By Proposition 26(4), for all $\lambda \in \mathbb{C}$, there exists $F \in \mathcal{F}_{\mathcal{B}}$ such that $(\lambda - (S_\infty + R_\infty) + F)^{-1} = K_\infty$, where K is an essentially bounded kernel, that is, $K_\infty \in \mathcal{I}_{\mathcal{B}}$. Also, when $\lambda \in \varrho(S_\infty + R_\infty)$, then this statement is true with $F = 0$. Now by (2), for all $\lambda \in \mathbb{C}$, $\lambda - (S_p + R_p) + F$ is Fredholm of index zero on $L^p[0,1]$. But

$$L^\infty[0,1] = \mathcal{R}(\lambda - (S_\infty + R_\infty) + F) \subseteq \mathcal{R}(\lambda - (S_p + R_p) + F).$$

Therefore $\lambda - (S_p + R_p) + F$ is invertible on $L^p[0,1]$. Certainly, we have $(\lambda - (S_p + R_p) + F)^{-1} = K_p$. This proves both (3) and (4).

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Received June 2, 1992
Revised version March 16, 1993 (2949)

STUDIA MATHEMATICA 106 (2) (1993)

Wavelet bases in $L^p(\mathbb{R})$

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Abstract. It is shown that an orthonormal wavelet basis for $L^2(\mathbb{R})$ associated with a multiresolution is an unconditional basis for $L^p(\mathbb{R})$, $1 < p < \infty$, provided the father wavelet is bounded and decays sufficiently rapidly at infinity.

1. Introduction. The purpose of this paper is to extend some of the results in [8] on unconditional bases (see e.g. [5]) in wavelet form $\{\psi(2^{-m}\bullet -k)\}_{m,k\in\mathbb{Z}}$ for the space $L^p(\mathbb{R};\mathbb{C})$, $1 < p < \infty$. Here ψ is a mother wavelet, that is, $\{2^{-m/2}\psi(2^{-m}\bullet -k)\}_{m,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R};\mathbb{C})$ (and $\bullet$ denotes a generic argument). The analysis in this paper is restricted to the one-dimensional case where there is also a father wavelet φ such that if V_m is the space spanned by $\{\varphi(2^{-m}\bullet -k)\}_{k\in\mathbb{Z}}$ and W_m is the space spanned by $\{\psi(2^{-m}\bullet -k)\}_{k\in\mathbb{Z}}$, then $V_{m-1} = V_m \oplus W_m$. In other words, the wavelets are associated with a multiresolution.

It is proved below that if φ and ψ are bounded and both decay sufficiently rapidly at infinity, then we get an unconditional basis for $L^p(\mathbb{R};\mathbb{C})$ for all $p \in (1,\infty)$. The main point of this paper is to show that no smoothness assumptions on φ and ψ (like those in [8] where it is required that $|\psi'(x)| \le C_p(1+|x|)^{-p}$ for every $p \ge 0$). are needed for this conclusion to hold. This means in particular that all the compactly supported wavelets constructed in [2] give rise to unconditional bases and we also get an alternative proof for the well-known fact that the Haar functions (for which $\varphi = \chi_{[0,1)}$ and $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$) form an unconditional basis (cf. [9, p. 207]).

2. Statement of results. First we define what we mean by a multiresolution or a multiresolution analysis as it is commonly called. We say that $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ is an *orthonormal multiresolution* of $L^2(\mathbb{R};\mathbb{C})$ provided the following four conditions hold.

1991 *Mathematics Subject Classification*: Primary 42C15; Secondary 46B15, 46E30.
Key words and phrases: basis, L^p, multiresolution, unconditional, wavelet.

(1) $\varphi \in L^2(\mathbb{R};\mathbb{C})$ and V_m is the closed subspace of $L^2(\mathbb{R};\mathbb{C})$ spanned by $\{\varphi(2^{-m}\bullet - k)\}_{k\in\mathbb{Z}}$, for each $m \in \mathbb{Z}$,

(2) $V_m \subset V_{m-1}$, $m \in \mathbb{Z}$,

(3) $\lim_{m\to-\infty} P_m f = f$ for every $f \in L^2(\mathbb{R};\mathbb{C})$, where P_m is the orthogonal projection of $L^2(\mathbb{R};\mathbb{C})$ onto V_m,

(4) $\{\varphi(\bullet - k)\}_{k\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R};\mathbb{C})$.

The function φ is then said to be the *father wavelet* or *scaling function*.

