[3] A. Bressan and F. Flores, *Multivalued Aumann integrals and controlled wave equations*, preprint S.I.S.S.A., Trieste 1992.
[4] L. Cesari, *Optimization. Theory and Applications*, Springer, New York 1983.
[5] P. R. Halmos, *The range of a vector measure*, Bull. Amer. Math. Soc. 54 (1948), 416–421.
[6] J. Lindenstrauss, *A short proof of Liapounoff's convexity theorem*, J. Math. Mech. 15 (1966), 971–972.
[7] A. A. Lyapunov, *On completely additive vector-valued functions*, Izv. Akad. Nauk SSSR Ser. Mat. 8 (1940), 465–478 (in Russian; French summary).
[8] C. Olech, *The Lyapunov theorem: its extensions and applications*, in: Methods of Nonconvex Analysis, A. Cellina (ed.), Lecture Notes in Math. 1446, Springer, 1989, 84–103.
[9] J. A. Yorke, *Another proof of the Liapunov convexity theorem*, SIAM J. Control 9 (1971), 351–353.

S.I.S.S.A.
VIA BEIRUT 4
TRIESTE 34014, ITALY

Received February 26, 1992
Revised version February 1, 1993 (2908)

STUDIA MATHEMATICA 106 (2) (1993)

Generalized inverses in C^*-algebras II

by

ROBIN HARTE (Belfast) and MOSTAFA MBEKHTA (Lille)

Abstract. Commutativity and continuity conditions for the Moore–Penrose inverse and the "conorm" are established in a C^*-algebra; moreover, spectral permanence and B^*-properties for the conorm are proved.

Suppose A is a ring, with identity 1 and invertible group A^{-1} (more generally, an "additive category"); then an element $a \in A$ will be called *regular* if it has a *generalized inverse* in A, $b \in A$ for which

$$a = aba . \tag{0.1}$$

It is clear that both products

$$ba = (ba)^2 \quad\text{and}\quad ab = (ab)^2 \tag{0.2}$$

are *idempotents* of A; in the presence of an *involution* $* : A \to A$ we can also enquire whether or not they are *self-adjoint*: when $(ba)^* = ba$ and $(ab)^* = ab$ then (provided also $b = bab$) the generalized inverse is called a *Moore–Penrose inverse* for A. If this exists then ([7], Theorem 5) it is uniquely determined, and lies in the *double commutant* of a and a^*; when A is a C^*-algebra then ([7], Theorem 6) every regular element has a Moore–Penrose inverse. We write a^+ for the Moore–Penrose inverse of $a \in A$; thus

$$a = aa^+a;\quad a^+ = a^+aa^+;\quad (a^+a)^* = a^+a;\quad (aa^+)^* = aa^+ . \tag{0.3}$$

By the uniqueness it is clear that

$$(a^*)^+ = (a^+)^* . \tag{0.4}$$

We recall also that, in a C^*-algebra A, necessary and sufficient for an element $a \in A$ to be regular (and hence have a Moore–Penrose inverse) is ([7], Theorems 2 and 8) that the range ideal be closed:

$$aA = \mathrm{cl}\, aA . \tag{0.5}$$

1991 *Mathematics Subject Classification*: Primary 46L05.

In this note we enquire for which elements $a \in aAa$ the Moore–Penrose inverse a^+ is a continuous function of a; this leads us to introduce the "conorm" of a normed algebra element.

If $0 \neq T : X \to Y$ is a bounded linear operator between normed spaces then its *reduced minimum modulus* is given by

(0.6) $$\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}(x, T^{-1}(0)) \geq 1\}.$$

For $T = 0$ this suggests (Kato [9], Ch. IV, §5) $\gamma(T) = \infty$, although Apostol [1] prefers $\gamma(0) = 0$. For example if X and Y are both complete then ([9], Theorem IV.5.2)

(0.7) $$\gamma(T) > 0 \Leftrightarrow T(X) = \operatorname{cl} T(X);$$

if T is invertible then

(0.8) $$\gamma(T) = \|T^{-1}\|^{-1}.$$

When A is a normed algebra then an element $a \in A$ acquires a "left" and a "right" conorm, the reduced minimum modulus of the operators L_a and R_a of multiplication by a on the normed space A:

