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Received October 29, 1992 (3017)

STUDIA MATHEMATICA 106 (2) (1993)

Properly semi-L-embedded complex spaces

by

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Abstract. We prove the existence of complex Banach spaces X such that every element F in the bidual X^{**} of X has a unique best approximation $\pi(F)$ in X, the equality $\|F\| = \|\pi(F)\| + \|F - \pi(F)\|$ holds for all F in X^{**}, but the mapping π is not linear.

1. Introduction. Semi-L-summands were introduced by Å. Lima [8] in connection with his study of subspaces of Banach spaces having the so-called "2-ball property". A *semi-L-summand* of a Banach space X is a subspace M of X such that every element x in X has a unique best approximation $\pi(x)$ in M and the equality $\|x\| = \|\pi(x)\| + \|x - \pi(x)\|$ holds for all x in X. If in addition the mapping π is linear, then M is said to be an *L-summand* of X, while otherwise M is called a *proper semi-L-summand* of X. A *semi-L-embedded* (respectively: *L-embedded*, *properly semi-L-embedded*) space is a Banach space X which is a semi-L-summand (respectively: L-summand, proper semi-L-summand) of the bidual X^{**} of X.

Real Banach spaces containing proper semi-L-summands are exhibited in the quoted paper [8]. The easiest example is the space of all real-valued affine functions on the triangle, the set of constant functions being then a proper semi-L-summand. Nonreflexive real or complex L-embedded spaces are also well known: l_1, the preduals of infinite-dimensional von Neumann algebras, and, more generally, the preduals of nonreflexive JBW*-triples [2] are examples of such spaces, and a complete information about them is to be found in [7]. Recently R. Payá and A. Rodríguez [11] have proved the existence of properly semi-L-embedded real spaces, the easiest example being the space of all real-valued continuous affine functions on a countable infinite product of copies of the triangle. More recently E. Behrends [3] has shown that a compact convex subset K of $\mathbb{C}^2$ with the property that $f(K)$ is a disk for every linear mapping $f : \mathbb{C}^2 \to \mathbb{C}$ need not have a center of symmetry, a fact which is equivalent to the existence of complex Banach spaces containing proper semi-L-summands [12].

1991 *Mathematics Subject Classification*: Primary 46B20.

Using Behrends' result and by adapting the techniques in [11] to the complex case, we shall prove in this note that also properly semi-L-embedded complex spaces do exist. This contains simultaneously the existence of properly semi-L-embedded real spaces and complex Banach spaces containing proper semi-L-summands.

All Banach spaces considered in this note will be assumed to be complex.

2. Some results involving numerical ranges. Let Y be a Banach space in which a fixed norm-one element u (the "distinguished element" of Y) has been selected, and let us denote by $D(Y,u)$ the set of all elements y^* in the dual space of Y satisfying $\|y^*\| = y^*(u) = 1$. The *numerical range* $V(y)$ and the *numerical radius* $v(y)$ of an element y in Y are defined by

$$V(y) := \{y^*(y) : y^* \in D(Y,u)\} \quad \text{and} \quad v(y) := \max\{|\lambda| : \lambda \in V(y)\},$$

respectively. The largest nonnegative number L satisfying $L\|y\| \le v(y)$ for all y in Y is called the *numerical index* of Y and is denoted by $n(Y)$. For elements ϕ in the bidual of Y we define the "small" numerical range $V_s(\phi)$ and the "small" numerical radius $v_s(\phi)$ by

$$V_s(\phi) := \{\phi(y^*) : y^* \in D(Y,u)\} \quad \text{and} \quad v_s(\phi) := \sup\{|\lambda| : \lambda \in V_s(\phi)\},$$

respectively. The largest nonnegative real number S satisfying $S\|\phi\| \le v_s(\phi)$ for all ϕ in Y^{**} is called the "small" numerical index of Y^{**} and is denoted by $n_s(Y^{**})$. It is proved in [9; Theorem 2.3] that, if K is a closed convex subset of $\mathbb{C}$ with nonempty interior and ϕ is in Y^{**} with $V_s(\phi) \subset K$, then there exists a net $\{y_\alpha\}$ in Y converging to ϕ in the w^*-topology and satisfying $V(y_\alpha) \subset K$ for all α. From this result we easily obtain the following proposition.