The definition of a multiresolution is frequently given in a slightly different form (but with exactly the same content; see for example [1], [3] and [6]–[8]), so that the fact that $\{\varphi(2^{-m}\bullet - k)\}_{k\in\mathbb{Z}}$ spans V_m is a consequence of the other assumptions. Condition (3) is often formulated as the requirement that $\bigcup_{m=-\infty}^{\infty} V_m$ is dense in $L^2(\mathbb{R};\mathbb{C})$ and it is combined with the assumption that $\bigcap_{m=-\infty}^{\infty} V_m = \{0\}$, which follows from the other conditions (see [1, p. 443]). It is not really essential that $\{\varphi(\bullet - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis, it would suffice (as is frequently done) to require that it is an unconditional basis, in which case it is a Riesz basis, but from such a basis one can construct an orthonormal one.

Since $\varphi \in V_0 \subset V_{-1}$ it follows that φ can be expressed with the help of the functions $\{\varphi(2\bullet - k)\}_{k\in\mathbb{Z}}$ which span V_{-1}, i.e.,

$$\varphi = 2\sum_{k\in\mathbb{Z}} \alpha(k)\varphi(2\bullet - k),$$

where

$$\alpha(k) = \int_{\mathbb{R}} \varphi(x)\overline{\varphi(2x-k)}\,dx. \tag{5}$$

We call this sequence α the *filter* associated with the multiresolution and it turns out to be crucial for the analysis and in particular for computations involving the wavelets (cf. [11]).

Having found the filter α we can define the *mother wavelet* ψ as follows:

$$\psi = 2\sum_{k\in\mathbb{Z}} (-1)^k \overline{\alpha(1-k)}\varphi(2\bullet - k). \tag{6}$$

It follows from these definitions that the sets

$$\{2^{-m/2}\psi(2^{-m}\bullet - k)\}_{m,k\in\mathbb{Z}}$$

and

$$\{2^{-m_0/2}\varphi(2^{-m_0}\bullet - k),\ 2^{-m/2}\psi(2^{-m}\bullet - k)\}_{m\le m_0,\,k\in\mathbb{Z}},$$

where $m_0 \in \mathbb{Z}$ is arbitrary, are orthonormal bases for $L^2(\mathbb{R};\mathbb{C})$ (see e.g. [3] or [8]).

One of the most useful properties of the wavelets is that, under appropriate assumptions, they form bases in a large number of other spaces as well (see [8]). Here we shall only consider the spaces $L^p(\mathbb{R};\mathbb{C})$ where $1 < p < \infty$.

THEOREM 1. *Let $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ be an orthonormal multiresolution of $L^2(\mathbb{R};\mathbb{C})$ and let ψ be the associated mother wavelet. Denote by $\psi_{m,k}$ and $\varphi_{m,k}$, for $m,k \in \mathbb{Z}$, the functions $2^{-m/2}\psi(2^{-m}\bullet - k)$ and $2^{-m/2}\varphi(2^{-m}\bullet - k)$ respectively. Assume that $\varphi \in L^\infty(\mathbb{R};\mathbb{C})$ and that*

$$\int_0^\infty x \operatorname*{ess\,sup}_{|y|\ge x} |\varphi(y)|\,dx < \infty. \tag{7}$$

Then the sets

$$\{2^{-m/2}\psi(2^{-m}\bullet - k)\}_{m,k\in\mathbb{Z}}$$

and

$$\{2^{-m_0/2}\varphi(2^{-m_0}\bullet - k),\ 2^{-m/2}\psi(2^{-m}\bullet - k)\}_{m\le m_0,\,k\in\mathbb{Z}},$$

where $m_0 \in \mathbb{Z}$ is arbitrary, are unconditional bases for $L^p(\mathbb{R};\mathbb{C})$ where $1 < p < \infty$.

Moreover, for every $p \in (1,\infty)$ and $m_0 \in \mathbb{Z}$ there exists a constant C_p such that for each $f \in L^p(\mathbb{R};\mathbb{C})$,

$$\frac{1}{C_p}\Big\|\Big(\sum_{m,k\in\mathbb{Z}} |\langle f,\psi_{m,k}\rangle|^2 |\psi_{m,k}(\bullet)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})} \le C_p\Big\|\Big(\sum_{m,k\in\mathbb{Z}} |\langle f,\psi_{m,k}\rangle|^2 |\psi_{m,k}(\bullet)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R})} \tag{8}$$

and

$$\begin{aligned} \frac{1}{C_p}\Big\|\Big(\sum_{k\in\mathbb{Z}} |\langle f,\varphi_{m_0,k}\rangle|^2 |\varphi_{m_0,k}(\bullet)|^2 &+ \sum_{\substack{m,k\in\mathbb{Z}\\ m\le m_0}} |\langle f,\psi_{m,k}\rangle|^2 |\psi_{m,k}(\bullet)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R})} \\ &\le \|f\|_{L^p(\mathbb{R})} \\ &\le C_p\Big\|\Big(\sum_{k\in\mathbb{Z}} |\langle f,\varphi_{m_0,k}\rangle|^2 |\varphi_{m_0,k}(\bullet)|^2 \\ &\quad + \sum_{\substack{m,k\in\mathbb{Z}\\ m\le m_0}} |\langle f,\psi_{m,k}\rangle|^2 |\psi_{m,k}(\bullet)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R})}, \end{aligned} \tag{9}$$

where $\langle f,g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx$.