1. Definition. The (*left*) *conorm* of an element $a \in A$ in a normed algebra A is given by

(1.1) $$\gamma(a) = \gamma_A^{\text{left}}(a) = \gamma(L_a) = \inf\{\|ax\| : \operatorname{dist}(x, a^{-1}(0)) \geq 1\},$$

where

(1.2) $$a^{-1}(0) = L_a^{-1}(0) = \{x \in A : ax = 0\}$$

is the right annihilator of a in A.

Similarly, the minimum modulus of the right multiplication R_a gives a "right conorm" for $a \in A$:

(1.3) $$\gamma_A^{\text{right}}(a) = \inf\{\|xa\| : \operatorname{dist}(x, a_{-1}(0)) \geq 1\},$$

where

(1.4) $$a_{-1}(0) = R_a^{-1}(0) = \{x \in A : xa = 0\}.$$

Whether or not the algebra is complete, the conorm of a regular element is positive:

2. Theorem. *If a and b are elements of a normed algebra A then*

(2.1) $$0 \neq a = aba \Rightarrow 1 \leq \|b\|\gamma(a) \leq \|ba\|\,\|ab\|.$$

If in particular $0 \neq a \in aAa$ is regular in a C^*-algebra A then

(2.2) $$\|a^+\|\gamma(a) = 1.$$

Proof. Suppose first that $T \in BL(X, Y)$ has a complemented null space in X, and that $P = P^2 \in BL(X, X)$ satisfies $P^{-1}(0) = T^{-1}(0)$; then we may define $T_P^\wedge : P(X) \to \operatorname{cl} T(X)$ by setting

(2.3) $$T_P^\wedge(Px) = Tx \quad \text{for each } x \in X.$$

Now we claim

(2.4) $$\|T_P^\wedge\| \leq \|T\| \leq \|P\|\,\|T_P^\wedge\|$$

and

(2.5) $$\gamma(T_P^\wedge) \leq \gamma(T) \leq \|P\|\gamma(T_P^\wedge).$$

For (2.4) argue

$$\sup \frac{\|Tx\|}{\|Px\|} = \sup \frac{\|TPx\|}{\|Px\|} \leq \sup \frac{\|Tx\|}{\|x\|} = \|T\| \leq \|P\| \sup \frac{\|Tx\|}{\|Px\|};$$

for (2.5) observe

(2.6) $$\operatorname{dist}(x, T^{-1}(0)) = \operatorname{dist}(Px, T^{-1}(0)) \leq \|Px\| \leq \|P\| \operatorname{dist}(x, T^{-1}(0)).$$

If in particular $T = TST$ has a generalized inverse $S \in BL(Y, X)$ then we may take $P = ST$, and apply also the analogue of (2.4) with S and $Q = TS$ in place of T and P; then (0.8) gives

(2.7) $$\gamma(T_P^\wedge) = \frac{1}{\|S_Q^\wedge\|}.$$

Now (2.4), (2.5) and (2.7) together give (2.1). When $a = aba$ in a normed algebra A then (2.1) applies with $T = L_a$ and $S = L_b$; to deduce (2.2) we need only show that if $b = a^+$ then

(2.8) $$\|aa^+\| = \|a^+a\| = 1,$$

giving $\|ST\| = \|TS\| = 1$ in (2.1). But if $p^* = p = p^2 \in A$ then $\|p\| = \|p^*p\| = \|p\|^2$. ■

In a C^*-algebra, the conorm can be represented as a sort of "spectral radius", and hence acquires "B^*" characteristics. Recall that the *spectrum* of a linear algebra element $a \in A$ is given by

(2.9) $$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}\},$$

and the *spectral radius* by

(2.10) $$|a|_\sigma = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Note that in general

(2.11) $$\lambda \notin \sigma(a) \Rightarrow |(a - \lambda)^{-1}|_\sigma \operatorname{dist}(\lambda, \sigma(a)) = 1.$$

