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Perturbation theory relative to a Banach algebra of operators

by

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Abstract. Let $\mathcal{B}$ be a Banach algebra of bounded linear operators on a Banach space X. Let S be a closed linear operator in X, and let R be a linear operator in X. In this paper the spectral and Fredholm theory relative to $\mathcal{B}$ of the perturbed operator $S+R$ is developed. In particular, the situation where R is S-inessential relative to $\mathcal{B}$ is studied. Several examples are given to illustrate the usefulness of these concepts.

1. Introduction. Let X be a Banach space, and let $\mathcal{B}$ be a fixed Banach algebra of operators on X with $\mathcal{B} \subseteq B(X)$, the algebra of all bounded linear operators on X. There are many interesting algebras $\mathcal{B}$ which occur in operator theory. In the last several years a useful Fredholm theory has been developed in some of these algebras; see [1], [2], [4], and [12, §5.8]. To give one concrete example, when X is a Banach lattice, the Banach algebra of all regular operators on X has been widely studied, and a Fredholm theory for this algebra was developed in [1].

In [6] we studied the spectral and Fredholm properties of a (in general) unbounded linear operator S which is affiliated with $\mathcal{B}$ in the sense that $(\lambda - S)^{-1} \in \mathcal{B}$ for some $\lambda \in \mathbb{C}$. In this paper we continue the study of such operators, specifically, we consider the perturbation theory of S relative to $\mathcal{B}$. We believe this theory to be a natural and useful generalization of the classical theory. By way of illustration, assume that one knows at the start that the resolvents $(\lambda - S)^{-1} \in \mathcal{B}$ for all λ in some nonempty open subset of $\mathbb{C}$. The operators in $\mathcal{B}$ all have some interesting property, so it is useful to know for what $\mu \in \mathbb{C}$ a perturbation $S+R$ of S has $(\mu - (S+R))^{-1} \in \mathcal{B}$.

The main theory we generalize here is the theory of relative boundedness and relative compactness of an operator with respect to S [13, Chapter 4, §1]. It should be noted that the classical theory works in a situation where the generalization does not, specifically when $\lambda - S$ has no bounded inverse

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for any λ. On the other hand, the generalization gives new information even for the case $\mathcal{B} = \mathcal{B}(X)$. For the generalization allows one to deal with Fredholm theory relative to the largest closed inessential ideal in $\mathcal{B}(X)$, while the ideal of compact operators is often properly contained in this largest ideal. This largest ideal is the closed ideal of inessential operators on X (as defined by D. Kleinecke in [14]). In addition, the generalization yields a useful perturbation theory relative to a large class of Banach algebras of operators.

We use the following notation throughout: S is a linear operator with domain $\mathcal{D}(S)$ in X. If R is any operator, then $\mathcal{N}(R)$ denotes the null space of R and $\mathcal{R}(R)$ denotes the range of R. An operator R with $\mathcal{D}(R)$ in X is *Fredholm* when R is closed, $\mathcal{R}(R)$ is closed and of finite codimension in X, and $\mathcal{N}(R)$ is finite-dimensional. We do *not* require that $\mathcal{D}(R)$ is dense in X. For any operator R,

$$\varrho(R) = \{\lambda \in \mathbb{C} : (\lambda - R)^{-1} \in \mathcal{B}(X)\}, \quad \sigma(R) = \mathbb{C} \setminus \varrho(R).$$

2. Certain Banach algebras of operators. There are many interesting Banach algebras of operators relative to which one can do Fredholm theory. All of the algebras $\mathcal{B}$ which we consider in this paper (and the algebras in [1], [2], [4], and [6]) have the properties: $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach subalgebra of $\mathcal{B}(X)$, $I \in \mathcal{B}$, and for some $M > 0$, $M\|T\|_{\mathcal{B}} \geq \|T\|$ (operator norm) for all $T \in \mathcal{B}$. We shall always assume that $\mathcal{B}$ has these properties. We also assume that $\mathcal{B}$ contains sufficiently many finite rank operators in the sense explained next. For $x \in X$ and $\alpha \in X'$ (the dual space of X), let $\alpha \otimes x$ denote the operator

$$(\alpha \otimes x)(y) = \alpha(y)x \quad (y \in X).$$

In order to have a useful Fredholm theory relative to $\mathcal{B}$, we assume throughout that $\mathcal{B}$ has the property:

(#) There exists an X-total subspace Y of X' such that $\alpha \otimes x \in \mathcal{B}$ for all $x \in X$ and $\alpha \in Y$.

Here Y *is X-total* means whenever $x \in X$ and $\alpha(x) = 0$ for all $\alpha \in Y$, then $x = 0$.

The following property is used repeatedly; a proof can be found in [15, Lemma 4.13, p. 41].

PROPOSITION 1. *Assume that Z is a subspace of X' and Z is X-total. If $\{x_1, \ldots, x_n\}$ is a linearly independent set in X, then $\exists\{\alpha_1, \ldots, \alpha_n\} \subseteq Z$ such that*

$$\alpha_k(x_j) = \delta_{k,j} \quad (\textit{Kronecker delta}), \quad 1 \leq k, j \leq n.$$

Let $\mathcal{F}_{\mathcal{B}}$ be the ideal of all $F \in \mathcal{B}$ such that F has finite-dimensional range. Note that by (#), $\alpha \otimes x \in \mathcal{F}_{\mathcal{B}}$ for all $x \in X$ and $\alpha \in Y$. Define

$$\widetilde{Y} = \{\alpha \in X' : \exists x \in X,\ x \neq 0,\ \text{with } \alpha \otimes x \in \mathcal{F}_{\mathcal{B}}\}.$$

By the definition of (#), $Y \subseteq \widetilde{Y}$. Now we prove that $\widetilde{Y}$ is a subspace of X' and that $\widetilde{Y}$ has a number of special useful properties.

PROPOSITION 2. *Let $\widetilde{Y}$ be as defined above.*

(1) *$\widetilde{Y}$ is an X-total subspace of X'.*
(2) *If $\alpha \in \widetilde{Y}$ and $x \in X$, then $\alpha \otimes x \in \mathcal{F}_{\mathcal{B}}$.*
(3) *If $\alpha \in \widetilde{Y}$ and $T \in \mathcal{B}$, then $\alpha \circ T \in \widetilde{Y}$.*
(4) *$\mathcal{F}_{\mathcal{B}} = \operatorname{span}\{\alpha \otimes x : \alpha \in \widetilde{Y},\ x \in X\}$.*

Proof. First we verify that $\widetilde{Y}$ is a subspace of X'. It suffices to show that $\widetilde{Y}$ is closed under addition. Assume $\alpha_1, \alpha_2 \in \widetilde{Y}$, and choose $x_1 \neq 0$ and $x_2 \neq 0$ in X such that $\alpha_k \otimes x_k \in \mathcal{F}_{\mathcal{B}}$ for $k = 1, 2$. Now $\exists\{\beta_1, \beta_2\} \subseteq Y$ such that $\beta_k(x_j) = \delta_{k,j}$. Then $(\alpha_1 + \alpha_2) \otimes x_1 = \alpha_1 \otimes x_1 + \alpha_2 \otimes x_1 = ((\beta_1 + \beta_2) \otimes x_1)(\alpha_1 \otimes x_1 + \alpha_2 \otimes x_2) \in \mathcal{F}_{\mathcal{B}}$. Therefore by definition $\alpha_1 + \alpha_2 \in \widetilde{Y}$. This proves (1).

To prove (2), assume $\alpha \in \widetilde{Y}$ and $x \in X$. By definition of $\widetilde{Y}$, $\exists z \neq 0$ such that $\alpha \otimes z \in \mathcal{F}_{\mathcal{B}}$. Choose $\beta \in Y$ such that $\beta(z) = 1$, and note that $\alpha \otimes x = (\beta \otimes x)(\alpha \otimes z) \in \mathcal{F}_{\mathcal{B}}$.

To prove (3), assume $\alpha \in \widetilde{Y}$ and $T \in \mathcal{B}$. Now $\alpha \otimes x \in \mathcal{F}_{\mathcal{B}}$ for some $x \neq 0$. Then $(\alpha \circ T) \otimes x = (\alpha \otimes x)T \in \mathcal{F}_{\mathcal{B}}$. Therefore $\alpha \circ T \in \widetilde{Y}$.

Finally, we prove (4). Assume $E \in \mathcal{F}_{\mathcal{B}}$, $E \neq 0$. Then E has the form $E = \sum_{k=1}^{n} \alpha_k \otimes x_k$ where $\{x_1, \ldots, x_n\}$ is a basis for $\mathcal{R}(E)$ and $\{\alpha_1, \ldots, \alpha_n\} \subseteq X'$. Fix k and choose $\beta \in Y$ such that $\beta(x_j) = 0$ for $j \neq k$ and $\beta(x_k) = 1$. Then $\alpha_k \otimes x_k = (\beta \otimes x_k)E \in \mathcal{F}_{\mathcal{B}}$, so by definition $\alpha_k \in \widetilde{Y}$. It follows by this argument and by (2) that $\mathcal{F}_{\mathcal{B}} = \operatorname{span}\{\alpha \otimes x : \alpha \in \widetilde{Y},\ x \in X\}$.

By Proposition 2 we may replace Y by $\widetilde{Y}$ in hypothesis (#). Accordingly, we assume for the remainder of the paper that

(##) Y has the properties stated for the space $\widetilde{Y}$ in Proposition 2.

COROLLARY 3. (1) *Let R be an operator in X and assume $F \in \mathcal{F}_{\mathcal{B}}$ with $\mathcal{R}(F) \subseteq \mathcal{D}(R)$. Then $RF \in \mathcal{F}_{\mathcal{B}}$. In particular, $\mathcal{F}_{\mathcal{B}}$ is a left ideal of $\mathcal{B}(X)$.*

(2) *Assume R is an operator in X with $R^{-1} = T \in \mathcal{B}(X)$. Also, assume $\exists V \in \mathcal{B}$ and $F \in \mathcal{F}_{\mathcal{B}}$ with $RV = I - F$. Then $R^{-1} \in \mathcal{B}$.*

Proof. By Proposition 2,

$$\mathcal{F}_{\mathcal{B}} = \operatorname{span}\{\alpha \otimes x : \alpha \in Y\ (= \widetilde{Y}),\ x \in X\}.$$

Thus F has the form $F = \sum_{k=1}^{n} \alpha_k \otimes x_k$ where $\alpha_k \in Y$. We may assume $\{\alpha_1, \ldots, \alpha_n\}$ is a linearly independent set. Choose $\{z_1, \ldots, z_n\} \subseteq X$ such that $\alpha_k(z_j) = \delta_{k,j}$, $1 \le k, j \le n$. Then for all j, $x_j = F(z_j) \in \mathcal{D}(R)$. Therefore $RF = \sum_{k=1}^n \alpha_k \otimes R(x_k) \in \mathcal{F}_\mathcal{B}$.

To prove (2), note that $V = TRV = T - TF$, and since by (1), $TF \in \mathcal{B}$, we have $T \in \mathcal{B}$.

We will need the following technical fact.