If one assumes that the wavelets are sufficiently regular, as is done for example in [8] where they are assumed to be r-regular, one need not assume that the wavelets arise from a multiresolution to get the first basis above; but the second basis (which presupposes the existence of a multiresolution) has many advantages in that the basis functions are much better localized because the dilation factor 2^{-m} is bounded away from zero.

It is clear that a necessary condition for the previous result to hold is that φ and ψ belong to $L^p(\mathbb{R};\mathbb{C})$ for all $p \in (1,\infty)$. In this sense the assumptions used above are not best possible, but for most practical purposes (7) is a quite reasonable requirement.

The proof is based on the following result.

PROPOSITION 2. *Let $(\{V_m\}_{m\in\mathbb{Z}},\varphi)$ be an orthonormal multiresolution of $L^2(\mathbb{R};\mathbb{C})$ and let ψ be the associated mother wavelet. Denote by $\psi_{m,k}$ and $\varphi_{m,k}$, for $m,k\in\mathbb{Z}$, the functions $2^{-m/2}\psi(2^{-m}\bullet - k)$ and $2^{-m/2}\varphi(2^{-m}\bullet - k)$ respectively. Assume that $\varphi \in L^\infty(\mathbb{R};\mathbb{C})$ and that (7) holds.*

Let $\tau : \mathbb{Z}^2 \to \{-1,1\}$ be an arbitrary function and define the operator T_τ : $L^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$ either by $T_\tau\psi_{m,k} = \tau(m,k)\psi_{m,k}$ for all $m,k\in\mathbb{Z}$ or by $T_\tau\psi_{m,k} = \tau(m,k)\psi_{m,k}$ for all $m \le m_0$, $k\in\mathbb{Z}$ and $T_\tau\varphi_{m_0,k} = \tau_{m_0+1,k}\varphi_{m_0,k}$, $k\in\mathbb{Z}$.

Then one can, for each $p\in(1,\infty)$, extend T_τ to a continuous operator from $L^p(\mathbb{R};\mathbb{C})$ into itself such that

$$\frac{1}{C_p}\|f\|_{L^p(\mathbb{R})} \le \|T_\tau f\|_{L^p(\mathbb{R})} \le C_p\|f\|_{L^p(\mathbb{R})}, \tag{10}$$

for some constant C_p independent of τ.

Once this result has been established, the rest of the proof is essentially the same as in [8], but for completeness it will be given below.

3. Proofs.

First we show that the mother wavelet ψ is bounded and satisfies the same moment condition as the father wavelet φ.

LEMMA 3. *Let $(\{V_m\}_{m\in\mathbb{Z}},\varphi)$ be an orthonormal multiresolution of $L^2(\mathbb{R};\mathbb{C})$ and let ψ be the associated mother wavelet. Assume that $\varphi\in L^\infty(\mathbb{R};\mathbb{C})$ and that (7) holds. Then $\psi\in L^\infty(\mathbb{R};\mathbb{C})$ and*

$$\int_0^\infty x \operatorname*{ess\,sup}_{|y|\ge x}|\psi(y)|\,dx < \infty. \tag{11}$$

Proof. Denote by φ_* the function

$$\varphi_* = \operatorname*{ess\,sup}_{|x|\ge|\bullet|-1}|\varphi(x)|.$$

The filter α associated with the multiresolution is the sequence $\alpha\in\ell^2(\mathbb{Z};\mathbb{C})$ defined by (5). We immediately see that

$$\begin{aligned} |\alpha(k)| &\le \Big(\int_{-\infty}^{k/3} + \int_{k/3}^{\infty}\Big)|\varphi(x)||\varphi(2x-k)|\,dx \\ &\le 2\|\varphi\|_{L^1(\mathbb{R})}\varphi_*(|k|/3+1), \qquad k\in\mathbb{Z}. \end{aligned} \tag{12}$$

This shows in particular that $\alpha\in\ell^1(\mathbb{Z};\mathbb{C})$ and therefore we conclude that $\psi\in L^\infty(\mathbb{R};\mathbb{C})$ because by (6),

$$|\psi(x)| \le 2\sum_{k\in\mathbb{Z}}|\alpha(1-k)||\varphi(2x-k)|, \qquad x\in\mathbb{R}.$$