When A is a complex Banach algebra then the spectral radius is less than or equal to the norm: for normal elements of a C^*-algebra there is equality. This includes "positive" elements ([10], Théorème 1.5):

3. THEOREM. *If $0 \neq a \in A$ is positive in a C^*-algebra A then*

$$\gamma(a) = \inf(\sigma(a) \setminus \{0\}). \tag{3.1}$$

Proof. We make two claims: there is implication

$$\inf(\sigma(a) \setminus \{0\}) > 0 \Rightarrow \gamma(a) \geq \inf(\sigma(a) \setminus \{0\}) \tag{3.2}$$

and

$$\gamma(a) > 0 \Rightarrow \inf(\sigma(a) \setminus \{0\}) \geq \gamma(a). \tag{3.3}$$

Towards (3.2) suppose $0 < \lambda < \inf(\sigma(a) \setminus \{0\})$ and write $b = (a - \lambda)^{-1}$; then $a - \lambda$ and b are both positive in A, so that ([6], Theorem 9.9.4; [8], Theorem 2.2) for arbitrary $x \in A$,

$$\|abx\|^2 = \|(x + \lambda bx)^*(x + \lambda bx)\| \geq \|\lambda^2 x^* b^* bx\| = \|\lambda bx\|^2.$$

Since $bA = A$ this gives $\gamma(a) \geq \lambda$ and hence (3.2). Towards (3.3) note that if $\gamma(a) > 0$ then by (0.7) the ideal aA is closed and hence $a \in aAa$ is regular, and has a Moore–Penrose inverse $a^+ \in A$. Since a is also positive and hence normal it actually commutes ([7], Theorem 10) with a^+, and is thus "simply polar" ([7], Theorem 9; [6], Definition 7.3.5). In particular, A is the direct sum of the ideals aA and $a^{-1}(0)$, and the restriction to aA of the multiplication L_a is invertible (inverse given by restricting L_{a^+}), while the restriction to $a^{-1}(0)$ is zero. If $0 < |\lambda| < \gamma(a)$ therefore both restrictions, and hence $a - \lambda$, are invertible, giving (3.3). ∎

By the spectral mapping theorem it follows that

$$a \geq 0 \Rightarrow \gamma(a)^2 = \gamma(a^2). \tag{3.4}$$

Theorem 3 gives the corresponding result for arbitrary elements ([10], Théorème 1.6):

4. THEOREM. *If $0 \neq a \in A$ is a non-zero C^*-algebra element then*

$$\gamma(a)^2 = \inf(\sigma(a^*a) \setminus \{0\}), \tag{4.1}$$

and hence

$$\gamma(a)^2 = \gamma(a^*a) = \gamma(aa^*) = \gamma(a^*)^2. \tag{4.2}$$

Also

$$\gamma(a) = \gamma(R_a); \tag{4.3}$$

if $A \subseteq B$ for a C^*-algebra B then

$$\gamma_A(a) = \gamma_B(a). \tag{4.4}$$

Proof. Recalling the square root equality

$$\|ax\| = \|(a^*a)^{1/2}x\| \quad \text{for each } x \in A, \tag{4.5}$$

together with (3.4), gives the first equality in (4.2), and hence (4.1), since Theorem 3 applies to the positive element a^*a. Since in general

$$1 - ba \in A^{-1} \Leftrightarrow 1 - ab \in A^{-1} \tag{4.6}$$

there is equality

$$\sigma(aa^*) \setminus \{0\} = \sigma(a^*a) \setminus \{0\}. \tag{4.7}$$

This with Theorem 3 gives the second equality in (4.2), and the third is just the first applied to a^*. Equality (4.3) follows from $\gamma(a^*) = \gamma(a)$, and finally the spectral permanence (4.4) is (4.1) together with the corresponding property of the spectrum σ. ∎