PROPOSITION 1. *$n_s(Y^{**}) = n(Y)$.*

Proof. Let ϕ be in Y^{**} and let ε be an arbitrary positive number. Then there exists a net $\{y_\alpha\}$ in Y converging to ϕ in the w^*-topology and satisfying $v(y_\alpha) \le v_s(\phi) + \varepsilon$ for all α. Since $n(Y)\|y_\alpha\| \le v(y_\alpha)$ for all α and the norm on Y^{**} is w^*-lower-semicontinuous, we have $n(Y)\|\phi\| \le v_s(\phi) + \varepsilon$. Now $n(Y) \le n_s(Y^{**})$, and the converse inequality is clear. ■

The following corollary is immediate.

COROLLARY 1. *$n(Y) = 1$ (if and) only if $v_s(\phi) = \|\phi\|$ for all ϕ in Y^{**}.*

If K is a bounded subset of a Banach space X and x^* is a norm-one element in X^*, then the diameter of $\mathrm{Re}\, x^*(K)$ measures the "width" of K in the direction determined by x^*. By a *set of constant width* in X we mean a bounded closed convex subset K of X such that $\mathrm{diam}(\mathrm{Re}\, x^*(K))$ remains constant when x^* runs over the unit sphere of X^*. This condition is equivalent to the fact that $K-K$ contains the open unit ball in X with center zero and radius the diameter of K [11; Proposition 1.1]. As a consequence, if X is a Banach space and K is a set of constant width in X^* not reduced to a point, then $|\!|\cdot|\!|$ defined on $\mathbb{C} \oplus X$ by

$$|\!|\lambda \oplus x|\!| := \sup\{|\lambda + t(x)| : t \in K\}$$

is a norm generating the usual product topology on $\mathbb{C} \oplus X$. Note also that, as a consequence of [10; Corollary 3.9], sets of constant width in dual Banach spaces are w^*-closed.

LEMMA 1. *Let X be a Banach space, K a set of constant width in X^* not reduced to a point, and consider the Banach space $Y := \mathbb{C} \oplus X$ under the norm*

$$|\!|\lambda \oplus x|\!| := \max\{|\lambda + t(x)| : t \in K\}.$$

Then the bidual norm of $|\!|\cdot|\!|$ on $Y^{**} = \mathbb{C} \oplus X^{**}$ (also denoted by $|\!|\cdot|\!|$) is given by

$$|\!|\lambda \oplus F|\!| = \sup\{|\lambda + F(t)| : t \in K\}.$$

Proof. Each $y = \lambda \oplus x$ in Y defines a continuous affine function $\widehat{y}$ from the (w^*-) compact convex set K into $\mathbb{C}$ by means of the formula $\widehat{y}(t) = \lambda + t(x)$ for all t in K. Moreover, by the definition of the norm $|\!|\cdot|\!|$ on Y, the mapping $y \to \widehat{y}$ is a linear isometry from Y into the Banach space $A(K,\mathbb{C})$ of all complex-valued continuous affine functions on K endowed with the supremum norm, which preserves the natural distinguished elements, namely $u := 1 \oplus 0$ for Y and the function of constant value one for $A(K,\mathbb{C})$. Since by the Hahn–Banach theorem such an isometry also preserves numerical ranges, we may apply [1; Corollary 2.11] and use the natural identification $Y^* = \mathbb{C} \oplus X^*$ to obtain for all $y = \lambda \oplus x$ in Y,

$$V(y) = V(\widehat{y}) = \widehat{y}(K) = \lambda + K(x) = (1 \oplus K)(y).$$

It follows from the Hahn–Banach separation theorem that $1 \oplus K = D(Y,u)$. Since clearly $n(Y) = 1$, the proof is concluded by applying Corollary 1. ■