Now we get for almost every $x \ge 0$,

$$\begin{aligned} |\psi(x)| &\le \Big(\sum_{k=-\infty}^{\lfloor 3x/2\rfloor} + \sum_{k=\lfloor 3x/2\rfloor+1}^{\infty}\Big)|\alpha(1-k)||\varphi(2x-k)| \\ &\le \|\alpha\|_{\ell^1(\mathbb{Z})}\varphi_*(x/2) + \sup_{k\le -\lfloor 3x/2\rfloor}|\alpha(k)|\sum_{k=\lfloor 3x/2\rfloor+1}^{\infty}|\varphi(2x-k)| \\ &\le \Big(\|\alpha\|_{\ell^1(\mathbb{Z})} + 2\|\varphi\|_{L^1(\mathbb{R})}\sum_{k\in\mathbb{Z}}\varphi_*(k)\Big)\varphi_*(|x|/2), \end{aligned}$$

by (12). Observe that this inequality holds for almost every $x < 0$ as well, and hence we get the desired conclusion. ■

Proof of Proposition 2. We shall only consider the case where T_τ is defined by $T_\tau\psi_{m,k} = \tau(m,k)\psi_{m,k}$ for all m, $k\in\mathbb{Z}$, because the proof of the second case is almost the same, the only difference is that the notation is messier. Moreover, we see that if we can prove the second inequality in (10), then we get the first as well, because T_τ^2 is the identity operator.

It follows from the orthonormality of $\psi_{m,k}$ that T_τ is an isometry from $L^2(\mathbb{R};\mathbb{C})$ into itself. Suppose for the moment that we know, in addition, that T_τ is of weak type $(1,1)$ uniformly in τ (see [10, p. 20]). Then we immediately see from the Marcinkiewicz interpolation theorem (see [10, Theorem I.5]) that T_τ can be extended to a bounded operator (uniformly in τ) on $L^p(\mathbb{R};\mathbb{C})$ when $1 < p < 2$, and since T_τ satisfies $\langle T_\tau f, g\rangle = \langle f, T_\tau g\rangle$ for all $f\in L^2(\mathbb{R};\mathbb{C})\cap L^p(\mathbb{R};\mathbb{C})$ and $g\in L^2(\mathbb{R};\mathbb{C})\cap L^q(\mathbb{R};\mathbb{C})$, this result can be extended to all $p\in(1,\infty)$. Thus we see that the only thing that remains to be proved is that T_τ is of weak type $(1,1)$ (uniformly for all τ).

We use the notation

$$I_{m,k} \stackrel{\text{def}}{=} [2^m k, 2^m(k+1)),$$

$$I^*_{m,k} \stackrel{\text{def}}{=} [2^m(k-1), 2^m(k+2)), \qquad m,k\in\mathbb{Z}.$$

Let $f\in L^1(\mathbb{R};\mathbb{C})\cap L^2(\mathbb{R};\mathbb{C})$ be arbitrary and fix a positive number ε. It follows from [10, Theorem I.4] that there exists a set $\Lambda\subset\mathbb{Z}^2$ such that the intervals $\{I_{m,k}\}_{(m,k)\in\Lambda}$ are disjoint,

(13) $$|f(x)| \le \varepsilon \quad \text{almost everywhere on} \quad F \stackrel{\text{def}}{=} \mathbb{R} \setminus \bigcup_{(m,k)\in\Lambda} I_{m,k},$$

and

(14) $$\varepsilon < 2^{-m} \int_{I_{m,k}} |f(x)|\,dx \le 2\varepsilon, \qquad (m,k)\in\Lambda.$$

An immediate consequence of (14) and of the fact that the intervals $\{I_{m,k}\}_{(m,k)\in\Lambda}$ are disjoint is that

(15) $$\sum_{(m,k)\in\Lambda} 2^m \le \frac{1}{\varepsilon}\|f\|_{L^1(\mathbb{R})}.$$

Let us denote by P_m the orthogonal projection in $L^2(\mathbb{R};\mathbb{C})$ onto the space V_m spanned by $\{\varphi(2^{-m}\bullet - k)\}_{k\in\mathbb{Z}}$ and define the functions g and h by

(16) $$\begin{aligned} g &= \chi_F f + \sum_{(m,k)\in\Lambda} P_m(\chi_{I_{m,k}} f), \\ h &= f - g = \sum_{(m,k)\in\Lambda} (\chi_{I_{m,k}} f - P_m(\chi_{I_{m,k}} f)). \end{aligned}$$

(In the second case we would define $g = \chi_F f + \sum_{(m,k)\in\Lambda, m<m_0} P_m(\chi_{I_{m,k}} f)$.)

