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Topological tensor products of a Fréchet–Schwartz space and a Banach space

by

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Abstract. We exhibit examples of countable injective inductive limits E of Banach spaces with compact linking maps (i.e. (DFS)-spaces) such that $E \otimes_\varepsilon X$ is not an inductive limit of normed spaces for some Banach space X. This solves in the negative open questions of Bierstedt, Meise and Hollstein. As a consequence we obtain Fréchet–Schwartz spaces F and Banach spaces X such that the problem of topologies of Grothendieck has a negative answer for $F \widehat{\otimes}_\pi X$. This solves in the negative a question of Taskinen. We also give examples of Fréchet–Schwartz spaces and (DFS)-spaces without the compact approximation property and with the compact approximation property but without the approximation property.

In this article we present a negative solution to the following problem stated in 1976 by Bierstedt and Meise [3]: Let $E = \mathrm{ind}_n E_n$ be an injective inductive limit of a sequence of Banach spaces with compact linking maps; i.e., a (DFS)-space. Is it true that $E \otimes_\varepsilon X = \mathrm{ind}_n (E_n \otimes_\varepsilon X)$ holds topologically for every Banach space X? This question and several partial positive answers of it are of relevance in connection with weighted inductive limits of spaces of holomorphic functions with values in a Banach space (see [1]) and also with vector-valued germs of holomorphic functions defined on compact subsets of Fréchet–Schwartz spaces (see [2]). The question was formulated again by Hollstein [6].

If E is a (DFS)-space, then its strong dual E'_b is a Fréchet–Schwartz space such that $(E'_b)'_b = E$. It turns out that the problem mentioned above is equivalent to a "dual" problem on the projective tensor product of Fréchet spaces, explicitly mentioned by Taskinen [13]. According to Taskinen [12], a pair (F, G) of Fréchet spaces is said to have the property (BB) if every bounded subset B of $F \widehat{\otimes}_\pi G$ is contained in the closure of the absolutely convex hull of the tensor product $C \otimes D$ of a bounded subset C of F and

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D of G. With this notation, Grothendieck's problem of topologies [5] asked if every pair of Fréchet spaces satisfies (BB) and was answered negatively by Taskinen [12, 14]. Taskinen asked in [13] if, for every Fréchet–Schwartz space F and for every Banach space X, the pair (F, X) has property (BB). In [1, 2.1] it is proved that if $E = \mathrm{ind}_n E_n$ is a (DFS)-space, then (E'_b, X) has (BB) for every Banach space X if and only if $E \otimes_\varepsilon X = \mathrm{ind}_n(E_n \otimes_\varepsilon X)$ for every Banach space X. Accordingly, in this paper we also exhibit examples of Fréchet–Schwartz spaces that give a negative answer to the problem of Taskinen [13].

In [2] we proved that *if $E = \mathrm{ind}_n E_n$ is a (DFS) such that the linking maps are approximable by finite rank operators, then $E \otimes_\varepsilon X = \mathrm{ind}_n(E_n \otimes_\varepsilon X)$ holds topologically for every Banach space X*. This result covers all the positive answers known before [13], [4].

Our counterexample is constructed as follows. First we show (in Proposition 1) that a (DFS)-space $E = \mathrm{ind}_n E_n$ such that $E \otimes_\varepsilon X = \mathrm{ind}_n(E_n \otimes_\varepsilon X)$ holds topologically for every Banach space X must satisfy the *compact approximation property*; i.e., for every compact subset K of E and every neighbourhood V of E there is a compact operator T on E such that $Tx - x \in V$ for every $x \in K$. In the main part of this paper we construct, using an example of Szankowski (cf. [9, pp. 107–110]), a (DFS)-space which does not satisfy the compact approximation property. In Proposition 4 we use an example of a Banach space with the compact approximation property but not the approximation property, due to Willis [15], to construct a (DFS)-space $E = \mathrm{ind}_n E_n$ without the approximation property but such that $E \otimes_\varepsilon X = \mathrm{ind}_n(E_n \otimes_\varepsilon X)$ holds topologically for every Banach space X.

