $t\in[0,1]$ such that T defined by (1.4) satisfies (1.3) and that its kneading sequence is generated by Q.

By the results of Section 3 the Markov diagram $(\mathcal{D},\to)$ of T has spectral radius ϱ. It follows easily from (4.1), (2.3) and (3.3), that $(\mathcal{D},\to)$ is irreducible. Since A is just the set I without the interiors of the maximal intervals on which T^k is monotone for each $k\geq 1$, it follows from the results of [4] that A is topologically transitive and that $h_{\text{top}}(T|A) = h_{\text{top}}(T|I) = \log\varrho$. Let $\pi(s)$ denote the topological pressure of the function $x\mapsto -s\log|T'(x)|$ on $(A,T|A)$. It is shown in [7] that the Hausdorff dimension $\mathrm{HD}(A)$ of A is the unique zero of $\pi(s)$ for $s\geq 0$. As $|T'|\equiv\beta$ on A, we get $\pi(s) = h_{\text{top}}(T|A)-s\log\beta$ by Theorem 9.7 of [8]. Hence $\mathrm{HD}(A) = h_{\text{top}}(T|A)/\log\beta$. Since $h_{\text{top}}(T|A)=\log\varrho$ the choice of β implies $\mathrm{HD}(A)=d$.

It remains to show that $\nu(A)=0$. To this end set $A_G = A\setminus\bigcup_{i=0}^\infty T^{-i}(G)$, where G is an open interval in I. Then A_G is a closed T-invariant subset of A. We show

$$A_G\neq A \;\Rightarrow\; \mathrm{HD}(A_G)<\mathrm{HD}(A). \tag{4.6}$$

With the same proof as for A we get $\mathrm{HD}(A_G) = h_{\text{top}}(T|A_G)/\log\beta$. Hence (4.6) follows if we show $h_{\text{top}}(T|A_G)<h_{\text{top}}(T|A)$. Since $h_{\text{top}}(T|A)=\log\varrho>0$, we can assume that $h_{\text{top}}(T|A_G)>0$. In [2] and [3] an isomorphism of $T|I$ to a shift space with finite alphabet is constructed which preserves invariant measures without atoms and, in particular, ergodic measures with nonzero entropy. Since shift spaces are expansive, there exist ergodic measures of maximal entropy on closed invariant subsets by Theorems 8.2 and 8.7 of [8]. This property then also holds for closed T-invariant subsets of I with positive entropy. Hence there are ergodic measures λ and μ of maximal entropy for $(A, T|A)$ and $(A_G, T|A_G)$ respectively. Furthermore, it follows from the results of [2] and [3] that λ is unique and has support A, as A is topologically transitive. If $A_G \neq A$ this implies that $\mu \neq \lambda$ and hence μ is not a measure of maximal entropy for $(A, T|A)$. Now Theorem 8.6 of [8] implies $h_{\text{top}}(T|A) > h_\mu = h_{\text{top}}(T|A_G)$, proving (4.6).

By Theorem 1 of [5] applied to $A \setminus \bigcup_{i=0}^\infty T^{-i}(K)$, where $K = \{0, T(0), T^2(0)\}$, there is an atomless d-conformal measure m on A. Suppose that $\nu(A) > 0$. By Lemma 10 there is $k \geq 2$ with $\alpha := |V_k|^d/m(V_k) < \frac{1}{2}\nu(A)$. For this k let U be an open interval which contains 0 as in Lemma 7. By Lemma 8 and (2.1) we have $T^{R_k}(0) < 0$ for all $k \geq 2$. Therefore $T^i(0) \notin F$ for $i \geq 0$, which gives $0 \in A$ and $A_U \neq A$. Let H be an open interval containing the closure of F such that $T^2(H) \subset I_1 = (0, T(0))$. Since the boundary of F is an orbit of period two which belongs to A and is contained in H, we get $A_H \neq A$. By (4.6) we have $\mathrm{HD}(A_G) < \mathrm{HD}(A)$ and hence $\nu(A_G) = 0$ for $G = U$ and $G = H$. Since $|T'| \equiv \beta$ on A, one easily shows that $\nu(A \cap T^{-1}(C)) = 0$ if $C \subset A$ and $\nu(C) = 0$ (cf. the proof of Theorem 6 in [5]). Therefore $\nu(A \cap \bigcup_{i=0}^\infty T^{-i}(A_G)) = 0$ for $G = U$ and $G = H$.

Set $B = A \setminus \bigcup_{i=0}^\infty T^{-i}(A_U \cup A_H \cup K)$. Since also countable sets have ν-measure zero, we get $\nu(A) = \nu(B)$. If $x \in B$ then $x \notin \bigcup_{i=0}^\infty T^{-i}(K)$ and hence $D_i(x)$ is defined for $i \geq 0$. If $x \in B$ then $T^j(x) \in U$ for infinitely many j. For these j we get $T^j(x) \in U \cap \bigcap_{i=0}^k T^{-i}(D_{i+j}(x))$ and hence $D_{i+j}(x) \in \{V_l : l \geq k\}$ for some $i \in \{0, 1, \ldots, k\}$ by Lemma 7 and the choice of U. Hence $D_j(x) \in \{V_l : l \geq k\}$ for infinitely many j. Furthermore, if $x \in B$ then $T^j(x) \in H$ for infinitely many j. By the choice of H this implies that $T^j(x)$, $T^{j+1}(x)$ and $T^{j+2}(x)$ are in I_1 and hence $D_j(x)$, $D_{j+1}(x)$ and $D_{j+2}(x)$ are subsets of I_1 by (3.1). One of them is then V_1 by Lemma 8(iv). Hence $D_j(x) = V_1$ for infinitely many j. Since each path in $(\mathcal{D}, \to)$ from V_1 to $\{V_l : l \geq k\}$ has to contain V_k, we conclude that $D_j(x) = V_k$ for infinitely many j if $x \in B$.

Fix $\delta > 0$. For each $x \in B$ there is an $n(x) > (\log|V_k| - \log\delta)/\log\beta$ with $D_{n(x)}(x) = V_k$. Set $W_x = Y_{n(x)+1}(x)$. Then $T^i(W_x)$ is contained in some element of $\mathcal{Y}$ for $0 \leq i \leq n(x)$. By Lemma 6 we have $T^{n(x)}(W_x) = V_k$. Since $V_k \cap F = \emptyset$ by Lemma 8(ii) and since $T(F) \subset F$, we have $T^i(W_x) \cap F = \emptyset$ for $0 \leq i < n(x)$. This implies $m(W_x) = \beta^{-dn(x)} m(V_k)$ and $|W_x| = \beta^{-n(x)}|V_k|$ using (4.3) and the mean value theorem. Therefore $|W_x|^d = \alpha m(W_x)$. Let $\mathcal{U}$ be a subset of $\{W_x : x \in B\}$ which still covers B and which has the property that no three different intervals in it have a common point. Then

$\sum_{W\in\mathcal{U}} |W|^d = \alpha \sum_{W\in\mathcal{U}} m(W) \leq 2\alpha$. Since $|W| < \delta$ for $W \in \mathcal{U}$ by the choice of $n(x)$ and since $\delta > 0$ was arbitrary, the definition of the Hausdorff measure implies that $\nu(B) \leq 2\alpha$. This contradicts the choice of k, since $\nu(B) = \nu(A)$. Hence $\nu(A) = 0$ and the theorem is proved. ∎

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