First we must show that $g \in L^2(\mathbb{R};\mathbb{C})$ and obtain an appropriate bound on the L^2-norm. By (13) we get

(17) $$\int_{\mathbb{R}} |\chi_F(x) f(x)|^2\,dx = \int_F |f(x)|^2\,dx \le \int_F \varepsilon |f(x)|\,dx \le \varepsilon \|f\|_{L^1(\mathbb{R})}.$$

Now we have

(18) $$P_m(\chi_{I_{m,k}} f) = \sum_{j\in\mathbb{Z}} \varphi(2^{-m}\bullet - j) a_{m,k,j},$$

where

$$a_{m,k,j} = 2^{-m} \int_{I_{m,k}} \overline{\varphi(2^{-m}x - j)} f(x)\,dx.$$

Again we denote by φ_* the function $\varphi_* = \operatorname{ess\,sup}_{|x|\ge|\bullet|-1} |\varphi(x)|$. Then it is easy to see from (14) that

(19) $$|a_{m,k,j}| \le 2\varepsilon \varphi_*(k-j).$$

From (18) we get

(20) $$\Big\| \sum_{(m,k)\in\Lambda} P_m(\chi_{I_{m,k}} f) \Big\|^2_{L^2(\mathbb{R})}$$

$$\le 2 \sum_{(m,k)\in\Lambda} \sum_{\substack{(m',k')\in\Lambda \\ m'\le m}} \sum_{j,j'\in\mathbb{Z}} |a_{m,k,j}||a_{m',k',j'}| \times \int_{\mathbb{R}} |\varphi(2^{-m}x - j)||\varphi(2^{-m'}x - j')|\,dx.$$

Fix the integers m and k. It follows from (19) that

(21) $$\begin{aligned} &\sum_{\substack{(m',k')\in\Lambda \\ m'\le m}} \sum_{j,j'\in\mathbb{Z}} |a_{m,k,j}||a_{m',k',j'}| \int_{\mathbb{R}} |\varphi(2^{-m}x - j)||\varphi(2^{-m'}x - j')|\,dx \\ &\le \sum_{\substack{(m',k')\in\Lambda \\ m'\le m}} \sum_{j,j'\in\mathbb{Z}} 4\varepsilon^2 \varphi_*(k-j)\varphi_*(k'-j') \times \int_{\mathbb{R}} |\varphi(2^{-m}x - j)||\varphi(2^{-m'}x - j')|\,dx \\ &= 4\varepsilon^2 \sum_{j,j'\in\mathbb{Z}} \varphi_*(j)\varphi_*(j') \times \sum_{\substack{(m',k')\in\Lambda \\ m'\le m}} \int_{\mathbb{R}} |\varphi(2^{-m}x - j - k)||\varphi(2^{-m'}x - j' - k')|\,dx. \end{aligned}$$

For fixed integers j and j' we get

(22) $$\begin{aligned} &\sum_{\substack{(m',k')\in\Lambda \\ m'\le m}} \int_{\mathbb{R}} |\varphi(2^{-m}x - j - k)||\varphi(2^{-m'}x - j' - k')|\,dx \\ &= \sum_{p\in\mathbb{Z}} \sum_{m'=-\infty}^{m} \sum_{\substack{k'=p2^{-m'+m} \\ (m',k')\in\Lambda}}^{(p+1)2^{-m'+m}-1} 2^{m'} \times \int_{\mathbb{R}} |\varphi(2^{-m+m'}(x + j' + k') - j - k)||\varphi(x)|\,dx \\ &\le \sum_{p\in\mathbb{Z}} \sum_{m'=-\infty}^{m} \sum_{\substack{k'=p2^{-m'+m} \\ (m',k')\in\Lambda}}^{(p+1)2^{-m'+m}-1} 2^{m'} \times \int_{\mathbb{R}} \varphi_*(2^{-m+m'}x + r + p)\varphi_*(x)\,dx, \end{aligned}$$

where $r = -j - k + \lceil 2^{-m+m'} j' \rceil$. If p and q are fixed nonnegative integers, then we have

$$(23)\quad \int_{\mathbb{R}} \varphi_*(2^q x + p)\varphi_*(x)\,dx = \Big(\int_{-\infty}^{-2^{-q-1}p} + \int_{-2^{-q-1}p}^{\infty}\Big)\varphi_*(2^q x+p)\varphi_*(x)\,dx$$
$$\le \|\varphi_*\|_{L^\infty(\mathbb{R})}\int_{2^{-q-1}p}^{\infty}\varphi_*(x)\,dx + \|\varphi_*\|_{L^1(\mathbb{R})}\varphi_*(p/2)$$
$$\le \|\varphi_*\|_{L^\infty(\mathbb{R})}\int_{|p|/2}^{\infty}\varphi_*(x)\,dx + \|\varphi_*\|_{L^1(\mathbb{R})}\varphi_*(p/2)\,,$$

and it is clear that the inequality we got holds for negative p as well.