PROPOSITION 4. *Assume D is a subspace of X and $E \in \mathcal{F}_\mathcal{B}$ with $E(D) \subseteq D$. Then $\exists F, G \in \mathcal{F}_\mathcal{B}$ with*

$$E = F + G, \quad G(D) = \{0\}, \quad \textit{and} \quad \mathcal{R}(F) \subseteq D.$$

Proof. If $E(D) = \{0\}$, then we are done. Otherwise, choose a basis $\{w_1, \ldots, w_n\}$ for $E(D)$. Choose $\{\beta_k\} \subseteq Y$ with $\beta_k(w_j) = \delta_{k,j}$ [Proposition 1]. Set $P = \sum_{k=1}^n \beta_k \otimes w_k \in \mathcal{F}_\mathcal{B}$, and note that P is a projection with range $E(D)$. Now $PE = E$ on D and $(I - P)E(D) = \{0\}$. Set $F = PE$ and $G = (I - P)E$. This is the desired decomposition of E.

Note that in this argument the collection $\{\beta_k\}$ can be chosen from any X-total subspace of X' in place of Y. We will have occasion to use this observation.

The proposition is used as follows. Suppose $V \in \mathcal{B}$ with $V(\lambda - S) = I - E$ on $\mathcal{D}(S)$, where $E \in \mathcal{F}_\mathcal{B}$ and $V(\mathcal{R}(\lambda - S)) \subseteq \mathcal{D}(S)$. Then setting $D = \mathcal{D}(S)$, we have by the proposition that $E = F + G$ where $\mathcal{R}(F) \subseteq D$ and $G(D) = \{0\}$. Thus, $V(\lambda - S) = I - F$ on $\mathcal{D}(S)$ and $\mathcal{R}(F) \subseteq \mathcal{D}(S)$.

We use the following notation concerning the operator S:

$$\varrho_\mathcal{B}(S) = \{\lambda : (\lambda - S)^{-1} \in \mathcal{B}\}, \quad \sigma_\mathcal{B}(S) = \mathbb{C} \setminus \varrho_\mathcal{B}(S).$$

If $\varrho_\mathcal{B}(S)$ is nonempty, then we say that S is *affiliated* with $\mathcal{B}$. This concept was defined and studied in [6].

Let $\mathcal{I}_\mathcal{B}$ be the largest closed ideal of $\mathcal{B}$ with the property that for any $T \in \mathcal{I}_\mathcal{B}$, $\sigma_\mathcal{B}(T)$ has no nonzero accumulation point. Certainly $\mathcal{F}_\mathcal{B} \subseteq \mathcal{I}_\mathcal{B}$. The set $\mathcal{I}_\mathcal{B}$ is the largest inessential ideal of $\mathcal{B}$; see [7, F.3.1, F.3.12]. In our situation, using [6, Proposition 4] and [7, Definition F.3.1, p. 35], we have $\mathcal{I}_\mathcal{B} = \mathrm{kh}(\mathcal{F}_\mathcal{B})$ (the intersection of all primitive ideals of $\mathcal{B}$ which contain $\mathcal{F}_\mathcal{B}$). In Fredholm theory relative to $\mathcal{B}$, the ideal $\mathcal{I}_\mathcal{B}$ can play the same role that the ideal of compact operators plays in Fredholm theory in $\mathcal{B}(X)$. When $\mathcal{B} = \mathcal{B}(X)$, then $\mathcal{I}_\mathcal{B}$ is the ideal of all inessential operators on X as defined by D. Kleinecke in [14]. Some information concerning inessential operators can be found in [14] and [16]. Following [6, Definition 5], we make the following definition.

DEFINITION 5. The operator $\lambda - S$ is *$\mathcal{B}$-Fredholm* if $\exists V, W \in \mathcal{B}$ and $\exists J, K \in \mathcal{I}_\mathcal{B}$ such that

$$(\lambda - S)W = I - J, \quad \text{and} \quad V(\lambda - S) = I - K \quad \text{on } \mathcal{D}(S).$$

The equation $(\lambda - S)W = I - J$ includes implicitly the information that $\mathcal{R}(W) \subseteq \mathcal{D}(S)$. The $\mathcal{B}$-Fredholm theory of operators affiliated with $\mathcal{B}$ is studied in [6]. When S is $\mathcal{B}$-Fredholm, we write $S \in \Phi_\mathcal{B}$. Also, when $S \in \Phi_\mathcal{B}$ and $\mathrm{ind}(S) = 0$ ($\mathrm{ind}(S)$ denoting the index of S), we write $S \in \Phi_\mathcal{B}^0$.

PROPOSITION 6. *The operator $(\lambda - S) \in \Phi_\mathcal{B} \Leftrightarrow$ there exist $V, W \in \mathcal{B}$ and $F, G \in \mathcal{F}_\mathcal{B}$ with*

$$(\lambda - S)W = I - G, \quad \textit{and} \quad V(\lambda - S) = I - F \quad \textit{on } \mathcal{D}(S).$$

Proof. Using the fact that $\mathcal{I}_\mathcal{B} = \mathrm{kh}(\mathcal{F}_\mathcal{B})$, it follows from [7, BA 2.4, p. 103] that if $L \in \mathcal{I}_\mathcal{B}$, then $\exists U_1, U_2 \in \mathcal{B}$ such that

$$(I - L)U_1 - I \in \mathcal{F}_\mathcal{B} \quad \text{and} \quad U_2(I - L) - I \in \mathcal{F}_\mathcal{B}.$$

Now suppose $\lambda - S \in \Phi_\mathcal{B}$, so there exist $V, W \in \mathcal{B}$ and $J, K \in \mathcal{I}_\mathcal{B}$ with

$$(\lambda - S)W = I - J, \quad \text{and} \quad V(\lambda - S) = I - K \quad \text{on } \mathcal{D}(S).$$

Since $J, K \in \mathcal{I}_\mathcal{B}$, as noted above, $\exists U_1, U_2 \in \mathcal{B}$ and $G, F \in \mathcal{F}_\mathcal{B}$ such that $(I - J)U_1 = I - G$ and $U_2(I - K) = I - F$. Therefore

$$(\lambda - S)WU_1 = I - G, \quad \text{and} \quad U_2V(\lambda - S) = I - F \quad \text{on } \mathcal{D}(S).$$

EXAMPLE 7. Now we describe briefly a particular Banach algebra of operators satisfying (#) which we use in §4. Let Ω be a locally compact Hausdorff space. Assume μ is a positive σ-finite regular Borel measure on Ω with the property that $\mu(U) > 0$ whenever U is a nonempty open subset of Ω. Let $C(\Omega)$ be the space of all *bounded* $\mathbb{C}$-valued continuous functions on Ω equipped with the sup-norm. If φ is any continuous function on Ω, let M_φ be the multiplication operator on $C(\Omega)$ determined by φ. When φ is bounded, then $M_\varphi \in \mathcal{B}(C(\Omega))$, and in general

$$\mathcal{D}(M_\varphi) = \{g \in C(\Omega) : \varphi g \in C(\Omega)\}.$$

Now let $\mathcal{C}$ be the set of all $\mathbb{C}$-valued functions $K(x, t)$ on $\Omega \times \Omega$ such that

$$x \to K(x, t) \text{ is a continuous bounded function of } \Omega \text{ into } L^1(\mu).$$

The functions in $\mathcal{C}$ determine an important class of integral operators in $\mathcal{B}(C(\Omega))$ according to the formula: For $K \in \mathcal{C}$,

$$K(f)(x) = \int_\Omega K(x, t) f(t)\, d\mu(t) \quad (f \in C(\Omega)).$$

Here K is used both to denote the kernel and the integral operator. The operator norm of K is

$$\|K\| = \sup_{x \in \Omega} \int_\Omega |K(x, t)|\, d\mu(t)$$

[12, Theorem 12.2, p. 303]. The set of all such operators forms a closed subalgebra of $\mathcal{B}(C(\Omega))$. This class of operators is studied in K. Jörgens' book [12, §12.2].

Denote by $\mathcal{J}$ the set of all integral operators which are determined by functions in $\mathcal{C}$. Let $\mathcal{K}$ be the closed subalgebra of $\mathcal{B}(C(\Omega))$ given by

$$\mathcal{K} = \{\lambda + K : \lambda \in \mathbb{C},\ K \in \mathcal{J}\}.$$

If $f \in C(\Omega)$ and $g \in L^1(\mu)$, then the kernel $f(x)g(t)$ is in $\mathcal{C}$. Thus, $\mathcal{K}$ satisfies (#) with $Y = L^1(\mu)$.

Next we enlarge $\mathcal{K}$ slightly. Let $\mathcal{B}$ be the set of all operators of the form

$$\mathcal{B} = \{M_\varphi + K : \varphi \in C(\Omega),\ K \in \mathcal{J}\}.$$

PROPOSITION 8. *The set $\mathcal{B}$ is a closed subalgebra of $\mathcal{B}(C(\Omega))$ which satisfies (#) with $Y = L^1(\mu)$.*

Proof. Since for $\varphi \in C(\Omega)$ and $K \in \mathcal{C}$, φK and $K\varphi$ are in $\mathcal{C}$, it is straightforward to verify that $\mathcal{B}$ is an algebra. Now $\mathcal{J}$ is an ideal of $\mathcal{B}$, and $\mathcal{J}$ is closed in $\mathcal{B}(C(\Omega))$ by [12, Exercise 12.4(b)]. Let $\overline{\mathcal{B}}$ be the closure of $\mathcal{B}$ in $\mathcal{B}(C(\Omega))$, so again, $\mathcal{J}$ is a closed ideal in $\overline{\mathcal{B}}$. Consider the map $\psi : C(\Omega) \to \overline{\mathcal{B}}/\mathcal{J}$ given by $\psi(\varphi) = M_\varphi + \mathcal{J}$ in $\overline{\mathcal{B}}/\mathcal{J}$. Since $C(\Omega)$ is a C^*-algebra, the image of the continuous homomorphism ψ is closed in $\overline{\mathcal{B}}/\mathcal{J}$ [9, Theorem 3.6, p. 72]. Therefore $\psi(C(\Omega)) = \overline{\mathcal{B}}/\mathcal{J}$. Thus, if $T \in \overline{\mathcal{B}}$, then $\exists \varphi \in C(\Omega)$ with $T - M_\varphi \in \mathcal{J}$. This proves that $\mathcal{B} = \overline{\mathcal{B}}$.

As a last bit of information concerning $\mathcal{K}$ and $\mathcal{B}$, we note a useful property of $\mathcal{I}_\mathcal{K}$ and $\mathcal{I}_\mathcal{B}$. We use the notation $\mathcal{W}(X)$ to denote the ideal of all weakly compact operators on X; see [10, IV.4].

PROPOSITION 9. $\mathcal{W}(C(\Omega)) \cap \mathcal{K} \subseteq \mathcal{I}_\mathcal{K}$; $\mathcal{W}(C(\Omega)) \cap \mathcal{B} \subseteq \mathcal{I}_\mathcal{B}$.