3. The main result. Now that Lemma 1 has been proved, the remaining part of the argument to show the existence of properly semi-L-embedded (complex) Banach spaces will be very similar to the one in [11] for the real case. We only need to consider a particular type of sets of constant width, which will play in the complex case the analogous role to that of general sets of constant width in the real case, and apply the fact that, according to Behrends' result [3], such "proper" particular sets do exist in $\mathbb{C}^2$ suitably normed. By a *round set* in a (complex) Banach space X we mean a bounded closed convex subset K of X such that, for every x^* in X^*, the closure $x^*(K)^-$ of $x^*(K)$ is a disk in $\mathbb{C}$ with radius $\frac{1}{2}\|x^*\|\,\mathrm{diam}(K)$. Examples of round sets are the so-called pseudoballs [4], namely closed convex subsets of X whose w^*-closure in X^{**} is a ball; before the above cited result of E. Behrends, no other examples were known.

LEMMA 2. *Let X be an L-embedded space, K a round set in X^* not reduced to a point, and consider the Banach space $Y := \mathbb{C} \oplus X$ under the norm*

$$\|\lambda \oplus x\| := \max\{|\lambda + t(x)| : t \in K\}.$$

Then Y is a semi-L-embedded space. Moreover, if for F in X^{**}, $\pi(F)$ denotes the best approximation of F in X and $\mu_K(F)$ denotes the center of the disk $F(K)^-$, then for $\lambda \oplus F$ in $Y^{**} = \mathbb{C} \oplus X^{**}$, $(\lambda + \mu_K(F - \pi(F))) \oplus \pi(F)$ is the best approximation of $\lambda \oplus F$ in Y.

Proof. Without loss of generality we may assume $\operatorname{diam}(K) = 2$. Then, since round sets are sets of constant width, we may apply Lemma 1 to obtain for every $\phi = \lambda \oplus F$ in Y^{**},

$$\|\phi\| = |\lambda + \mu(F)| + \|F\|.$$

Now the fact that Y is a semi-L-embedded space, as well as the information about the metric projection $Y^{**} \to Y$ claimed in the statement, follows as in the proof of the implication (iii)⇒(i) in [11; Theorem 1.4]. ■

In our complex context, the argument in [11; Lemma 1.6] gives that pseudoballs in a Banach space Z are those round sets K in Z for which the mapping $z^* \to \mu_K(z^*)$ (= the center of the disk $z^*(K)^-$) from Z^* into $\mathbb{C}$ is linear. Moreover, as in the proof of [11; Proposition 1.7], if M is a closed subspace of Z, if $q : Z \to Z/M$ denotes the quotient mapping, and if K is a round set in Z, then $q(K)^-$ is a round set in Z/M, and it is a pseudoball if and only if μ_K is linear on the polar M^0 of M in Z^*.

Now recall that an *M-ideal* of a Banach space is a closed subspace whose polar is an L-summand in the dual Banach space, and that a Banach space is called *M-embedded* if it is an M-ideal in its bidual. The dual E^* of an M-embedded space E is an L-embedded space and the metric projection $\pi : E^{***} \to E^*$ is nothing but the Dixmier duality projection on E^{***} corresponding to the decomposition $E^{***} = E^* \oplus E^0$ [5]. Reading Lemma 2 in the particular case of X being the dual of an M-embedded space E, we see that Y is L-embedded if and only if μ_K $(: E^{***} \to \mathbb{C})$ is linear on the polar E^0 of E in E^{***}. Therefore we have:

LEMMA 3. *Let E be an M-embedded space and K a round set in E^{**}, and consider the Banach space $Y := \mathbb{C} \oplus E^*$ under the norm*

$$\|\lambda \oplus x\| := \max\{|\lambda + t(x)| : t \in K\}.$$

Then Y is a semi-L-embedded space, and it is an L-embedded space if and only if $q(K)^-$ is a pseudoball in E^{**}/E, where $q : E^{**} \to E^{**}/E$ denotes the quotient mapping.