Recall that the intervals $\{I_{m',k'}\}_{(m',k')\in\Lambda}$ are disjoint, the length of $I_{m',k'}$ is $2^{m'}$, and that if $m' \le m$ and $p2^{-m'+m} \le k' < (p+1)2^{-m'+m}$, then $I_{m',k'} \subset I_{m,p}$. It follows that for each integer p we have

$$\sum_{m'=-\infty}^{m}\ \sum_{\substack{k'=p2^{-m'+m}\\(m',k')\in\Lambda}}^{(p+1)2^{-m'+m}-1} 2^{m'} \le 2^m\,.$$

If we use this estimate together with (23) in (22) we get

$$\sum_{\substack{(m',k')\in\Lambda\\ m'\le m}}\int_{\mathbb{R}}\varphi_*(2^{-m}x-j-k)\varphi_*(2^{-m'}x-j'-k')\,dx$$
$$\le 2^m\Big(\|\varphi_*\|_{L^\infty(\mathbb{R})}\sum_{p\in\mathbb{Z}}\int_{|p|/2}^{\infty}\varphi_*(x)\,dx + \|\varphi_*\|_{L^1(\mathbb{R})}\sum_{p\in\mathbb{Z}}^{\infty}\varphi_*(p/2)\Big)\,.$$

Our assumptions imply the series $\sum_{p\in\mathbb{Z}}^{\infty}\int_{|p|/2}^{\infty}\varphi_*(x)\,dx$ and $\sum_{p\in\mathbb{Z}}^{\infty}\varphi_*(p/2)$ converge and therefore it follows from (15), (20), (21), and from the inequality above that there exists a constant c_1 such that

$$(24)\qquad \Big\|\sum_{(m,k)\in\Lambda} P_m(\chi_{I_{m,k}}f)\Big\|^2_{L^2(\mathbb{R})} \le c_1\varepsilon\|f\|_{L^1(\mathbb{R})}\,.$$

Since T_τ is an isometry on $L^2(\mathbb{R};\mathbb{C})$ it follows from (16) and (17) that

$$\|T_\tau g\|^2_{L^2(\mathbb{R})} \le 2(1+c_1)\varepsilon\|f\|_{L^1(\mathbb{R})}\,,$$

and therefore

$$(25)\qquad \mathrm{m}(\{x\in\mathbb{R} \mid |(T_\tau g)(x)| \ge \varepsilon/2\}) \le \frac{1}{\varepsilon}8(1+c_1)\|f\|_{L^1(\mathbb{R})}\,.$$

We proceed to consider the second term in the decomposition $T_\tau f = T_\tau g + T_\tau h$. By definition we have

$$T_\tau h = \sum_{m',k'\in\mathbb{Z}}\tau(m',k')\langle h,\psi_{m',k'}\rangle\psi_{m',k'}$$
$$= \sum_{m',k'\in\mathbb{Z}}\tau(m',k')\sum_{m,k\in\Lambda}\langle(\chi_{I_{m,k}}f - P_m(\chi_{I_{m,k}}f)),\psi_{m',k'}\rangle\psi_{m',k'}\,.$$

The set $\{\psi_{m',k'}\}_{m'\le m,k\in\mathbb{Z}}$ is an orthonormal basis for the orthogonal complement of V_m in $L^2(\mathbb{R};\mathbb{C})$ and hence for each (m,k) we have

$$\langle(\chi_{I_{m,k}}f - P_m(\chi_{I_{m,k}}f)),\psi_{m',k'}\rangle = \begin{cases}\langle\chi_{I_{m,k}}f,\psi_{m',k'}\rangle, & m'\le m,\\ 0, & m'>m.\end{cases}$$

Thus

$$(26)\qquad T_\tau h = \sum_{m,k\in\Lambda}\ \sum_{\substack{m',k'\in\mathbb{Z}\\ m'\le m}}\tau(m',k')\langle\chi_{I_{m,k}}f,\psi_{m',k'}\rangle\psi_{m',k'}\,.$$

We denote by F^* the set $\mathbb{R}\setminus\bigcup_{(m',k')\in\Lambda}I^*_{m',k'}$ and we will calculate an estimate for $\int_{F^*}|(T_\tau h)(x)|\,dx$. We have

$$(27)\qquad \int_{\mathbb{R}\setminus I^*_{m,k}}\Big|\sum_{\substack{m',k'\in\mathbb{Z}\\ m'\le m}}\tau(m',k')\langle\chi_{I_{m,k}}f,\psi_{m',k'}\rangle\psi_{m',k'}(x)\Big|\,dx$$
$$\le \int_{\mathbb{R}\setminus I^*_{m,k}}\sum_{\substack{m',k'\in\mathbb{Z}\\ m'\le m}}|\langle\chi_{I_{m,k}}f,\psi_{m',k'}\rangle||\psi_{m',k'}(x)|\,dx$$
$$\le \int_{I_{m,k}}|f(y)|\sum_{\substack{m',k'\in\mathbb{Z}\\ m'\le m}}\Big(\int_{\mathbb{R}\setminus I^*_{m,k}}|\psi_{m',k'}(y)||\psi_{m',k'}(x)|\,dx\Big)\,dy$$
$$\le \int_{I_{m,k}}|f(y)|\,dy\sum_{\substack{m',k'\in\mathbb{Z}\\ m'\le m}}\operatorname*{ess\,sup}_{y\in I_{m,k}}|\psi_{m',k'}(y)|\int_{\mathbb{R}\setminus I^*_{m,k}}|\psi_{m',k'}(x)|\,dx.$$