Our notation for locally convex (l.c.) spaces is standard; see, e.g., Jarchow [7], Pérez Carreras and Bonet [10]. If X is a Banach space, B_X denotes its closed unit ball. For a l.c. space E, $\mathit{FIN}(E)$ denotes the set of all finite-dimensional subspaces of E. If E is a l.c. space, $\mathcal{U}_0(E)$ and $\mathcal{B}(E)$ stand for the families of all absolutely convex closed 0-neighbourhoods and bounded sets in E respectively. If $V \in \mathcal{U}_0(E)$ we denote by p_V the Minkowski functional of V. If E and X are l.c. spaces, and $A \subset X'$ as well as $C \subset E$ are absolutely convex subsets, we denote by $W(A, C)$ the following subset of $E \otimes_\varepsilon X$ (considered canonically as a subspace of $L(X', E)$):

$$W(A, C) := \{z \in E \otimes_\varepsilon X : z(A) \subset C\}.$$

In what follows we will use the spaces C_p $(1 < p < \infty)$ of Johnson as defined, e.g., in [7] except that we assume $C'_p = C_q$ $(1/p + 1/q = 1)$. This amounts to choosing a sequence $(F_k)_{k\in\mathbb{N}}$ of finite-dimensional Banach spaces which is dense in the set of all finite-dimensional Banach spaces endowed with the Banach–Mazur distance and letting C_p be the l_p-direct sum of $\bigoplus_k F_k \times \bigoplus_k F'_k$. The space C_p $(1 < p < \infty)$ is reflexive and has a Schauder basis (cf. [8]). We recall that a l.c. space E is called *quasinormable* if

$$\forall U \in \mathcal{U}_0(E)\ \exists V \in \mathcal{U}_0(E)\ \forall \varepsilon > 0\ \exists B \in \mathcal{B}(E): \quad V \subset \varepsilon U + B.$$

Every (DF)-space is quasinormable (cf. [7]). A l.c. space is called *quasinormable by operators* (cf. [11]) if

$$\forall U \in \mathcal{U}_0(E)\ \exists V \in \mathcal{U}_0(E)\ \forall \varepsilon > 0\ \exists P \in L(E, E): \quad P(V) \in \mathcal{B}(E) \text{ and } (I - P)(V) \subset \varepsilon U.$$

These spaces are thoroughly studied in [11].

PROPOSITION 1. *Let $E = \mathrm{ind}_n E_n$ be a (DFS)-space such that $E \otimes_\varepsilon X = \mathrm{ind}_n(E_n \otimes_\varepsilon X)$ holds topologically for every Banach space X. Then E satisfies the compact approximation property and it is quasinormable by operators.*

Proof. We divide the proof in two steps.

Step 1. We show that our assumptions on E imply

$$\forall U \in \mathcal{U}_0(E)\ \exists V \in \mathcal{U}_0(E)\ \forall \varepsilon > 0\ \exists B \in \mathcal{B}(E)\ \forall M \in \mathit{FIN}(E)\ \exists P_M \in L(E, E):$$

$$\text{(i)}\quad P_M(M \cap V) \subset B,$$
$$\text{(ii)}\quad (I - P_M)(M \cap V) \subset \varepsilon U.$$

Indeed, we take $X := C_2$. By hypothesis $E \otimes_\varepsilon X$ is a (DF)-space, hence quasinormable. Accordingly we have

(1) $$\forall U \in \mathcal{U}_0(E)\ \exists V \in \mathcal{U}_0(E)\ (V \subset U)\ \forall \varepsilon > 0\ \exists B \in \mathcal{B}(E): \quad W(B_{X'}, V) \subset W(B_{X'}, B) + \varepsilon W(B_{X'}, U).$$