Our last lemma is

LEMMA 4. *Let K be a set of constant width in the dual of a Banach space Z, and assume that $K(z)$ is a disk in $\mathbb{C}$ for every z in Z. Then K is a round set in Z^*.*

Proof. We may assume that $\operatorname{diam}(K) = 2$, and it is enough to prove that $F(K)^-$ is a disk in $\mathbb{C}$ for every norm-one element F in Z^{**}. For such an F let $\{z_\alpha\}$ be a net in the closed unit ball of Z converging to F in the w^*-topology of Z^{**}. By assumption, for each α, $K(z_\alpha)$ is a disk in $\mathbb{C}$ (say, with center λ_α) with radius ≤ 1. If λ is any limit point of the (bounded) net $\{\lambda_\alpha\}$ and t is in K, then from the inequality $|t(z_\alpha) - \lambda_\alpha| \leq 1$ we obtain $|F(t) - \lambda| \leq 1$. Now $F(K)^-$ is a set of constant width in $\mathbb{C}$ (hence diametrically complete [10; Corollary 3.9]) with diameter two and is contained in a disk of radius one. It follows that $F(K)^-$ is a disk, as required. ■

Now we conclude the proof of the main result.

THEOREM 1. *There exists a properly semi-L-embedded (complex) space.*

Proof. By [3], there exists a compact convex subset J of $\mathbb{C}^2$ without a center of symmetry and with the property that $f(J)$ is a disk for every linear mapping $f : \mathbb{C}^2 \to \mathbb{C}$. As is well known and easy to see, such a set J has nonempty interior in $\mathbb{C}^2$ and $J - J$ is absolutely convex. It follows that there exists a (unique) norm $\|\cdot\|$ on $\mathbb{C}^2$ such that $J - J = \{w \in \mathbb{C}^2 : \|w\| \leq 1\}$. Moreover, denoting by T the Banach space $(\mathbb{C}^2, \|\cdot\|)$, it is clear that J is a round set in T which is not a ball. Let $E = c_0(T)$ be the Banach space of all null sequences in T so that E^{**} equals $l_\infty(T)$ (the space of all bounded sequences in T), and write $K := \{\{w_n\} \in l_\infty(T) : w_n \in J \text{ for all } n \text{ in } \mathbb{N}\}$. Then E is an M-embedded space [5; Theorem 3.4] and it is easy to see that K is a set of constant width in E^{**}. Moreover, for f in $E^* = l_1(T^*)$ (the space of absolutely convergent series in the dual T^* of T), $K(f)$ is a ball in $\mathbb{C}$, hence by Lemma 4, K is actually a round set in E^{**}. Now Lemma 3 gives us that $Y := \mathbb{C} \oplus E^*$ with the norm

$$\|\lambda \oplus x\| := \max\{|\lambda + t(x)| : t \in K\}$$

is a semi-L-embedded space. To prove that Y is a properly semi-L-embedded space, again by Lemma 3 it is enough to show that $q(K)^-$ is not a pseudoball in E^{**}/E, where $q : E^{**} \to E^{**}/E$ denotes the quotient mapping. Recall that every finite-dimensional Banach space has the so-called "intersection property" [6], and that this property passes from a given Banach space to any l_∞-sum of copies of that space and also to any quotient by an M-ideal [6; Proposition 1.1]. Now assume that $q(K)^-$ is a pseudoball in E^{**}/E. Since E^{**}/E has the intersection property, $q(K)^-$ is a ball in

E^{**}/E [4; Theorems 4.3 and 3.4], hence J is a ball in T [11; Lemma 1.9], a contradiction. ∎

Acknowledgements. The author wishes to express his gratitude to D. Yost for helpful discussions about the topics developed in this note.

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Received November 17, 1992
Revised version February 21, 1993 (3022)

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