Next we fix $m'\le m$ and derive the estimate

$$(28)\qquad \sum_{k'\in\mathbb{Z}}\operatorname*{ess\,sup}_{y\in I_{m,k}}|\psi_{m',k'}(y)|\int_{\mathbb{R}\setminus I^*_{m,k}}|\psi_{m',k'}(x)|\,dx$$
$$= \sum_{k'\in\mathbb{Z}}\operatorname*{ess\,sup}_{y\in I_{m,k}}|\psi(2^{-m'}y-k')|\Big(\int_{-\infty}^{2^m(k-1)} + \int_{2^m(k+2)}^{\infty}\Big)2^{-m'}|\psi(2^{-m'}x-k')|\,dx$$

$$= \sum_{k'\in\mathbb{Z}} \operatorname*{ess\,sup}_{y\in 2^{-m'}I_{m,k-k'}} |\psi(y)| \Big(\int_{-\infty}^{2^{-m'+m}(k-1)-k'} + \int_{2^{-m'+m}(k+2)-k'}^{\infty} \Big) |\psi(x)|\,dx$$

$$= \sum_{k'\in\mathbb{Z}} \operatorname*{ess\,sup}_{y\in[k',2^{-m'+m}+k')} |\psi(y)| \Big(\int_{-\infty}^{k'-2^{-m'+m}} + \int_{2^{-m'+m+1}+k'}^{\infty} \Big) |\psi(x)|\,dx.$$

Let $\psi_* = \operatorname{ess\,sup}_{|y|\ge|\bullet|-1}|\psi(y)|$ and assume that $k' \ge 2^{-m'+m-1}$. Then we have

$$\operatorname*{ess\,sup}_{y\in[k',2^{-m'+m}+k')} |\psi(y)| \int_{-\infty}^{k'-2^{-m'+m}} |\psi(x)|\,dx \le \psi_*(k')\|\psi\|_{L^1(\mathbb{R})}.$$

If $-2^{-m'+m+1} \le k' < 2^{-m'+m-1}$ then

$$\operatorname*{ess\,sup}_{y\in[k',2^{-m'+m}+k')} |\psi(y)| \int_{-\infty}^{k'-2^{-m'+m}} |\psi(x)|\,dx \le \|\psi\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{-2^{-m'+m-1}} |\psi(x)|\,dx,$$

and if $k' < -2^{-m'+m+1}$ then

$$\operatorname*{ess\,sup}_{y\in[k',2^{-m'+m}+k')} |\psi(y)| \int_{-\infty}^{k'-2^{-m'+m}} |\psi(x)|\,dx \le \psi_*(k'+2^{-m'+m})\|\psi\|_{L^1(\mathbb{R})}.$$

By breaking the second integral into parts in a similar manner we conclude that

$$(29)\quad \sum_{k'\in\mathbb{Z}} \operatorname*{ess\,sup}_{y\in[k',2^{-m'+m}+k')} |\psi(y)| \Big(\int_{-\infty}^{k'-2^{-m'+m}} + \int_{2^{-m'+m+1}+k'}^{\infty} \Big) |\psi(x)|\,dx$$
$$\le 4\|\psi\|_{L^1(\mathbb{R})} \sum_{k'=\lceil 2^{-m'+m-1}\rceil}^{\infty} \psi_*(k')$$
$$+ 2\|\psi\|_{L^\infty(\mathbb{R})} 2^{-m'+m+2} \int_{-\infty}^{-2^{-m'+m-1}} |\psi(x)|\,dx.$$

Now it is easy to check that

$$\sum_{m'=-\infty}^{m} \sum_{k'=\lceil 2^{-m'+m-1}\rceil}^{\infty} \psi_*(k') = \sum_{k'=1}^{\infty} \psi_*(k')(\lfloor \log_2(k')\rfloor + 2)$$
$$\le 2\int_0^\infty (x+1)\psi_*(x)\,dx,$$

and also that

$$\sum_{m'=-\infty}^{m} 2^{-m'+m} \int_{-\infty}^{-2^{-m'+m-1}} |\psi(x)|\,dx$$
$$\le \int_0^\infty \psi_*(x) \sum_{m'=\lfloor \log_2(x)\rfloor - m+1}^{m} 2^{-m'+m}\,dx \le 4\int_0^\infty x\psi_*(x)\,dx.$$