Proof. Define $\mathcal{M} = \mathcal{W}(C(\Omega)) \cap \mathcal{B}$. Then $\mathcal{M}$ is a closed ideal of $\mathcal{B}$ [10, pp. 483–484], and if $T \in \mathcal{M}$, then $\sigma(T)$ has no nonzero accumulation point [10, Corollary 5, p. 494]. Now since $\mathcal{B}$ is a closed subalgebra of $\mathcal{B}(C(\Omega))$, $\operatorname{bndry}(\sigma_\mathcal{B}(T)) \subseteq \sigma(T) \subseteq \sigma_\mathcal{B}(T)$ [8, Proposition 12, p. 25]. Therefore $\sigma_\mathcal{B}(T) = \sigma(T)$. Thus, by definition $\mathcal{M}$ is an inessential ideal of $\mathcal{B}$, and so $\mathcal{M} \subseteq \mathcal{I}_\mathcal{B}$ [7, Theorem R.2.6, p. 58].

3. S-inessential operators. Fix a Banach algebra of operator $\mathcal{B}$ which satisfies (#) and (##). Throughout this section S is an operator in X with domain $\mathcal{D}(S)$. It is assumed that S is affiliated with $\mathcal{B}$ ($\varrho_\mathcal{B}(S)$ is nonempty), and that $\mathcal{D}(S)$ is Y-total. The main purpose here is to study perturbations of S by some operator R in X having one of the following properties.

DEFINITION 10. Let R be an operator in X with $\mathcal{D}(S) \subseteq \mathcal{D}(R)$.

(1) R is *S-bounded relative to $\mathcal{B}$* if for some $\lambda \in \varrho_\mathcal{B}(S)$, $R(\lambda - S)^{-1} \in \mathcal{B}$.

(2) R is *S-inessential relative to $\mathcal{B}$* if for some $\lambda \in \varrho_\mathcal{B}(S)$, $R(\lambda - S)^{-1} \in \mathcal{I}_\mathcal{B}$.

We prove a useful note.

NOTE 11. *Assume R is S-inessential [or S-bounded] relative to $\mathcal{B}$. If $V \in \mathcal{B}$ with $(\lambda - S)V \in \mathcal{B}$ for some λ, then $RV \in \mathcal{I}_\mathcal{B}$ [$RV \in \mathcal{B}$]. In particular, for any $\lambda \in \varrho_\mathcal{B}(S)$, $R(\lambda - S)^{-1} \in \mathcal{I}_\mathcal{B}$ [$R(\lambda - S)^{-1} \in \mathcal{B}$].*

Proof. Assume $\mu \in \varrho_\mathcal{B}(S)$ and $R(\mu - S)^{-1} \in \mathcal{I}_\mathcal{B}$. Assume $(\lambda - S)V \in \mathcal{B}$. Then $RV = R(\mu - S)^{-1}(\mu - S)V = R(\mu - S)^{-1}((\mu - \lambda) + (\lambda - S))V \in \mathcal{I}_\mathcal{B}$.

Mainly we are interested in $\mathcal{B}$-Fredholm properties of the perturbed operator $S + R$ where R is S-inessential relative to $\mathcal{B}$. There is a rather technical result which we need in the situation where $\lambda - S$ is $\mathcal{B}$-Fredholm. We establish this result now. There is an interesting biproduct of the argument which we state as Theorem 15.

PROPOSITION 12. *Assume $\lambda - S$ is Fredholm with $\operatorname{ind}(\lambda - S) \le 0$, and $\exists V \in \mathcal{B}$ and $G \in \mathcal{F}_\mathcal{B}$ with $(\lambda - S)V = I - G$. Then $\exists F \in \mathcal{F}_\mathcal{B}$ and $W \in \mathcal{B}$ such that:*

(i) $\mathcal{N}(\lambda - S + F) = \{0\}$;
(ii) $W(\lambda - S + F) = I$ *on* $\mathcal{D}(S)$;
(iii) $\mathcal{R}(W) \subseteq \mathcal{D}(S)$;
(iv) $(\lambda - S)W = I - E$ *where* $E \in \mathcal{F}_\mathcal{B}$.

Proof. Define $N = \mathcal{N}(\lambda - S)$ and choose M finite-dimensional with $X = M \oplus \mathcal{R}(\lambda - S)$. By hypothesis, $\dim(M) \ge \dim(N)$. Choose a basis $\{z_1, \ldots, z_n\}$ for N and a linearly independent subset $\{x_1, \ldots, x_n\} \subseteq M$. Choose $\{\alpha_1, \ldots, \alpha_n\} \subseteq Y$ with $\alpha_k(z_j) = \delta_{k,j}$. Set $F = \sum_{k=1}^n \alpha_k \otimes x_k \in \mathcal{F}_\mathcal{B}$. Suppose $x \in \mathcal{N}(\lambda - S + F)$, so $(\lambda - S)x = -Fx$. Since $Fx \in M$ we have $(\lambda - S)x = -Fx = 0$. Therefore $x \in N$ and $0 = \sum_{k=1}^n \alpha_k(x)x_k$. This implies $\alpha_k(x) = 0$ for $1 \le k \le n$. Now $x = \sum_{k=1}^n \lambda_k z_k$, so $\lambda_k = \alpha_k(x) = 0$ for $1 \le k \le n$, and thus, $x = 0$. This proves (i).

The operator $\lambda - S + F$ is Fredholm so we can choose a finite-dimensional subspace P of X with $X = P \oplus \mathcal{R}(\lambda - S + F)$. Define $W(x) = 0$ for $x \in P$ and $W((\lambda - S + F)u) = u$ for $u \in \mathcal{D}(S)$. Since $\lambda - S + F$ is a closed operator, the Open Mapping Theorem implies that W is a bounded operator on X. By definition $\mathcal{R}(W) \subseteq \mathcal{D}(S)$ and $W(\lambda - S + F) = I$ on $\mathcal{D}(S)$.

Let V and G be as in the statement of the proposition. Then

$$V = W(\lambda - S + F)V = W(I - G + FV).$$

Since $W(-G + FV) \in \mathcal{F}_\mathcal{B}$ by Corollary 3, this implies $W \in \mathcal{B}$. Finally,

$$(\lambda - S)W = (\lambda - S)(V + WG - WFV) = I - G + (\lambda - S)W(G - FV).$$

Since $\mathcal{R}(W) \subseteq \mathcal{D}(S)$, by Corollary 3, $(\lambda - S)W(G - FV) \in \mathcal{F}_\mathcal{B}$.

We need a result similar to Proposition 12 for the case where $\operatorname{ind}(\lambda - S) > 0$. To achieve this we construct adjoints of all the operators involved as follows. Define a bilinear form on $X \times Y$ by setting

$$\langle x, \alpha\rangle = \alpha(x) \quad (x \in X,\ \alpha \in Y).$$

For each $T \in \mathcal{B}$, let

$$T^\dagger(\alpha) = \alpha \circ T \quad \text{for all } \alpha \in Y.$$

Note that by Proposition 2, $T^\dagger$ maps Y into Y. It is straightforward to verify that for each $x_0 \in X$, $x_0 \neq 0$, $Y \otimes x_0 = \{\alpha \otimes x_0 : \alpha \in Y\}$ is a minimal right ideal of $\mathcal{B}$, and therefore closed. Fix $x_0 \in X$, $x_0 \neq 0$, and define a norm on Y by $\|\alpha\|_Y = \|\alpha \otimes x_0\|_{\mathcal{B}}$. We have the following facts:

(1) $(Y, \|\cdot\|_Y)$ is a Banach space;
(2) $T^\dagger \in \mathcal{B}(Y)$ for all $T \in \mathcal{B}$;
(3) $\langle x, \alpha\rangle$ is a bounded bilinear form on $X \times Y$.

To see (2), for $T \in \mathcal{B}$, $\alpha \in Y$, $\|T^\dagger(\alpha)\|_Y = \|(\alpha \otimes x_0)T\|_{\mathcal{B}} \le \|\alpha \otimes x_0\|_{\mathcal{B}}\|T\|_{\mathcal{B}} = \|\alpha\|_Y\|T\|_{\mathcal{B}}$. Also, (3) follows from the computation for $x \in X$, $\alpha \in Y$:

$$\begin{aligned}|\langle x, \alpha\rangle|\|x_0\| &= |\alpha(x)|\|x_0\| = \|(\alpha \otimes x_0)(x)\|\\ &\le \|\alpha \otimes x_0\|\|x\| \le M\|\alpha \otimes x_0\|_{\mathcal{B}}\|x\| \le M\|\alpha\|_Y\|x\|.\end{aligned}$$

It follows that $\mathcal{B}$ is a subalgebra of the Jörgens algebra $\mathcal{A}\langle X, Y\rangle$; see [12, pp. 43–45].

PROPOSITION 13. *$\mathcal{B}^\dagger = \{T^\dagger : T \in \mathcal{B}\}$ is a Banach algebra in the norm $\|T^\dagger\|' = \|T\|_{\mathcal{B}}$, and $\mathcal{B}^\dagger \subseteq \mathcal{B}(Y)$. We let $\mathcal{F}_{\mathcal{B}^\dagger} = (\mathcal{F}_{\mathcal{B}})^\dagger$. Then the preceding results apply to $\mathcal{B}^\dagger$ (here the roles of X and Y are reversed).*

When S is an operator in X affiliated with $\mathcal{B}$, then $S^\dagger$, as defined in [6], is an operator in Y affiliated with $\mathcal{B}^\dagger$.

Now we are in a position to prove a useful result concerning $\mathcal{B}$-Fredholm operators. When $\mathcal{B} = \mathcal{B}(X)$, this theorem is a standard result which is established by a direct construction [17, Theorem 1.1, p. 162]. It is perhaps surprising that an elaborate argument seems necessary to establish this result in the general situation.

THEOREM 14. *Assume $\lambda - S \in \Phi_{\mathcal{B}}$. There exists $W \in \mathcal{B}$ with $\mathcal{R}(W) \subseteq \mathcal{D}(S)$, and $\exists F, G \in \mathcal{F}_{\mathcal{B}}$ with $\mathcal{R}(G) \subseteq \mathcal{D}(S)$ such that*

$$(\lambda - S)W = I - F, \quad \text{and} \quad W(\lambda - S) = I - G \quad \text{on } \mathcal{D}(S).$$

Proof. First assume $\operatorname{ind}(\lambda - S) \le 0$. Then the result follows from Proposition 12. Now suppose $\operatorname{ind}(\lambda - S) > 0$. Form $\mathcal{B}^\dagger$ and $S^\dagger$ as in the discussion above. By [6, Theorem 19], $\sigma_{\mathcal{B}}(S) = \sigma_{\mathcal{B}^\dagger}(S^\dagger)$, so $S^\dagger$ is affiliated with $\mathcal{B}^\dagger$. Now $\lambda - S \in \Phi_{\mathcal{B}}$. Assume for convenience that $T = S^{-1} \in \mathcal{B}$. We have as in [6, Theorem 6] that T is $\mathcal{B}$-Fredholm. This easily implies that $T^\dagger$ is $\mathcal{B}^\dagger$-Fredholm. Then again applying [6, Theorem 6], $\lambda - S^\dagger$ is $\mathcal{B}^\dagger$-Fredholm. In addition, by [6, Theorem 14], $\operatorname{ind}(\lambda - S^\dagger) = -\operatorname{ind}(\lambda - S) < 0$.