In the special case $A = BL(X, X)$ of operators, (4.1) is given by Apostol ([1], page 280). Theorem 4 shows that "spectral permanence" in C^*-algebras extends to regularity and in particular the Moore–Penrose inverse:

$$a \in A \cap aBa \Rightarrow a \in aAa \text{ with } a^+ \in A. \tag{4.8}$$

This is an improvement on Theorem 5 of [7], which gives (4.8) for W^*-algebras; (4.8) of course follows also from the formulae of Groetsch ([4]; [3], Corollary 1, §2.2). Another simple corollary is that (if $a \neq 0$)

$$\gamma(a) = \|a\| \Leftrightarrow \frac{a}{\|a\|} \text{ partial isometry}. \tag{4.9}$$

To investigate the continuity of the Moore–Penrose inverse and the conorm, we need a simple observation:

5. THEOREM. *If $a \in aAa$ and $b \in bAb$ have generalized inverses in a C^*-algebra A then there is equality*

$$\begin{aligned} b^+ - a^+ = & -b^+(b-a)a^+ + b^+b^{*+}(b^* - a^*)(1 - aa^+) \\ & + (1 - b^+b)(b^* - a^*)a^{*+}a^+, \end{aligned} \tag{5.1}$$

and, if $a \neq 0 \neq b$, implication

$$\|b^+b - a^+a\| < 1 \Rightarrow |\gamma(b) - \gamma(a)| \leq \|b - a\|. \tag{5.2}$$

Proof. Towards (5.1) write

$$b^+ - a^+ + b^+(b-a)a^+ = b^+(1 - aa^+) - (1 - b^+b)a^+,$$

and use (0.3) and (0.4) to see that

$$b^+b^{*+}(b^* - a^*)(1 - aa^+) = b^+(1 - aa^+)$$

and

$$(1 - b^+b)(b^* - a^*)a^{*+}a^+ = -(1 - b^+b)a^+.$$

For (5.2) write $c = (1 + a^+a - b^+b)^{-1}$ and argue

$$\|bx\| \geq \|aa^+acx\| - \|b - a\| \, \|a^+acx\| \geq (\gamma(a) - \|b - a\|) \operatorname{dist}(x, b^{-1}(0)),$$

since, by (2.6) and (2.8),

$$\mathrm{dist}(a^+acx, a^{-1}(0)) = \|a^+acx\| = \|b^+bx\| = \mathrm{dist}(x, b^{-1}(0)).$$

This gives $\gamma(b) \geq \gamma(a) - \|b-a\|$, which is half of (5.2), and similarly the other half. ■

Various conditions are equivalent to the convergence of a sequence of Moore–Penrose inverses ([10], Théorème 2.2):

6. THEOREM. *If $0 \neq a \in aAa$ and $0 \neq a_n \in a_nAa_n$ are regular in a C^*-algebra A, with $\|a_n - a\| \to 0$, then the following are equivalent:*

(6.1) $$\|a_n^+ - a^+\| \to 0;$$

(6.2) $$\gamma(a_n) \to \gamma(a);$$

(6.3) $$\sup_n \|a_n^+\| < \infty.$$

Proof. If (6.1) holds then both sequences of projections also converge:

(6.4) $$a_n^+a_n \to a^+a$$

and

(6.5) $$a_na_n^+ \to aa^+;$$

conversely, if either (6.4) or (6.5) is valid then (5.2) gives convergence (6.2) for the conorms. If (6.2) holds then (since $\gamma(a) > 0$) the sequence $\gamma(a_n)$ is bounded below, and (6.3) follows from the equality (2.2). Finally, (6.1) follows from (6.3) and the equality (5.1). ■

In the finite-dimensional case (matrix algebra), Theorem 6 recovers a result of Penrose ([11], Theorem 3.5), which says that, if (a_n) converges to a, then (a_n^+) converges to (a^+) if and only if eventually $\mathrm{rank}(a_n) = \mathrm{rank}(a)$. We can also see that the only normal elements at which the Moore–Penrose inverse is continuous are invertible: more generally, if the conorm γ is continuous at $a \in A$ then, using (6.4) and (6.5),