If we combine the last two inequalities with (11), (27), (28), and (29), then we see that there exists a constant c_2 such that

$$\int_{\mathbb{R}\setminus I^*_{m,k}} \Big| \sum_{\substack{m',k'\in\mathbb{Z}\\ m'\le m}} \tau(m',k')\langle \chi_{I_{m,k}} f, \psi_{m',k'}\rangle \psi_{m',k'}(x)\Big|\,dx \le c_2 \int_{I_{m,k}} |f(y)|\,dy,$$

and therefore by (26) in particular that

$$\int_{F^*} |(T_\tau h)(x)|\,dx \le c_2\|f\|_{L^1(\mathbb{R})}.$$

Recall that by (15) we have $\mathrm{m}(\mathbb{R}\setminus F^*) \le 3\|f\|_{L^1(\mathbb{R})}/\varepsilon$ and hence

$$\mathrm{m}(\{x\in\mathbb{R} \mid |(T_\tau h)(x)| \ge \varepsilon/2\}) \le \frac{1}{\varepsilon}(2c_2+3)\|f\|_{L^1(\mathbb{R})}.$$

When we combine this result with (25) and with the fact that $T_\tau f = T_\tau g + T_\tau h$ we see that T_τ is of weak type $(1,1)$, uniformly in τ and the proof is complete. ∎

Now we can establish the main result.

Proof of Theorem 1. Fix a number $p\in(1,\infty)$ and choose an arbitrary function f that belongs to $L^p(\mathbb{R};\mathbb{C})\cap L^2(\mathbb{R};\mathbb{C})$.

We denote by $\{\phi_j\}_{j=0}^\infty$ either $\{2^{-m/2}\psi(2^{-m}\bullet - k)\}_{m,k\in\mathbb{Z}}$ or $\{2^{-m_0/2}\times \varphi(2^{-m_0}\bullet - k), 2^{-m/2}\psi(2^{-m}\bullet - k)\}_{m\le m_0,k\in\mathbb{Z}}$. Thus f can be expressed as

$$f = \sum_{j=0}^{\infty} \langle f,\phi_j\rangle\phi_j.$$

In particular, it follows that for almost every $x\in\mathbb{R}$ the series $\sum_{j=0}^\infty |\langle f,\phi_j\rangle|^2 \times |\phi_j(x)|^2$ converges. It is a consequence of the theory of Rademacher series (see [12, Theorem 8.4, p. 213]) that there exists a constant c_p such that

$$\frac{1}{c_p}\Big(\sum_{j=0}^{\infty}|\langle f,\phi_j\rangle|^2|\phi_j(x)|^2\Big)^{p/2} \le \int_0^1 \Big|\sum_{j=0}^{\infty}\langle f,\phi_j\rangle\phi_j(x)\,\mathrm{sign}(\sin(2^{j+1}\pi\tau))\Big|^p\,d\tau$$
$$\le c_p\Big(\sum_{j=0}^{\infty}|\langle f,\phi_j\rangle|^2|\phi_j(x)|^2\Big)^{p/2},$$

for almost every $x \in \mathbb{R}$. (Here we take e.g. $\operatorname{sign}(0) = 1$.) If we integrate over $\mathbb{R}$ and denote by T_τ the operator that takes f to $\sum_{j=0}^{\infty} \langle f, \phi_j \rangle \phi_j(\bullet) \times \operatorname{sign}(\sin(2^{j+1}\pi\tau))$, then we conclude that

$$\frac{1}{c_p} \Big\| \Big(\sum_{j=0}^{\infty} |\langle f, \phi_j \rangle|^2 |\phi_j(\bullet)|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R})}^p \le \int_0^1 \|T_\tau f\|_{L^p}^p \, d\tau \le c_p \Big\| \Big(\sum_{j=0}^{\infty} |\langle f, \phi_j \rangle|^2 |\phi_j(\bullet)|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R})}^p.$$

When we combine these inequalities with (10) (the identification of the real number τ with a mapping: $\mathbb{Z} \times \mathbb{Z} \to \{-1, 1\}$ is obtained by considering the mapping $\tau \to \{\operatorname{sign}(\sin(2^{j+1}\pi\tau))\}_{j=0}^{\infty}$ and reordering the indices) we conclude that (8) and (9) hold for our function f. Since $L^p(\mathbb{R}; \mathbb{C}) \cap L^2(\mathbb{R}; \mathbb{C})$ is dense in $L^p(\mathbb{R}; \mathbb{C})$ we see that these conclusions hold for all functions in $L^p(\mathbb{R}; \mathbb{C})$.

The statement about the unconditional bases now follows from [5, Theorem 7.1] because it is clear that both sets span dense sets in $L^p(\mathbb{R}; \mathbb{C})$, and the proof is complete. ■

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Received August 31, 1992 (2990)
Revised version January 18, 1993