Apply Proposition 12 to $\lambda - S^\dagger$, with the result: $\exists W \in \mathcal{B}$ and $F, G \in \mathcal{F}_{\mathcal{B}}$ such that

$$\begin{aligned}(\lambda - S^\dagger)W^\dagger &= I - G^\dagger,\\ W^\dagger(\lambda - S^\dagger) &= I - F^\dagger \quad \text{on } \mathcal{D}(S^\dagger).\end{aligned}$$

Implicit in the top equation is the fact that $\mathcal{R}(W^\dagger) \subseteq \mathcal{D}(S^\dagger)$. For $x \in \mathcal{D}(S)$ and $\alpha \in Y$,

$$\begin{aligned}\langle W(\lambda - S)x, \alpha\rangle &= \langle(\lambda - S)x, W^\dagger\alpha\rangle = \langle x, (\lambda - S^\dagger)W^\dagger\alpha\rangle\\ &= \langle x, (I - G^\dagger)\alpha\rangle = \langle(I - G)x, \alpha\rangle.\end{aligned}$$

Therefore $W(\lambda - S) = I - G$ on $\mathcal{D}(S)$. The fact that we may assume $\mathcal{R}(G) \subseteq \mathcal{D}(S)$ follows from Proposition 4 once we establish $\mathcal{R}(W) \subseteq \mathcal{D}(S)$. Assume $S^{-1} = T \in \mathcal{B}$. Then $W^\dagger S^\dagger = W^\dagger((\lambda - S^\dagger) - \lambda) = I - F^\dagger - \lambda W^\dagger$ on $\mathcal{D}(S^\dagger)$. Therefore $W^\dagger = T^\dagger - F^\dagger T^\dagger - \lambda W^\dagger T^\dagger$, so $W = T - TF - \lambda TW$. It follows that $\mathcal{R}(W) \subseteq \mathcal{R}(T) \subseteq \mathcal{D}(S)$. For any $x \in X$, $\alpha \in \mathcal{D}(S^\dagger)$,

$$\begin{aligned}\langle(\lambda - S)Wx, \alpha\rangle &= \langle Wx, (\lambda - S^\dagger)\alpha\rangle = \langle x, W^\dagger(\lambda - S^\dagger)\alpha\rangle\\ &= \langle x, (I - F^\dagger)\alpha\rangle = \langle(I - F)x, \alpha\rangle.\end{aligned}$$

Thus, $(\lambda - S)W = I - F$ because $\mathcal{D}(S^\dagger)$ is X-total [6, Proposition 12].

THEOREM 15. *Assume $\lambda - S \in \Phi_{\mathcal{B}}$.*

(1) *If $\operatorname{ind}(\lambda - S) \le 0$, then $\exists W \in \mathcal{B}$ and $\exists F \in \mathcal{F}_{\mathcal{B}}$ such that*

$$W(\lambda - S + F) = I \quad \text{on } \mathcal{D}(S).$$

(2) *If $\operatorname{ind}(\lambda - S) \ge 0$, then $\exists W \in \mathcal{B}$ and $\exists F \in \mathcal{F}_{\mathcal{B}}$ such that*

$$(\lambda - S + F)W = I.$$

(3) *If $\operatorname{ind}(\lambda - S) = 0$, then $\exists F \in \mathcal{F}_{\mathcal{B}}$ such that $\lambda - S + F$ has an inverse in $\mathcal{B}$.*

Proof. Part (1) is part of Proposition 12, and part (2) follows by applying this same proposition to the adjoint $\lambda - S^\dagger$.

To prove (3), we have again by Proposition 12 that $\exists F \in \mathcal{F}_{\mathcal{B}}$ such that $\mathcal{N}(\lambda - S + F) = \{0\}$. Now $\operatorname{ind}(\lambda - S + F) = 0$, so it follows that $\lambda - S + F$ has an inverse in $\mathcal{B}(X)$. In addition, $\lambda - S + F \in \Phi_{\mathcal{B}}$, so $\lambda - S + F$ has an inverse in $\mathcal{B}$ by Corollary 3(2).

The following technical lemma is used in the proof of the next theorem.

LEMMA 16. *Let R be an operator with $\mathcal{D}(S) \subseteq \mathcal{D}(R)$. Assume $\mu_0 \in \varrho_{\mathcal{B}}(S)$, and $W \in \mathcal{B}$ with $RW \in \mathcal{I}_{\mathcal{B}}$. There exists $G \in \mathcal{F}_{\mathcal{B}}$ of the form $G = H(\mu_0 - S)^{-1}$ where $H \in \mathcal{F}_{\mathcal{B}}$ such that $\mathcal{R}(G) \subseteq \mathcal{D}(S)$ and $I - RW - RG$ has an inverse in $\mathcal{B}$.*

Proof. Since $RW \in \mathcal{I}_{\mathcal{B}}$, $\mathcal{R}(I-RW)$ is closed and has finite codimension. Also, $\mathcal{R}(I-RW)+\mathcal{R}(RW)=X$, so we can choose $\{z_1,\ldots,z_n\} \subseteq \mathcal{D}(S) \subseteq \mathcal{D}(R)$ such that $\{R(z_1),\ldots,R(z_n)\}$ is linearly independent and

$$(1)\qquad \mathcal{R}(I-RW) \oplus \operatorname{span}\{R(z_1),\ldots,R(z_n)\} = X.$$

Now note that the set $\{\beta\circ(\mu_0-S)^{-1} : \beta \in Y\}$ is X-total. Assume $\mathcal{N}(I-RW)$ has a basis $\{x_1,\ldots,x_n\}$. By Proposition 1 we can choose $\{\beta_1,\ldots,\beta_n\} \subseteq Y$ with the property that $\alpha_k = \beta_k \circ (\mu_0-S)^{-1}$ takes the values

$$(2)\qquad \alpha_k(x_j) = \delta_{k,j} \quad (1 \le k, j \le n).$$

Let $H = \sum_{k=1}^n \beta_k \otimes z_k \in \mathcal{F}_{\mathcal{B}}$. Set $G = H(\mu_0-S)^{-1}$, so

$$RG = \sum_{k=1}^{n} \alpha_k \otimes R(z_k).$$

Note that $\mathcal{R}(G) \subseteq \mathcal{D}(S)$.

It remains to verify that $I-RW-RG$ has an inverse in $\mathcal{B}$. Since this operator is in $\Phi^0_{\mathcal{B}}$, by Corollary 3(2) it suffices to show that it has an inverse in $\mathcal{B}(X)$. Furthermore, it is enough to show that $\mathcal{N}(I-RW-RG) = \{0\}$. Suppose $(I-RW-RG)x=0$. Then

$$(I-RW)x = RGx \in \operatorname{span}\{R(z_1),\ldots,R(z_n)\}.$$

Therefore by (1), $(I-RW)x = 0 = RGx$. Since $RGx=0$, $\alpha_k(x)=0$ for $1 \le k \le n$. Also, $x \in \mathcal{N}(I-RW) = \operatorname{span}\{x_1,\ldots,x_n\}$, and so by (2) we have $x=0$.

Now we prove the main result of the paper. The result depends heavily on the properties of a $\mathcal{B}$-Fredholm operator as stated in Theorem 14.

THEOREM 17. *Assume $\lambda - S \in \Phi_{\mathcal{B}}$ and R is S-inessential relative to $\mathcal{B}$. Then $\lambda-(S+R) \in \Phi_{\mathcal{B}}$ and $\operatorname{ind}(\lambda-S) = \operatorname{ind}(\lambda-(S+R))$. Furthermore, $S+R$ is closed.*

Proof. First by Theorem 14 there exist $E, F \in \mathcal{F}_{\mathcal{B}}$ and $W \in \mathcal{B}$ with $\mathcal{R}(F) \subseteq \mathcal{D}(S)$, $\mathcal{R}(W) \subseteq \mathcal{D}(S)$ and

$$(\lambda-S)W = I-E, \quad W(\lambda-S) = I-F \quad \text{on } \mathcal{D}(S).$$

As verified in Note 11, $RW \in \mathcal{I}_{\mathcal{B}}$. We note that

$$(1)\qquad (\lambda-(S+R))W = I-E-RW.$$

Choose G and H as in Lemma 16 with $\mathcal{R}(G) \subseteq \mathcal{D}(S)$ and with $I-R(W+G)$ having inverse $V \in \mathcal{B}$. On $\mathcal{D}(S)$,

$$(2)\qquad (W+G)(\lambda-S-R) = I-F-(W+G)R+G(\lambda-S).$$

Also, on $\mathcal{D}(S)$,

$$\begin{aligned}(I+(W+G)VR)&(I-(W+G)R)\\ &= I-(W+G)R+(W+G)VR(I-(W+G)R)\\ &= I-(W+G)R+(W+G)V(I-R(W+G))R\\ &= I-(W+G)R+(W+G)R = I.\end{aligned}$$

Combining this equality with (2) we have on $\mathcal{D}(S)$,

$$\begin{aligned}(3)\qquad (I+(W+G)VR)&(W+G)(\lambda-(S+R))\\ &= (I+(W+G)VR)(I-F-(W+G)R+G(\lambda-S))\\ &= I+(I+(W+G))VR(-F+G(\lambda-S)).\end{aligned}$$

Now $\mathcal{R}(G) \subseteq \mathcal{D}(R)$ and $RW \in \mathcal{B}$ so $(I+(W+G)VR)(W+G) \in \mathcal{B}$ by Corollary 3(1). Also $\mathcal{R}(F) \subseteq \mathcal{D}(S)$, so again by Corollary 3, we have $(I-(W+G)VR)F \in \mathcal{F}_{\mathcal{B}}$. By Lemma 16, $G = H(\mu_0-S)^{-1}$ where $H \in \mathcal{F}_{\mathcal{B}}$, so on $\mathcal{D}(S)$,

$$\begin{aligned}G(\lambda-S) = H(\mu_0-S)^{-1}(\lambda-S) &= H(\mu_0-S)^{-1}[(\mu_0-S)+(\lambda-\mu_0)]\\ &= H+(\lambda-\mu_0)H(\mu_0-S)^{-1} \in \mathcal{F}_{\mathcal{B}}.\end{aligned}$$

Thus, by (3), $\exists T \in \mathcal{B}$ and $J \in \mathcal{F}_{\mathcal{B}}$ with $T(\lambda-(S+R)) = I-J$ on $\mathcal{D}(S)$. Together with (1), this equality shows that $\lambda-(S+R) \in \Phi_{\mathcal{B}}$.

Now we consider the index. Let $\mathcal{D}(S)$ have the usual graph norm, and then $\lambda - S : \mathcal{D}(S) \to X$ is a bounded Fredholm operator, $W : X \to \mathcal{D}(S)$ is a bounded Fredholm operator, and $(\lambda-S)W = I-E$, so $\operatorname{ind}(\lambda-S)+\operatorname{ind}(W) = 0$. Also, R is S-bounded so again, $\lambda-(S+R) : \mathcal{D}(S) \to X$ is a bounded Fredholm operator and by (1), $\operatorname{ind}(\lambda-(S+R))+\operatorname{ind}(W) = 0$. Hence, $\operatorname{ind}(\lambda-S) = \operatorname{ind}(\lambda-(S+R))$.