Given $M \in \mathit{FIN}(E)$ we write $M = M' \oplus N$ with $N \subset \ker p_V$ and $M' \cap \ker p_V = \{0\}$. We select $k \in \mathbb{N}$ such that for the kth coordinate F_k of $X' = C_2$ we can find an isomorphism $T : F_k \to (M', p_V)$ satisfying $\|T\| \le 1$ and $\|T^{-1}\| \le 2$. We denote by $i_{M'} : (M', p_V) \to E$ the canonical inclusion (which is continuous since p_V is a norm on M') and we define $R : X' \to E$ by $R((x_n)_{n\in\mathbb{N}}) := i_{M'}(Tx_k)$. Clearly $R \in E \otimes X$ and $R \in W(B_{X'}, V)$ since $\|T\| \le 1$. By (1) we can find $S : X' \to E$ with finite rank such that $S \in W(B_{X'}, B)$ and $R - S \in \varepsilon W(B_{X'}, U)$.

Define $Q : M \to E$ by $Q(x + y) := S(j_k(T^{-1}(x)))$ for $x \in M'$, $y \in N$, where $j_k : F_k \to X'$ is the canonical inclusion. If $a \in M \cap V$ and $a = b + c$, $b \in M'$, $c \in N \subset \ker p_V$, then $p_V(b) \le 1$; hence $\|T^{-1}(b)\|_{F_k} \le 2$ and $j_k(T^{-1}(b)) \in 2B_{X'}$. Accordingly $Q(M \cap V) \subset 2B$, since $S \in W(B_{X'}, B)$. On the other hand, if $x = x_1 + x_2 \in M \cap V$, $x_1 \in M'$, $x_2 \in N$, we get

$$x - Qx = x - S(j_k(T^{-1}(x_1))) = (R - S)(j_k(T^{-1}(x_1))) + x_2 \in 2\varepsilon U.$$

To conclude, if P_M is any continuous extension of Q, we have obtained (i) and (ii).

Step 2. First we show that E is quasinormable by operators, i.e.,

(2) $\forall U \in \mathcal{U}_0(E)\ \exists V \in \mathcal{U}_0(E)\ \forall \varepsilon > 0\ \exists P \in L(E,E)$:

(a) $P(V) \in \mathcal{B}(E)$,

(b) $(I-P)(V) \subset \varepsilon U$.

To see this define $J := FIN(E)$ and consider any ultrafilter $\mathcal{D}$ on J containing the filter generated by the natural order of J. We apply our first step and define $P : E \to E$ by setting $P(x) := \lim_{\mathcal{D}} P_i(x)$, the limit taken for those $i \in J$ such that $x \in i$. Since B is compact in E, $P(x)$ is a well-defined element in E and (i) and (ii) of Step 1 now imply conditions (a) and (b). In particular, it follows from (a) that $P \in L(E,E)$.

We finally check the compact approximation property. Given $U \in \mathcal{U}_0(E)$ and $K \in \mathcal{B}(E)$ we apply (2) to find $V \in \mathcal{U}_0(E)$ and we select $\varepsilon > 0$ with $\varepsilon K \subset V$. Again we apply (2) for this $\varepsilon > 0$ to find $P \in L(E,E)$ with $P(V) \in \mathcal{B}(E)$, which implies that P is a compact operator and if $x \in K$, $\varepsilon x \in V$, hence $(I-P)(\varepsilon x) \in \varepsilon U$; i.e., $x - Px \in U$. This completes the proof. ■

We construct now a (DFS)-space without the compact approximation property. To this end, we recall the necessary notations and facts from [9, Section 1.g].