(6.6) $$0 \neq a \in aAa \text{ and } a \in \mathrm{cl}(A^{-1}) \Rightarrow a \in A^{-1}.$$

We can improve Theorem 6: the regularity of a actually follows from the convergence $a_n \to a$ of a regular sequence (a_n) satisfying the condition (6.3). We begin by showing that the conorm is everywhere upper semi-continuous:

7. THEOREM. *The conorm is upper semi-continuous on $A \setminus \{0\}$:*

(7.1) $$\|a_n - a\| \to 0 \Rightarrow \limsup_n \gamma(a_n) \leq \gamma(a).$$

Proof. We claim that for each $k > 0$,

(7.2) $$\{a \in A : \gamma(a) \geq k\} \text{ is closed in } A,$$

and show first that the analogue of (7.2) holds in the subset A^+ of positive elements of A. If $a \in A^+$ then $\gamma(a) \geq k$ if and only if

(7.3) $$]0, k[\subseteq \mathbb{C} \setminus \sigma(a)$$

and hence by (2.11),

(7.4) $$0 < \lambda < k \Rightarrow \|(a-\lambda)^{-1}\| = |(a-\lambda)^{-1}|_\sigma \leq \frac{1}{\min(\lambda, k-\lambda)}.$$

If, more generally, $a \in A^+$ lies in the closure of the set of positive elements satisfying (7.3) then whenever $0 < \lambda < k$ there is $b \in A^+$ with $b - \lambda \in A^{-1}$ and $\|b - a\| < \min(\lambda, k-\lambda)$, giving

$$a - \lambda = (b-\lambda)(1 + (b-\lambda)^{-1}(a-b)) \in A^{-1}.$$

This means that (7.3) holds for a. (7.2) follows: the set of elements $a \in A$ with $\gamma(a) \geq k$ is the counterimage, under the continuous mapping $x \mapsto x^*x$, of the set of positive elements $b \in A^+$ with $\gamma(b) \geq k^2$. Now (7.1) is clear. ■

In the special case $A = BL(X, X)$ of operators, (7.3) is given by Apostol ([1], Corollary 1.2). Theorem 7 gives the improved version of Theorem 6:

8. THEOREM. *If $a \in A$ and (a_n) in A with $a_n \to a$ and $a_n \in a_nAa_n$ for each $n \in \mathbb{N}$ then*

(8.1) $$\liminf_n \|a_n^+\| < \infty \Rightarrow a \in aAa,$$

and hence

(8.2) $$\sup_n \|a_n^+\| < \infty \Rightarrow \exists a^+ = \lim_n a_n^+.$$

Proof. Observe, using (2.2),

$$\gamma(a) \geq \limsup_n \gamma(a_n) \geq 1/\liminf_n \|a_n^+\| > 0.$$

This makes $a \in A$ regular, giving (8.1); for (8.2) apply Theorem 6. ■

We shall call an element $a \in A$ *semi-invertible* if it has either a left inverse or a right inverse; for the full algebra of bounded operators on Hilbert space these are the only non-trivial continuity points of the conorm. We look first at general normed spaces:

9. THEOREM. *The reduced minimum modulus is continuous on the open sets of bounded below and of almost open operators between a pair of normed spaces. If $T : X \to Y$ is a bounded linear operator between normed spaces for which*

(9.1) $$\gamma(T) > 0 \quad \textit{and} \quad T^{-1}(0) \neq \{0\} \quad \textit{and} \quad \mathrm{cl}\, T(X) \neq Y,$$

then T is not a continuity point of the reduced minimum modulus γ.