Now we check that $S+R$ is closed. It suffices to show that $U = \lambda-(S+R)$ is closed. By the argument above using (1) and (3) we have that there exist $K, J \in \mathcal{F}_{\mathcal{B}}$ with $\mathcal{R}(J) \subseteq \mathcal{D}(U)$ $(=\mathcal{D}(S))$ and $W, T \in \mathcal{B}$ with $\mathcal{R}(T) \subseteq \mathcal{D}(U)$ such that

$$UW = I-K, \quad \text{and} \quad TU = I-J \quad \text{on } \mathcal{D}(U).$$

Write $J = \sum_{k=1}^n \alpha_k \otimes x_k$ where $\{x_1,\ldots,x_n\}$ is a linearly independent set and $\{\alpha_1,\ldots,\alpha_n\} \subseteq Y$. Choose a maximal linearly independent subset of $\{\alpha_1,\ldots,\alpha_n\}$; we may assume it is $\{\alpha_1,\ldots,\alpha_m\}$. Then $\mathcal{N}(J) = \bigcap_{k=1}^m \ker(\alpha_k)$. Since $\mathcal{D}(U)$ is Y-total, $\exists\{y_1,\ldots,y_m\} \subseteq \mathcal{D}(U)$ such that $\alpha_k(y_j) = \delta_{k,j}$ for $1 \le k, j \le m$. Set $P = \sum_{k=1}^m \alpha_k \otimes y_k \in \mathcal{F}_{\mathcal{B}}$, and note that $\mathcal{N}(P) = \mathcal{N}(J)$, $P = P^2$, and $\mathcal{R}(P) \subseteq \mathcal{D}(U)$. Now since $(I-J)(I-P) = I-P$,

$$(4)\qquad U(I-P)W = I-D, \quad TU(I-P) = I-P \quad \text{on } \mathcal{D}(U)$$

where $D = K + UPW \in \mathcal{F}_{\mathcal{B}}$.

Next we verify

(5) $\mathcal{R}(U(I-P))$ is closed, and $\mathcal{N}(T)\cap\mathcal{R}(U(I-P))=\{0\}$.

First, for all $x\in X$ by (4), $U(I-P)Wx=(I-D)x$, and so $\mathcal{R}(I-D)\subseteq\mathcal{R}(U(I-P))$. Thus $\mathcal{R}(U(I-P))$ is a finite-dimensional extension of the closed subspace $\mathcal{R}(I-D)$, and so it is closed. Next suppose $x\in\mathcal{N}(T)\cap\mathcal{R}(U(I-P))$. Then $x=U(I-P)y$ for some $y\in\mathcal{D}(U)$, and therefore $(I-P)y=T(U(I-P))y=Tx=0$ by (4). Since $(I-P)y=0$, we have $x=0$.

Now $U=U(I-P)+UP$ and UP is bounded, so it suffices to verify that $U(I-P)$ is closed. Assume $\{x_n\}\subseteq\mathcal{D}(U)$, $x_n\to x_0$, and $U(I-P)x_n\to y_0$. Then $(I-P)x_n\to(I-P)x_0$, and by (4), $(I-P)x_n=TU(I-P)x_n\to Ty_0$. Therefore $(I-P)x_0=T(y_0)$, and as $\mathcal{R}(P)\subseteq\mathcal{D}(U)$ and $\mathcal{R}(T)\subseteq\mathcal{D}(U)$, we have $x_0\in\mathcal{D}(U)$. By (5), $\mathcal{R}(U(I-P))$ is closed so $y_0\in\mathcal{R}(U(I-P))$. Finally, $T(U(I-P)x_0-y_0)=(I-P)x_0-T(y_0)=0$, so by (5), $U(I-P)x_0=y_0$.

There is a special interesting case where the conclusions of Theorem 17 hold, but where the proof is short and elementary. Assume R has the three properties:

(i) $R\in\mathcal{B}$;
(ii) $R(\mu-S)^{-1}\in\mathcal{I}_{\mathcal{B}}$ for some $\mu\in\varrho_{\mathcal{B}}(S)$;
(iii) $(\delta-S)^{-1}R\in\mathcal{I}_{\mathcal{B}}$ for some $\delta\in\varrho_{\mathcal{B}}(S)$.

In this case, assuming $V,W\in\mathcal{B}$, $F,G\in\mathcal{F}_{\mathcal{B}}$ with $(\lambda-S)W=I-F$, and $V(\lambda-S)=I-G$ on $\mathcal{D}(S)$, we have

$$(\lambda-(S+R))W=I-F-RW,$$

and

$$V(\lambda-(S+R))=I-G-VR\quad\text{on }\mathcal{D}(S).$$

But by (i)–(iii), RW and VR are in $\mathcal{I}_{\mathcal{B}}$. Therefore $\lambda-(S+R)$ is $\mathcal{B}$-Fredholm. The rest of Theorem 17 is also easy to verify.

Now we consider S-bounded perturbations of S. For $T\in\mathcal{B}$ we use the notation $r_{\mathcal{B}}(T)$ for the spectral radius of T in the Banach algebra $\mathcal{B}$.

THEOREM 18. *Assume R is S-bounded relative to $\mathcal{B}$. Also, assume $\lambda-S\in\Phi_{\mathcal{B}}$, so by Theorem 14, $\exists W\in\mathcal{B}$, $\exists F,G\in\mathcal{F}_{\mathcal{B}}$, with $\mathcal{R}(G)\subseteq\mathcal{D}(S)$, such that*

$$(\lambda-S)W=I-F,\quad\text{and}\quad W(\lambda-S)=I-G\quad\text{on }\mathcal{D}(S).$$

If $r_{\mathcal{B}}(RW)<1$, then $\lambda-(S+R)\in\Phi_{\mathcal{B}}$ and $\operatorname{ind}(\lambda-(S+R))=\operatorname{ind}(\lambda-S)$.

Proof. Since $r_{\mathcal{B}}(RW)<1$, by a standard Banach algebra result we have $(I-RW)^{-1}\equiv Z\in\mathcal{B}$. Therefore

(1) $$(\lambda-(S+R))WZ=(I-F-RW)Z=I-FZ.$$

Also, for $x\in\mathcal{D}(R)$,

$$\begin{aligned}(I+WZR)(I-WR)x&=x-WRx+WZR(x-WRx)\\&=x-WRx+WZ(I-RW)Rx\\&=x-WRx+WRx=x.\end{aligned}$$

Note that $(I+WZR)W\in\mathcal{B}$. For $x\in\mathcal{D}(S)$,

$$\begin{aligned}(I+WZR)W(\lambda-(S+R))x&=(I+WZR)(I-G-WR)x\\&=(I-(I+WZR)G)x.\end{aligned}$$

Therefore

(2) $$(I+WZR)W(\lambda-(S+R))=I-(I+WZR)G\quad\text{on }\mathcal{D}(S).$$

Since $\mathcal{R}(G)\subseteq\mathcal{D}(R)$, it follows by Corollary 3(1) that $(I+WZR)G\in\mathcal{F}_{\mathcal{B}}$. Then (1) and (2) imply that $\lambda-(S+R)\in\Phi_{\mathcal{B}}$.

As to the index, $\lambda-S$ is a bounded Fredholm operator from $\mathcal{D}(S)$ (with the graph norm) into X and W is a Fredholm operator from X into $\mathcal{D}(S)$. By the usual index theorem for bounded Fredholm operators,

$$0=\operatorname{ind}(I-F)=\operatorname{ind}((\lambda-S)W)=\operatorname{ind}(\lambda-S)+\operatorname{ind}(W).$$

Similarly, using (1),

$$\begin{aligned}0&=\operatorname{ind}(I-FZ)=\operatorname{ind}((\lambda-(S+R))WZ)\\&=\operatorname{ind}(\lambda-(S+R))+\operatorname{ind}(W)+\operatorname{ind}(Z)=\operatorname{ind}(\lambda-(S+R))+\operatorname{ind}(W).\end{aligned}$$

This proves $\operatorname{ind}(\lambda-(S+R))=\operatorname{ind}(\lambda-S)$.

In general it is useful to know that $\sigma(S)=\sigma_{\mathcal{B}}(S)$. Next we look at a situation where the very strong conclusion that $\sigma(S+R)=\sigma_{\mathcal{B}}(S+R)$ holds for all R which are S-inessential relative to $\mathcal{B}$.

We use the following notation for the Weyl spectrum of an operator R:

$$\sigma_{\mathrm{W}}(R)=\{\lambda:\lambda-R\text{ is not Fredholm of index zero on }X\};$$
$$\sigma_{\mathrm{W},\mathcal{B}}(R)=\{\lambda:\lambda-R\notin\Phi_{\mathcal{B}}^0\}.$$

When R is S-inessential relative to B, then by Theorem 17,

$$\sigma_{\mathrm{W}}(S+R)=\sigma_{\mathrm{W}}(S)\quad\text{and}\quad\sigma_{\mathrm{W},\mathcal{B}}(S+R)=\sigma_{\mathrm{W},\mathcal{B}}(S).$$

THEOREM 19. *Assume $\sigma_{\mathrm{W}}(S)=\sigma_{\mathrm{W},\mathcal{B}}(S)$ (this holds if $\sigma_{\mathrm{W}}(S)=\sigma(S)=\sigma_{\mathcal{B}}(S)$). If R is S-inessential relative to B, then*

$$\sigma(S+R)=\sigma_{\mathcal{B}}(S+R).$$

Proof. It is always true that $\sigma(S+R)\subseteq\sigma_{\mathcal{B}}(S+R)$. Assume $\lambda\notin\sigma(S+R)$, so $\lambda\notin\sigma_{\mathrm{W}}(S+R)=\sigma_{\mathrm{W}}(S)=\sigma_{\mathrm{W},\mathcal{B}}(S)=\sigma_{\mathrm{W},\mathcal{B}}(S+R)$ [Theorem 17]. Therefore $\lambda-(S+R)\in\Phi_{\mathcal{B}}$ and $\lambda-(S+R)$ has an inverse in $\mathcal{B}(X)$. By Corollary 3(2), $\lambda\notin\sigma_{\mathcal{B}}(S+R)$.

Concerning the condition $\sigma_W(S) = \sigma(S) = \sigma_{\mathcal{B}}(S)$, it follows from this that $\sigma_{W,\mathcal{B}}(S) \subseteq \sigma_W(S)$. The reverse inclusion always holds.

4. Examples. In this section we present three examples where the results in §3 apply. These examples are for the purpose of illustrating the types of possible applications. We need a preliminary proposition concerning compactness properties of an integral operator on $C(\Omega)$. Here Ω is a normal locally compact space and the measure μ on Ω is a σ-finite regular Borel measure.

PROPOSITION 20. *Let $J \in \mathcal{C}$.*

(1) *If J has the property that given any $\varepsilon > 0$, $\exists \Delta$, a compact subset of Ω, with*

$$\sup_{x \in \Delta^c} \int_\Omega |J(x,t)|\, d\mu(t) \le \varepsilon ,$$

then the operator J is compact on $C(\Omega)$.