Let $1 \le p < 2$ and let X be the space of all sequences $x = (x_4, x_5, x_6, \ldots)$ so that

$$\|x\| := \Big(\sum_{n=2}^{\infty} \sum_{B \in \Delta_n} \Big(\sum_{j \in B} |x_j|^2\Big)^{p/2}\Big)^{1/p} < \infty,$$

where Δ_n is a suitable partition of $\sigma_n := \{2^n, 2^n+1, \ldots, 2^{n+1}-1\}$ (see [9, 1.g.5]). It is easy to see that a fundamental system of absolutely convex compact subsets in X is given by $\{K(\alpha) : \alpha = (\alpha_n)_{n=4}^{\infty}$ converges to 0, $\alpha_n > 0$, $n = 4, 5, \ldots\}$, where

$$K(\alpha) := \{x \in X : \|r_n(x)\| \le \alpha_n,\ n = 4, 5, \ldots\}$$

and $r_n(x) := (0, \ldots, 0, x_n, x_{n+1}, \ldots)$, $n = 4, 5, \ldots$ We denote by $\{e_j\}_{j=4}^{\infty}$ the unit vector basis of X and by $\{e_j^*\}_{j=4}^{\infty}$ the corresponding biorthogonal functionals in X'. We let Z be the closed subspace of X spanned by the sequence

$$z_i := e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}, \qquad i = 2, 3, \ldots$$

Now it is possible (see [9, 1.g.4]) to find a sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets of Z such that, defining $z_i^* := \frac{1}{2}(e_{2i}^* - e_{2i+1}^*) \in Z'$, the following holds:

(i) $z_j^*(z_j) = 1$, $j = 2, 3, \ldots$,

(ii) $|\beta_n(T) - \beta_{n-1}(T)| \le \sup\{\|Tx\| : x \in F_n\}$, for $n = 1, 2, \ldots$, and for every $T : Z \to Z$ linear, where $\beta_0(T) := 0$ and

$$\beta_n(T) := 2^{-n} \sum_{i \in \sigma_n} z_i^*(Tz_i), \qquad n = 1, 2, \ldots,$$

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, where $\gamma_n := \sup\{\|x\| : x \in F_n\}$.

THEOREM 2. *There are (DFS)-spaces and Fréchet–Schwartz spaces without the compact approximation property.*

Proof. Let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers tending to ∞ so that $C := \sum_{n=1}^{\infty} \eta_n \gamma_n < \infty$, and put $K := \{0\} \cup \bigcup_{n=1}^{\infty} (\eta_n\gamma_n)^{-1} F_n$. Clearly, K is a compact subset of Z. Take a decreasing sequence $\alpha^1 = (\alpha_n^1)_{n=4}^{\infty}$ of positive numbers which converges to zero such that $K \subset K_1 := K(\alpha^1) \cap Z$. Find another decreasing sequence $\alpha^2 = (\alpha_n^2)_{n=4}^{\infty}$ convergent to zero with $\lim_k \alpha_k^1/\alpha_{2k+3}^2 = 0$. Defining $K_2 := K(\alpha^2) \cap Z$ it follows easily that K_1 is a compact subset of Z_{K_2} (the linear span of K_2 endowed with the norm associated with the Minkowski functional of K_2). By induction we find a sequence $\{\alpha^n\}_{n\in\mathbb{N}}$ of decreasing sequences convergent to zero such that

(1) $$\lim_k \frac{\alpha_k^n}{\alpha_{2k+3}^{n+1}} = 0.$$

We put $K_n := K(\alpha^n) \cap Z$ and $E := \operatorname{ind}_n Z_{K_n}$, which is a (DFS)-space. Clearly the natural inclusion $E \hookrightarrow Z$ is continuous. If E has the compact approximation property then, in particular, for every $\varepsilon > 0$ there exists a compact operator T in E such that

(2) $$\sup\{\|Tx - x\| : x \in K_1\} \le \varepsilon.$$

By (ii) and the definition of K, $\beta(S) := \lim_n \beta_n(S)$ exists for every $S \in L(E,E)$ and