Proof. If $T : X \to Y$ is either bounded below or almost open there is implication (cf. Lemma B.3.11 of [12])

$$\|S-T\| < \gamma(T) \Rightarrow |\gamma(S)-\gamma(T)| \le \|S-T\|\,; \tag{9.2}$$

for if $x \in X$ is arbitrary then

$$\|S-T\| < \gamma(T) \Rightarrow (\gamma(T) - \|S-T\|)\|x\| \le \|Sx\|\,,$$

giving $\gamma(T) - \|S-T\| \le \gamma(S)$, while at the same time

$$\gamma(S)\|x\| \le \|Sx\| \le \|Tx\| + \|S-T\|\,\|x\|\,,$$

giving $\gamma(S) - \|S-T\| \le \gamma(T)$. This proves that the reduced minimum modulus is continuous on the open set of bounded below operators, and hence by duality on the almost open operators (note ([9], Theorem IV.1.8) that when $\gamma(T) > 0$ then $\gamma(T^\dagger) = \gamma(T)$). For the last part suppose $x \in X$, $y \in Y$, $f \in X^\dagger$ and $g \in Y^\dagger$ satisfy

$$gT = 0 = Tx \quad\text{and}\quad f(x) \ne 0 \ne g(y)\,.$$

If we put, for each $\varepsilon > 0$,

$$T_\varepsilon = T - \varepsilon f \odot y : z \mapsto Tz - \varepsilon f(z)y\,,$$

then $\|T_\varepsilon - T\| \to 0$ and $T_\varepsilon^{-1}(0) \subseteq f^{-1}(0)$, for if $z \in X$ is arbitrary then

$$T_\varepsilon z = 0 \Rightarrow Tz = \varepsilon f(z)y \Rightarrow 0 = g(Tz) = \varepsilon f(z)g(y)\,.$$

Now $\mathrm{dist}(x, f^{-1}(0)) \le \mathrm{dist}(x, T_\varepsilon^{-1}(0))$, giving

$$\mathrm{dist}(x, f^{-1}(0))\gamma(T_\varepsilon) \le \|T_\varepsilon x\| = \|(T_\varepsilon - T)x\| = \varepsilon|f(x)|\,\|y\| \to 0\,.$$

We conclude that, as $\varepsilon \to 0$, $T_\varepsilon \to T$ and $\gamma(T_\varepsilon) \to 0 \ne \gamma(T)$. ∎

Theorem 9 says the conorm will always be continuous at semi-invertible elements, and the upper semi-continuity of Theorem 7 says that the conorm is continuous at elements with no generalized inverses. When A is the bounded operators then this is all; in more general algebras we can expect the situation to be more complicated. For example, if

$$a = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in A = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$$

with $vu = 1 \ne uv$ in D then the conorm on A will be continuous at a, which will be regular but not semi-invertible.

We conclude by considering when the Moore–Penrose inverse a^+ commutes with an element $a \in A$; as we have seen this occurs ([11], Theorem 10) when a is normal, and of course also when a is invertible. Kerry Brock [2] has found the answer for bounded operators:

10. Theorem. *If $a \in aAa$ is a regular element of a C^*-algebra A then the following are equivalent*:

$$aa^+ = a^+a\,; \tag{10.1}$$

$$a^{-1}(0) = a^{*-1}(0)\,; \tag{10.2}$$

$$a_{-1}(0) = a^*_{-1}(0)\,; \tag{10.3}$$

$$a \in A^{-1}a^*\,; \tag{10.4}$$

$$a \in a^*A^{-1}\,. \tag{10.5}$$

Proof. The argument divides neatly in two: if $a \in aAa$ in a C^*-algebra then

$$a^* \in A^{-1}a^+ \quad\text{and}\quad a^* \in a^+A^{-1}\,; \tag{10.6}$$

if $a = aba \in A$ with no restriction on A then there is implication

$$ba^2 = a = a^2b \Leftrightarrow ba = ab\,, \tag{10.7}$$

while if in addition $b = bab$ then

$$a \in Ab \Rightarrow b^{-1}(0) \subseteq a^{-1}(0) \Rightarrow a = a^2b \tag{10.8}$$

and

$$ab = ba \Leftrightarrow a \in A^{-1}b \text{ and } a \in bA^{-1}\,. \tag{10.9}$$