(2) *Assume $J \in L^\infty(\Omega \times \Omega)$ and $f \in L^1$. Set $K(x,t) = J(x,t)f(t)$. Then the operator K is weakly compact on $C(\Omega)$.*

Proof. To prove (1) we use the compactness criterion in Jörgens' book [12, Theorem 12.3, p. 305]. Let $\varepsilon > 0$ be given, and choose Δ compact such that

$$\sup_{x \in \Delta^c} \int_\Omega |J(x,t)|\, d\mu(t) < \varepsilon/2 .$$

For each $y \in \Delta$, let

$$Vy = \Big\{ x \in \Omega : \int_\Omega |J(x,t) - J(y,t)|\, d\mu(t) < \varepsilon \Big\} .$$

Take $\{Vy_1, \ldots, Vy_n\}$ a finite cover for Δ. Fix $y_{n+1} \in \Delta^c$. Note that

$$\int_\Omega |J(x,t) - J(y_{n+1},t)|\, d\mu(t) < \varepsilon$$

for all $x \in \Delta^c$ (by the choice of Δ). Set $Vy_{n+1} = \Delta^c$. Then $\{Vy_j : 1 \le j \le n+1\}$ satisfy the requirement in [12] for compactness of J, that is,

$$\int_\Omega |J(x,t) - J(y_j,t)|\, d\mu(t) < \varepsilon \quad \text{for all } x \in Vy_j ,\ 1 \le j \le n+1 .$$

To prove (2), let $P : C(\Omega) \to L^1$ be defined by $P(g) = fg$, $g \in C(\Omega)$; and let $Q : L^1 \to C(\Omega)$ be defined by $Q(h)(x) = \int_\Omega J(x,t)h(t)\, d\mu(t)$. The operator Q takes values in $C(\Omega)$ since $J(x,t)h(t) \in \mathcal{C}$, and $Q(h)$ is just this kernel applied to the constant function 1. Both P and Q are bounded operators, but more, P is weakly compact by [10, Theorem 6, p. 494]. Therefore $K = QP$ is a weakly compact operator on $C(\Omega)$ [10, Theorem 5, p. 484].

Assume J is a kernel defined on $\Omega \times \Omega$ which determines a bounded operator from L^1 into $C(\Omega)$, $f \in L^1$, and $K(x,t) = J(x,t)f(t)$ is a kernel in $\mathcal{C}$. Then K is weakly compact on $C(\Omega)$. This more general result has the same proof as given above for (2). As a specific example, let $\Omega = [0,\infty)$ with Lebesgue measure, and set

$$J(x,t) = \chi_{[0,x]}(t) \quad (x, t \ge 0) .$$

Certainly J maps $L^1([0,\infty))$ continuously into $C[0,\infty)$, and for any $f \in L^1([0,\infty))$, it is easy to check that

$$K(x,t) = \chi_{[0,x]}(t) f(t) \in \mathcal{C} .$$

EXAMPLE I. Fix $\Omega = [0,\infty)$ equipped with Lebesgue measure. We work in the Banach algebra of operators $\mathcal{K}$ described in Example 7. Also, the space $\mathcal{C}$ is as in this same example. Let $C^1[0,\infty)$ be the subspace of all $f \in C[0,\infty)$ such that $f'(x)$ exists for all $x \in [0,\infty)$. Let $D : \mathcal{D}(D) \to C[0,\infty)$ be the differentiation operator:

$$\mathcal{D}(D) = \{ f \in C^1[0,\infty) : f' \in C[0,\infty) \text{ and } f(0) = 0 \} ,$$
$$D(f) = f' \quad \text{for } f \in \mathcal{D}(D) .$$

We now derive a result for differential operators of the form

$$L = D^n + \varphi_{n-1} D^{n-1} + \ldots + \varphi_1 D + \varphi_0$$

where $\varphi_k \in C[0,\infty)$ for all k and

$$\lim_{x \to \infty} \varphi_k(x) = a_k \in \mathbb{C} \quad \text{for } 0 \le k \le n-1 .$$

Let S be the operator with constant coefficients $\{a_k\}$,

$$S = D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0 ,$$

and let

$$p_S(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 .$$

Set $R = L - S$. The operators L, S, and R all have domains the set $\mathcal{D}(D^n)$, where $\mathcal{D}(D^n)$ is determined in the usual way from $\mathcal{D}(D)$.

THEOREM 21. *Let S, L, and R be as above. If for some λ, $p_S(z) - \lambda = \prod_{k=1}^n (z - \lambda_k)$ where $\operatorname{Re}(\lambda_k) < 0$ for $1 \le k \le n$, then R is S-inessential relative to $\mathcal{K}$. We denote the set of all such λ by Γ.*

Proof. For any $\lambda \in \mathbb{C}$, consider the integral operator

$$T_\lambda(g)(x) = \int_0^x e^{\lambda(x-t)} g(t)\, dt .$$

T_λ maps continuous functions on $[0,\infty)$ into continuous functions on $[0,\infty)$. When $\mathrm{Re}(\lambda) < 0$, then $T_\lambda \in \mathcal{B}(C[0,\infty))$, and it is straightforward to verify that $T_\lambda = (D-\lambda)^{-1}$. The operator T_λ is bounded in this case since $e^{\lambda y} \in L^1[0,\infty)$ when $\mathrm{Re}(\lambda) < 0$. In fact, the kernel

$$K_\lambda(x,t) = e^{\lambda(x-t)}\chi_{[0,x]}(t)$$

that determines T_λ has the property $K_\lambda \in \mathcal{C}$.

Now assume λ and $\{\lambda_1,\ldots,\lambda_n\}$ are as stated in the theorem. Note that

$$D(T_{\lambda_j}) = I + \lambda_j T_{\lambda_j} \quad \text{for } 1 \le j \le n.$$

For $1 \le m \le n-1$,

$$D^m\Big(\prod_{k=1}^{n} T_{\lambda_k}\Big) = D^m\Big(\prod_{k=1}^{m} T_{\lambda_k}\Big)\Big(\prod_{k=m+1}^{n} T_{\lambda_k}\Big) = \prod_{k=1}^{m}(I+\lambda_k T_{\lambda_k})\Big(\prod_{k=m+1}^{n} T_{\lambda_k}\Big).$$

It follows that $D^m(\prod_{k=1}^n T_{\lambda_k})$ is an integral operator with kernel J_m in $\mathcal{C}$. The operator $S-\lambda$ has inverse $(S-\lambda)^{-1} = (\prod_{k=1}^n T_{\lambda_k})$, and

$$R(S-\lambda)^{-1} = \sum_{m=0}^{n-1}(\varphi_m - a_m)D^m\Big(\prod_{k=1}^{n} T_{\lambda_k}\Big) = \sum_{m=0}^{n-1}(\varphi_m - a_m)J_m.$$

Since $\lim_{x\to\infty}(\varphi_m(x) - a_m) = 0$ for all m and $J_m \in \mathcal{C}$, it follows by Proposition 20 and Proposition 9 that $R(S-\lambda)^{-1} \in \mathcal{I}_{\mathcal{K}}$. This proves that R is S-inessential relative to $\mathcal{K}$.

It is easy to check that $D-\mu$ is always one-to-one on $\mathcal{D}(D)$ for arbitrary $\mu \in \mathbb{C}$. It follows that $S-\lambda$ is one-to-one on $\mathcal{D}(S)$ for all $\lambda \in \mathbb{C}$. If $S-\lambda$ is Fredholm and of index zero, then it follows that $S-\lambda$ has a bounded inverse. This proves $\sigma_{\mathrm{W}}(S) = \sigma(S)$. Now automatically $\sigma(S) \subseteq \sigma_{\mathcal{K}}(S)$, and by the argument in the proof of Theorem 21, $\sigma_{\mathcal{K}}(S) \subseteq \Gamma^c$. It is straightforward to check that $\sigma(S) = \Gamma^c$. Thus, $\sigma_{\mathrm{W}}(S) = \sigma(S) = \sigma_{\mathcal{K}}(S)$. Therefore Theorem 19 applies, so $\sigma(L) = \sigma_{\mathcal{K}}(L)$. We summarize this in the following result.

THEOREM 22. *Let S, L, and R be as above, so $L = S + R$. Then*

$$\sigma_{\mathrm{W}}(S) = \sigma(S) = \sigma_{\mathcal{K}}(S), \qquad \sigma(L) = \sigma_{\mathcal{K}}(L),$$

and $\sigma_{\mathrm{W},\mathcal{K}}(L) = \sigma_{\mathrm{W},\mathcal{K}}(S) = \sigma_{\mathrm{W}}(S) = \sigma(S) = \Gamma^c$.

EXAMPLE II. In this example we consider operators of the form $M_\varphi + J$ where M_φ is a multiplication operator (perhaps unbounded), and J is an integral operator determined by a kernel in $\mathcal{C}$. The Banach algebra of operators involved is the algebra $\mathcal{B}$ described in Example 7. The underlying space is $C(\Omega)$ where Ω is locally compact and normal and has no isolated points.

Let φ be a continuous function on Ω. Let $\mathcal{D}(M_\varphi) = \{f \in C(\Omega) : \varphi f \in C(\Omega)\}$, and define $M_\varphi(f) = \varphi f$ for $f \in \mathcal{D}(M_\varphi)$.

PROPOSITION 23. $\sigma(M_\varphi) = \sigma_{\mathcal{B}}(M_\varphi) = \sigma_{\mathrm{W}}(M_\varphi) =$ *the closure of the range of φ on Ω.*

Proof. This is all fairly elementary. We prove that $\sigma(M_\varphi) = \sigma_{\mathrm{W}}(M_\varphi)$. Suppose $\mathcal{N}(M_\varphi) \ne \{0\}$. Fix $f \in \mathcal{N}(M_\varphi)$, $f \ne 0$. Let $U = \{\omega \in \Omega : f(\omega) \ne 0\}$. Then U is a nonempty open subset of Ω. Now $V \equiv \{g \in C(\Omega) : (\text{support of } g) \subseteq \overline{U}\} \subseteq \mathcal{N}(M_\varphi)$ since $\varphi \equiv 0$ on $\overline{U}$. Since Ω has no isolated points, U must be infinite. It follows that V, hence $\mathcal{N}(M_\varphi)$, is infinite-dimensional.

Now if $\lambda \notin \sigma_{\mathrm{W}}(M_\varphi)$, then $\lambda - M_\varphi = M_{\lambda-\varphi}$ is Fredholm of index zero. But as argued above, $\mathcal{N}(M_{\lambda-\varphi})$ is either $\{0\}$ or infinite-dimensional. Therefore $\lambda - M_\varphi$ is invertible.

THEOREM 24. *Assume φ is a $\mathbb{C}$-valued continuous function on Ω, and let M_φ be the multiplication operator described above. Assume φ has the property that for some $\lambda \notin \{\varphi(\omega) : \omega \in \Omega\}^-$, $(\lambda - \varphi)^{-1} \in L^1(\mu)$. Let J be a kernel in $\mathcal{C}$ such that J is essentially bounded. Then J is M_φ-inessential relative to $\mathcal{B}$, and*

$$\sigma_{\mathrm{W}}(M_\varphi + J) = \sigma_{\mathrm{W}}(M_\varphi) = \sigma_{\mathrm{W},\mathcal{B}}(M_\varphi) = \sigma_{\mathrm{W},\mathcal{B}}(M_\varphi + J)$$
$$= \{\textit{closure of the range of } \varphi \textit{ on } \Omega\},$$

and $\sigma(M_\varphi + J) = \sigma_{\mathcal{B}}(M_\varphi + J)$.