(3) $$|\beta(S)| \le C \sup\{\|Sx\| : x \in K\}.$$

Obviously, by (i), if I is the identity operator on E, then $\beta_n(I) = 1$, $n \in \mathbb{N}$; thus $\beta(I) = 1$. Let us see that $\beta(T) = 0$ for every compact operator T in E; that implies

$$\sup\{\|Tx - x\| : x \in K\} \ge C^{-1}|\beta(I-T)| = C^{-1},$$

which contradicts (2) for $\varepsilon < C^{-1}$. To do this, if T is a compact operator in E, there are $n \in \mathbb{N}$ and $M > 0$ such that $T(K_{n+1}) \subset MK_n$. Since $\frac{1}{6}\alpha_{4i+3}^{n+1} z_i \in K_{n+1}$ by definition of z_i and K_{n+1}, the element $y = (y_j)_{j=4}^{\infty} :=$

$T(\frac{1}{6}\alpha_{4i+3}^{n+1}z_i)$ belongs to MK_n, and we have

$$|z_i^*(Tz_i)| = \frac{6}{\alpha_{4i+3}^{n+1}}\left|z_i^*\left(T\left(\frac{\alpha_{4i+3}^{n+1}}{6}z_i\right)\right)\right| = \frac{6}{\alpha_{4i+3}^{n+1}}\left|\frac{y_{2i}}{2} - \frac{y_{2i+1}}{2}\right|$$
$$\le \frac{3}{\alpha_{4i+3}^{n+1}}(|y_{2i}| + |y_{2i+1}|) \le \frac{6M\alpha_{2i}^n}{\alpha_{4i+3}^{n+1}}, \quad i = 2, 3, \ldots$$

Therefore, by (1), $\lim_i z_i^*(Tz_i) = 0$ and we obtain $\beta(T) = 0$, concluding that E does not have the compact approximation property.

The dual $F = E_b'$ is a Fréchet–Schwartz space which does not have the compact approximation property. ■

As a consequence we obtain the negative solution to the problems of Bierstedt, Meise, Hollstein and Taskinen.

COROLLARY 3. (1) *There are Fréchet–Schwartz spaces F such that* (F, X) *does not have property* (BB) *for some reflexive Banach space X with Schauder basis.*

(2) *There are (DFS)-spaces $E = \text{ind}_n E_n$ such that $E \otimes_\varepsilon X = \text{ind}_n(E_n \otimes_\varepsilon X)$ does not hold topologically for some reflexive Banach space X with Schauder basis.*

According to Proposition 1 and the positive result [2, Theorem 1], mentioned in the introduction, it is a natural question whether a (DFS)-space $E = \text{ind}_n E_n$ such that $E \otimes_\varepsilon X = \text{ind}_n(E_n \otimes_\varepsilon X)$ holds topologically for every Banach space X must satisfy the approximation property. The answer is negative.

PROPOSITION 4. *There are (DFS)-spaces $E = \text{ind}_n E_n$ without the approximation property for which $E \otimes_\varepsilon X = \text{ind}_n(E_n \otimes_\varepsilon X)$ holds topologically for every Banach space X.*

Proof. By Willis [15] there are Banach spaces Z without the approximation property having the compact approximation property. Then there is an absolutely convex compact subset K_1 of Z such that the identity I_Z of Z cannot be uniformly approximated on K_1 by finite rank operators in Z. Since E has the compact approximation property, there is a sequence $\{P_n\}_{n\in\mathbb{N}}$ of compact operators in Z such that

$$\widetilde{K}_n := (I_Z - P_n)(K_1) \subset \frac{1}{2^n}B_Z \quad \forall n \in \mathbb{N}.$$

Now take an absolutely convex compact subset K_2 of Z which satisfies

(i) $\bigcup_{n\in\mathbb{N}}((n/2^n)B_Z \cap n\widetilde{K}_n) \subset K_2$,
(ii) $\forall n \in \mathbb{N}\ \exists\lambda_n > 0$: $P_n(B_Z) \subset \lambda_n K_2$,
(iii) the natural inclusion $Z_{K_1} \hookrightarrow Z_{K_2}$ is compact.