Proof. Applying Proposition 20(2), we have that when $(\lambda-\varphi)^{-1} \in L^1(\mu)$, then the kernel $J(x,t)(\lambda - \varphi(t))^{-1}$ determines a weakly compact operator on $C(\Omega)$. By Proposition 9 this operator is in $\mathcal{I}_{\mathcal{B}}$. The theorem follows by applying Proposition 23, Theorem 17, and Theorem 19.

As indicated following Proposition 20, Theorem 24 will apply to a more general class of integral operators J. For example, when $\Omega = [0,\infty)$ with Lebesgue measure and $\varphi(x) = x^2$, then the Theorem holds for the operator T on $C[0,\infty)$ given by

$$T(g)(x) = x^2 g(x) + \int_0^x g(t)\,dt.$$

EXAMPLE III. In this example we consider a Banach algebra of operators acting on $L^\infty = L^\infty(\mu)$ where μ is a *finite* measure defined on a σ-algebra of subsets of a fixed set Ω. To avoid trivialities we assume that $L^\infty(\mu)$ is infinite-dimensional and μ is a continuous measure ($\mu(\{x\}) = 0$ for all $x \in \Omega$). In this example we let $\mathcal{B}$ be the algebra of all operators of the form $\lambda I + K_\infty$ where I is the identity operator on L^∞, $\lambda \in \mathbb{C}$, and K_∞ is the integral operator on L^∞ determined by a kernel K in $L^\infty(\Omega\times\Omega, \mu\times\mu)$. For $\lambda I + K_\infty \in \mathcal{B}$, let $\|\lambda I + K_\infty\|_{\mathcal{B}} = |\lambda| + \|K\|_\infty$. Clearly $\mathcal{B}$ is a Banach subalgebra of $\mathcal{B}(L^\infty)$. Also, $\mathcal{B}$ satisfies (#) and

$$\mathcal{F}_\mathcal{B} = \operatorname{span}\{\psi(x)\varphi(t) : \varphi, \psi \in L^\infty(\mu)\}.$$

Let K be an essentially bounded kernel. Operators on $L^p(\mu)$ determined by K have a number of special properties, we list two:

(1) For all p, $1 \le p \le \infty$, K determines a bounded operator K_p on $L^p(\mu)$,

$$K_p(f)(x) = \int_\Omega K(x,t) f(t)\, d\mu(t).$$

Also, K_p is a Hille–Tamarkin operator for all p; see [12, §11.3]. In particular, K_p is compact for $1 < p < \infty$, and weakly compact for $p = 1, \infty$ [12, §11].

(2) For all p, $1 \le p \le \infty$, $K_p(L^p) \subseteq L^\infty$, and if $\{f_n\} \subseteq L^p$, $\|f_n - f\|_p \to 0$, then $\|K_p(f_n) - K_p(f)\|_\infty \to 0$. Also, if $\{f_n\} \subseteq L^\infty$, $\|f_n\|_\infty \le M$ for $n \ge 1$ and $f_n \to f$ a.e. on Ω, then applying the Dominated Convergence Theorem it follows that $\|K_\infty(f_n) - K_\infty(f)\|_\infty \to 0$.

Let $\mathcal{H}_\infty$ be the set of all integral operators K_∞ on $L^\infty(\mu)$ determined by kernels K for which the norm

$$\tau_\infty(K) = \operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega |K(x,t)|\, d\mu(t) < \infty.$$

Then $(\mathcal{H}_\infty, \tau_\infty)$ is a Banach algebra of operators, the Hille–Tamarkin operators on $L^\infty(\mu)$; see [12, §11] where the notation $\mathcal{H}_{\infty\infty}$ is used for $\mathcal{H}_\infty$. From [12, §11] we have that $\mathcal{H}_\infty$ is a closed subalgebra of $\mathcal{B}(L^\infty)$ and that $\sigma_{\mathcal{H}_\infty}(K_\infty) = \sigma(K_\infty)$ for all $K_\infty \in \mathcal{H}_\infty$. Clearly $\{K_\infty : K \in L^\infty(\mu \times \mu)\}$ is a subalgebra of $\mathcal{H}_\infty$.

PROPOSITION 25. *Let $\mathcal{B}$ and $\mathcal{H}_\infty$ be as above.*

(1) *$\{K_\infty : K \in L^\infty(\mu \times \mu)\}$ is a right ideal of $\mathcal{H}_\infty$.*
(2) *$\sigma_\mathcal{B}(\lambda + K_\infty) = \sigma(\lambda + K_\infty)$ for all operators $\lambda + K_\infty \in \mathcal{B}$.*
(3) *$\mathcal{I}_\mathcal{B} = \{K_\infty : K \in L^\infty(\mu \times \mu)\}$.*

Proof. First assume J and K are kernels with $\tau_\infty(J) < \infty$ and $K \in L^\infty(\mu \times \mu)$. Consider the kernel defined almost everywhere by

$$(J * K)(x,t) = \int_\Omega J(x,z) K(z,t)\, d\mu(z).$$

For almost all (x,t),

$$|(J*K)(x,t)| \le \int_\Omega |J(x,z)||K(z,t)|\, d\mu(z) \le \int_\Omega |J(x,z)|\, d\mu(z) \|K\|_\infty.$$

Therefore $\|J * K\|_\infty \le \tau_\infty(J)\|K\|_\infty < \infty$. Since $(J*K)_\infty = J_\infty K_\infty$, this proves (1).

For $T \in \mathcal{H}_\infty$, it follows from [12, Theorem 11.11, p. 292] that $\sigma_{\mathcal{H}_\infty}(T) = \sigma(T)$. Also, when $T \in \mathcal{B}$, then (1) easily implies that $\sigma_\mathcal{B}(T) = \sigma_{\mathcal{H}_\infty}(T)$. Therefore (2) follows.

Now we prove (3). For $K \in L^\infty(\mu \times \mu)$, let $K^*(x,t) = K(t,x)$. The integral operator $(K^*)_1$ on $L^1(\mu)$ determined by K^* is a Hille–Tamarkin operator. Therefore $\sigma((K^*)_1)$ has no nonzero accumulation points [12, Theorem 11.9, p. 289]. Since K_∞ is the adjoint of $(K^*)_1$, it follows that $\sigma(K_\infty)$ has no nonzero accumulation point. Applying (2) we have

$$\sigma_\mathcal{B}(K_\infty) \text{ has no nonzero accumulation point for all } K \in L^\infty(\mu \times \mu).$$

From the definition of $\mathcal{I}_\mathcal{B}$, this implies

$$\mathcal{I}_\mathcal{B} = \{K_\infty : K \in L^\infty(\mu \times \mu)\}.$$

Now we look at a situation where an operator S on L^∞ has $(\lambda - S)^{-1} \in \mathcal{B}$ for all $\lambda \in \varrho(S)$. It is interesting to know when the resolvents of some perturbation $S + R$ of S also have this very strong property.

PROPOSITION 26. *Assume S is an operator on $L^\infty(\mu)$ with $S^{-1} \in \mathcal{I}_\mathcal{B}$, so $S^{-1} = K_\infty$ where K is an essentially bounded kernel.*

(1) *$\varrho_\mathcal{B}(S) = \varrho(S)$.*
(2) *If $\lambda \in \varrho(S)$, then $(\lambda - S)^{-1} \in \mathcal{I}_\mathcal{B}$.*
(3) *For all $\lambda \in \mathbb{C}$, $\lambda - S \in \Phi^0_\mathcal{B}$.*
(4) *Assume that R is an operator on L^∞ such that R is S-inessential relative to $\mathcal{B}$.*
  (i) *$\sigma_\mathcal{B}(S + R) = \sigma(S + R)$.*
  (ii) *For all $\lambda \in \mathbb{C}$, $\lambda - (S+R) \in \Phi^0_\mathcal{B}$, and $\exists F \in \mathcal{F}_\mathcal{B}$ such that $\lambda - (S+R) + F$ has an inverse in $\mathcal{I}_\mathcal{B}$.*
(5) *If for some $p \ge 1$, S has a closed extension S_p on $L^p(\mu)$ with $\mathcal{N}(S_p) = \{0\}$, then $S_p^{-1} = K_p$ and $\varrho(S_p) = \varrho_\mathcal{B}(S)$ (here K_p is the operator on $L^p(\mu)$ determined by the kernel K).*

Proof. (1) Since $S^{-1} \in \mathcal{I}_\mathcal{B}$, $\sigma_\mathcal{B}(S^{-1})$ has no nonzero accumulation point. Also note that $\sigma(S^{-1}) \subseteq \sigma_\mathcal{B}(S^{-1})$. Consider the continuous embedding $\varphi : \mathcal{B} \to \mathcal{B}(L^\infty)$ given by $\varphi(T) = T$, $T \in \mathcal{B}$. From [4, Theorem 4.5] it follows that any isolated point in $\sigma_\mathcal{B}(T)$ is in $\sigma(\varphi(T)) = \sigma(T)$. Thus when $\sigma_\mathcal{B}(T)$ is countable, it is the closure of the set of all its isolated points, so in this case $\sigma(T) = \sigma_\mathcal{B}(T)$. Applying this argument to S^{-1}, we have $\sigma(S^{-1}) = \sigma_\mathcal{B}(S^{-1})$. Then by [6, Theorem 2(2)], $\sigma_\mathcal{B}(S) = \sigma(S)$. This proves (1).

(2) Assume that $\lambda \in \varrho(S) = \varrho_\mathcal{B}(S)$. By the usual resolvent equation, $(\lambda - S)^{-1} + S^{-1} = \lambda(\lambda - S)^{-1}S^{-1}$. Since $(\lambda - S)^{-1} \in \mathcal{B}$ and $S^{-1} \in \mathcal{I}_\mathcal{B}$, this equation shows $(\lambda - S)^{-1} \in \mathcal{I}_\mathcal{B}$.

(3) The operator $(\lambda - S^{-1}) \in \Phi^0_\mathcal{B}$ for all $\lambda \ne 0$ (S^{-1} is weakly compact on L^∞). It follows from [6, Theorem 8] that $\lambda - S \in \Phi^0_\mathcal{B}$ for all $\lambda \in \mathbb{C}$.

(4) Assume R is S-inessential relative to $\mathcal{B}$, so $RS^{-1} \in \mathcal{I}_{\mathcal{B}}$. By (3) and Theorem 17, $\sigma_{\mathrm{W},\mathcal{B}}(S+R) = \sigma_{\mathrm{W},\mathcal{B}}(S)$, which is empty. Thus, $\lambda-(S+R) \in \Phi^0_{\mathcal{B}}$ for all $\lambda \in \mathbb{C}$. Also, $\sigma_{\mathrm{W}}(S) \subseteq \sigma_{\mathrm{W},\mathcal{B}}(S)$ (always), so $\sigma_{\mathrm{W}}(S) = \sigma_{\mathrm{W},\mathcal{B}}(S)$. It follows from Theorem 19 that $\sigma_{\mathcal{B}}(S+R) = \sigma(S+R)$. Finally, applying Theorem 15, $\exists F \in \mathcal{F}_{\mathcal{B}}$ such that $\lambda - (S+R) + F$ has an inverse in $\mathcal{B}$ (of course if $\lambda \in \varrho_{\mathcal{B}}(S+R)$, then $F = 0$ will suffice). Let W be this inverse. Then

$$S^{-1} = W(\lambda - (S+R) + F)S^{-1} = W(\lambda S^{-1} - I - RS^{-1} + FS^{-1});$$

solving for W and using the hypothesis that $S^{-1} \in \mathcal{I}_{\mathcal{B}}$, we have $W \in \mathcal{I}_{\mathcal{B}}$.