From this we obtain

(a) $P_n^{-1}(K_2) \in \mathcal{U}_0(Z)$,
(b) $(I_Z - P_n)(K_1) \subset (1/n)K_2$, $\forall n \in \mathbb{N}$.

Proceeding by induction we construct a sequence $\{K_n\}_{n\in\mathbb{N}}$ of absolutely convex compact subsets of Z such that, for every $n \in \mathbb{N}$,

(1) $Z_{K_n} \hookrightarrow Z_{K_{n+1}}$ is compact,
(2) there is a sequence $\{Q_k^n\}_{k\in\mathbb{N}}$ of compact operators in Z with
 (a) $(Q_k^n)^{-1}(K_{n+1}) \in \mathcal{U}_0(Z)$,
 (b) $(I_Z - Q_k^n)(K_n) \subset (1/k)K_{n+1}$, $\forall k \in \mathbb{N}$.

We define $E_n := Z_{K_n}$ and $E := \text{ind}_n E_n$. Clearly E is a (DFS)-space without the approximation property (see e.g. [7, 18.5.8]). On the other hand, it is easy to see that (2) above implies that the injection from E_n into E_{n+1} can be approximated in the operator norm by restrictions to E_n of continuous linear maps from E into E_{n+1}. Now the proof of [2, Theorem 1] shows that $E \otimes_\varepsilon X = \text{ind}_n(E_n \otimes_\varepsilon X)$ holds topologically for every Banach space X. ■

COROLLARY 5. *There are Fréchet–Schwartz spaces and (DFS)-spaces which have the compact approximation property but not the approximation property.*

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Properly semi-L-embedded complex spaces

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Abstract. We prove the existence of complex Banach spaces X such that every element F in the bidual X^{**} of X has a unique best approximation $\pi(F)$ in X, the equality $\|F\| = \|\pi(F)\| + \|F - \pi(F)\|$ holds for all F in X^{**}, but the mapping π is not linear.

1. Introduction. Semi-L-summands were introduced by Å. Lima [8] in connection with his study of subspaces of Banach spaces having the so-called "2-ball property". A *semi-L-summand* of a Banach space X is a subspace M of X such that every element x in X has a unique best approximation $\pi(x)$ in M and the equality $\|x\| = \|\pi(x)\| + \|x - \pi(x)\|$ holds for all x in X. If in addition the mapping π is linear, then M is said to be an *L-summand* of X, while otherwise M is called a *proper semi-L-summand* of X. A *semi-L-embedded* (respectively: *L-embedded*, *properly semi-L-embedded*) space is a Banach space X which is a semi-L-summand (respectively: L-summand, proper semi-L-summand) of the bidual X^{**} of X.

Real Banach spaces containing proper semi-L-summands are exhibited in the quoted paper [8]. The easiest example is the space of all real-valued affine functions on the triangle, the set of constant functions being then a proper semi-L-summand. Nonreflexive real or complex L-embedded spaces are also well known: l_1, the preduals of infinite-dimensional von Neumann algebras, and, more generally, the preduals of nonreflexive JBW*-triples [2] are examples of such spaces, and a complete information about them is to be found in [7]. Recently R. Payá and A. Rodríguez [11] have proved the existence of properly semi-L-embedded real spaces, the easiest example being the space of all real-valued continuous affine functions on a countable infinite product of copies of the triangle. More recently E. Behrends [3] has shown that a compact convex subset K of $\mathbb{C}^2$ with the property that $f(K)$ is a disk for every linear mapping $f : \mathbb{C}^2 \to \mathbb{C}$ need not have a center of symmetry, a fact which is equivalent to the existence of complex Banach spaces containing proper semi-L-summands [12].

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