(5) Let S_p be a closed extension of S on L^p with $\mathcal{N}(S_p) = \{0\}$. For $g \in L^p$, choose $\{g_n\} \subseteq L^\infty$ with $\|g_n - g\|_p \to 0$. Then $\|K_p(g_n) - K_p(g)\|_p \to 0$, $\{K_p(g_n)\} \subseteq \mathcal{D}(S) \subseteq \mathcal{D}(S_p)$, and $\|S_p(K_p(g_n)) - g\|_p \to 0$. Since S_p is closed, $K_p(g) \in \mathcal{D}(S_p)$ and $S_p(K_p(g)) = g$. Now if $u \in \mathcal{D}(S_p)$,

$$S_p(K_pS_p(u) - u) = 0$$

so $K_pS_p(u) = u$ for all $u \in \mathcal{D}(S_p)$.

Next we consider a specific case where Proposition 26 applies. We take as the underlying space $L^\infty[0,1]$, the measure being Lebesgue measure. Let $\mathrm{AC}[0,1]$ denote the absolutely continuous functions on $[0,1]$. Define

$$\mathcal{D}(S_\infty) = \{u \in \mathrm{AC}[0,1] : u' \in \mathrm{AC}[0,1],\ u'' \in L^\infty[0,1],\ \text{and } u(0) = u(1) = 0\},$$

$$S_\infty(u) = -u'' \quad \text{for } u \in \mathcal{D}(S_\infty).$$

Let $K(x,t)$ be the bounded kernel on $[0,1] \times [0,1]$,

$$K(x,t) = (1-x)t\chi_{[0,x]}(t) + x(1-t)\chi_{(x,1]}(t).$$

It is easy to see that $S_\infty^{-1} = K_\infty$. Also note that S_∞ has a closed extension S_p on $L^p[0,1]$, $1 \le p \le \infty$, and $\mathcal{N}(S_p) = \{0\}$ ($\mathcal{D}(S_p)$ is defined analogously to $\mathcal{D}(S_\infty)$ with the requirement that $u'' \in L^p[0,1]$). In fact, by direct computation one easily verifies that $S_p^{-1} = K_p$ for all p. Thus all the conclusions of Proposition 26 hold for S_∞ (in place of S), and part (5) of that result holds for S_p. Also note that the operator $g \to (S_p^{-1}(g))'$ is determined by the bounded kernel $-t\chi_{[0,x]}(t) + (1-t)\chi_{(x,1)}(t)$.

Now we consider the spectral and Fredholm properties of an integro-differential operator W on $L^p[0,1]$ of the form

$$W(u)(x) = -u''(x) + \varphi(x)u'(x) + \psi(x)u(x) + \int_0^1 H(x,t)u'(t)\,dt + \int_0^1 J(x,t)u(t)\,dt.$$

Applying the theory developed in this paper, we prove under mild assumptions that for all $\lambda \in \varrho(W)$, $(\lambda - W)^{-1} \in \mathcal{I}_{\mathcal{B}}$, that is, $(\lambda - W)^{-1}$ is an integral operator determined by an essentially bounded kernel. Also, $\lambda - W$ is Fredholm of index zero for all λ and has Fredholm inverses in $\mathcal{I}_{\mathcal{B}}$.

THEOREM 27. *Fix p, $1 \le p < \infty$. Assume $\varphi, \psi \in L^\infty[0,1]$. Let H and J be kernels with $\tau_\infty(H) < \infty$ and $\tau_\infty(J) < \infty$ and which determine bounded integral operators H_j and J_j on $L^j[0,1]$. For $j = p$ and ∞, let*

$$\mathcal{D}(R_j) = \{u \in \mathrm{AC}[0,1] : u' \in L^j[0,1]\},$$

and define

$$R_j(u) = \varphi u' + \psi u + H_j(u') + J_j(u) \quad (u \in \mathcal{D}(R_j)).$$

Let S_j, $j = p, \infty$, be the second derivative operator defined above.

(1) *$R_\infty S_\infty^{-1} \in \mathcal{I}_{\mathcal{B}}$; $R_pS_p^{-1}$ is compact on $L^p[0,1]$ for $1 < p < \infty$.*

(2) *$\varrho_{\mathcal{B}}(S_\infty + R_\infty) = \varrho(S_j + R_j)$; for all $\lambda \in \mathbb{C}$, $\lambda - (S_\infty + R_\infty) \in \Phi^0_{\mathcal{B}}$, and $\lambda - (S_p + R_p)$ is Fredholm of index zero on $L^p[0,1]$.*

(3) *For $j = p$ and ∞, when $\lambda \in \varrho(S_j + R_j)$, then $(\lambda - (S_j + R_j))^{-1} = K_j$ for some $K \in \mathcal{I}_{\mathcal{B}}$.*

(4) *For $j = p$ and ∞, for all $\lambda \in \mathbb{C}$, there exists $F \in \mathcal{F}_{\mathcal{B}}$ such that $(\lambda - (S_j + R_j) + F)^{-1} = K_j$ for some $K \in L^\infty$.*

Proof. The operator S_j^{-1} is determined by a bounded kernel M, and as noted above, $g \to (S_j^{-1}(g))'$ is determined by some bounded kernel N. Therefore

$$R_j(S_j^{-1}g) = (\varphi N)_j(g) + (\psi M)_j(g) + H_jN_j(g) + J_jM_j(g).$$

By Proposition 25(1), $H * N$ and $J * M$ are essentially bounded kernels. This proves $R_\infty S_\infty^{-1} \in \mathcal{I}_{\mathcal{B}}$ (Proposition 25), and the second statements in (1) follow from the fact that $R_pS_p^{-1}$ is Hille–Tamarkin for $1 < p$; see [12, §11]. For the case $p = 1$, it can be verified directly that M_1 and N_1 are compact on $L^1(\mu)$.

By Proposition 26(4) we have that $\varrho_{\mathcal{B}}(S_\infty + R_\infty) = \varrho(S_\infty + R_\infty)$ and $\lambda - (S_\infty + R_\infty) \in \Phi^0_{\mathcal{B}}$ for all λ. Also, since $R_pS_p^{-1}$ is compact, we have $\sigma_{\mathrm{W}}(S_p + R_p) = \sigma_{\mathrm{W}}(S_p)$ which is empty. Thus for all λ, $\lambda - (S_p + R_p)$ is Fredholm of index zero on $L^p[0,1]$.

We use these facts to prove $\varrho(S_\infty + R_\infty) = \varrho(S_p + R_p)$. If $\lambda \in \sigma(S_\infty + R_\infty)$, then $\mathcal{N}(\lambda - (S_\infty + R_\infty)) \neq \{0\}$, so $\mathcal{N}(\lambda - (S_p + R_p)) \neq \{0\}$. It follows that $\lambda \in \sigma(S_p + R_p)$. Conversely, if $\lambda \in \varrho(S_\infty + R_\infty)$, then

$$L^\infty[0,1] = \mathcal{R}(\lambda - (S_\infty + R_\infty)) \subseteq \mathcal{R}(\lambda - (S_p + R_p)).$$

Thus in this case, $\lambda \in \varrho(S_p + R_p)$ since $\lambda - (S_p + R_p)$ is Fredholm of index zero. We have proved that $\varrho(S_\infty + R_\infty) = \varrho(S_p + R_p)$, which completes the proof of (2).

By Proposition 26(4), for all $\lambda \in \mathbb{C}$, there exists $F \in \mathcal{F}_{\mathcal{B}}$ such that $(\lambda - (S_\infty + R_\infty) + F)^{-1} = K_\infty$, where K is an essentially bounded kernel, that is, $K_\infty \in \mathcal{I}_{\mathcal{B}}$. Also, when $\lambda \in \varrho(S_\infty + R_\infty)$, then this statement is true with $F = 0$. Now by (2), for all $\lambda \in \mathbb{C}$, $\lambda - (S_p + R_p) + F$ is Fredholm of index zero on $L^p[0,1]$. But

$$L^\infty[0,1] = \mathcal{R}(\lambda - (S_\infty + R_\infty) + F) \subseteq \mathcal{R}(\lambda - (S_p + R_p) + F).$$

Therefore $\lambda - (S_p + R_p) + F$ is invertible on $L^p[0,1]$. Certainly, we have $(\lambda - (S_p + R_p) + F)^{-1} = K_p$. This proves both (3) and (4).

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Wavelet bases in $L^p(\mathbb{R})$

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Abstract. It is shown that an orthonormal wavelet basis for $L^2(\mathbb{R})$ associated with a multiresolution is an unconditional basis for $L^p(\mathbb{R})$, $1 < p < \infty$, provided the father wavelet is bounded and decays sufficiently rapidly at infinity.

1. Introduction. The purpose of this paper is to extend some of the results in [8] on unconditional bases (see e.g. [5]) in wavelet form $\{\psi(2^{-m}\bullet -k)\}_{m,k\in\mathbb{Z}}$ for the space $L^p(\mathbb{R};\mathbb{C})$, $1 < p < \infty$. Here ψ is a mother wavelet, that is, $\{2^{-m/2}\psi(2^{-m}\bullet -k)\}_{m,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R};\mathbb{C})$ (and $\bullet$ denotes a generic argument). The analysis in this paper is restricted to the one-dimensional case where there is also a father wavelet φ such that if V_m is the space spanned by $\{\varphi(2^{-m}\bullet -k)\}_{k\in\mathbb{Z}}$ and W_m is the space spanned by $\{\psi(2^{-m}\bullet -k)\}_{k\in\mathbb{Z}}$, then $V_{m-1} = V_m \oplus W_m$. In other words, the wavelets are associated with a multiresolution.

It is proved below that if φ and ψ are bounded and both decay sufficiently rapidly at infinity, then we get an unconditional basis for $L^p(\mathbb{R};\mathbb{C})$ for all $p \in (1,\infty)$. The main point of this paper is to show that no smoothness assumptions on φ and ψ (like those in [8] where it is required that $|\psi'(x)| \le C_p(1+|x|)^{-p}$ for every $p \ge 0$). are needed for this conclusion to hold. This means in particular that all the compactly supported wavelets constructed in [2] give rise to unconditional bases and we also get an alternative proof for the well-known fact that the Haar functions (for which $\varphi = \chi_{[0,1)}$ and $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$) form an unconditional basis (cf. [9, p. 207]).

2. Statement of results. First we define what we mean by a multiresolution or a multiresolution analysis as it is commonly called. We say that $(\{V_m\}_{m\in\mathbb{Z}}, \varphi)$ is an *orthonormal multiresolution* of $L^2(\mathbb{R};\mathbb{C})$ provided the following four conditions hold.

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