These are quickly checked: for example $aa^+ = (aa^+)^* = a^{+*}a^*$, giving $a^* = a^*aa^+ \in Aa^+$, and hence

$$a^* = (a^*a + 1 - a^+a)a^+ \text{ with } a^+a^{+*} + 1 - a^+a = (a^*a + 1 - a^+a)^{-1}\,, \tag{10.10}$$

giving (10.6). For (10.7) note that if $aba = a$ with $ba = ab$ then $ba^2 = aba = a$; conversely, if $a = ba^2 = a^2b$ then $ba = ba^2b = ab$. The first implication of (10.8) is immediate; if $b^{-1}(0) \subseteq a^{-1}(0)$ and $b = bab$ then $1 - ab \in a^{-1}(0)$, giving the second. Finally, if $ba = ab$ then

$$a = (a^2 + 1 - a^+a)b \text{ with } b^2 + 1 - a^+a = (a^2 + 1 - a^+a)^{-1}\,, \tag{10.11}$$

giving (10.9). ∎

There are also "one-sided" versions of Theorem 10, combining (10.6) and (10.8). It is sufficient for example for

$$a = a^+a^2 \tag{10.12}$$

that a be either left invertible, or *quasinormal* ([5], Problem 108) in the sense that

$$aa^*a = a^*a^2\,; \tag{10.13}$$

equivalently [5], a has a commuting "polar decomposition".

References

[1] C. Apostol, *The reduced minimum modulus*, Michigan Math. J. 32 (1985), 279–294.

[2] K. G. Brock, *A note on commutativity of a linear operator and its Moore–Penrose inverse*, Numer. Funct. Anal. Optim. 11 (1990), 673–678.

[3] S. R. Caradus, *Generalized Inverses and Operator Theory*, Queen's Papers in Pure and Appl. Math. 50, Queen's Univ., Kingston, Ont., 1978.

[4] C. W. Groetsch, *Representations of the generalized inverse*, J. Math. Anal. Appl. 49 (1975), 154–157.

[5] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, New York 1967.

[6] R. E. Harte, *Invertibility and Singularity*, Dekker, New York 1988.

[7] R. E. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math. 103 (1992), 71–77.

[8] R. E. Harte and M. Ó Searcóid, *Positive elements and the B^* condition*, Math. Z. 193 (1986), 1–9.

[9] T. Kato, *Perturbation Theory for Linear Operators*, Springer, New York 1966.

[10] M. Mbekhta, *Conorme et inverse généralisé dans les C^*-algèbres*, Canad. Math. Bull. 35 (1992), 515–522.

[11] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. 51 (1955), 406–413.

[12] A. Pietsch, *Operator Ideals*, North-Holland, 1980.

QUEEN'S UNIVERSITY
BELFAST BT7 1NN, UK

UNIVERSITÉ DE LILLE I
59655 VILLENEUVE D'ASCQ CEDEX, FRANCE

Received March 3, 1992
Revised version March 17, 1993 (2915)

STUDIA MATHEMATICA 106 (2) (1993)

On continuity properties of functions with conditions on the mean oscillation

by

HUGO AIMAR and LILIANA FORZANI (Santa Fe)

Abstract. In this paper we study distribution and continuity properties of functions satisfying a vanishing mean oscillation property with a lag mapping on a space of homogeneous type.

Since the initial works by F. John and L. Nirenberg and J. Moser in 1961, the study of regularity of functions with properties on their mean oscillation over balls was developed by S. Campanato, G. Meyers, S. Spanne and A. P. Calderón. Extensions from the euclidean setting to spaces of homogeneous type were considered by N. Burger, R. Macías and C. Segovia and one of the authors.

In 1967, J. Moser in his paper on Harnack's inequality for parabolic equations introduces a BMO type condition with a time lag. In 1985, E. Fabes and N. Garofalo, applying an extension of Calderón's method as stated by U. Neri obtained a John–Nirenberg type lemma for this parabolic case. In 1988 one of us proved an extension of these results to the setting of spaces of homogeneous type that can be applied to degenerate parabolic equations. Related results come from the analysis of one-sided maximal functions and weights; in a recent paper F. Martín-Reyes and A. de la Torre prove a John–Nirenberg type lemma for one-sided BMO functions.

In this paper we study distribution and continuity property of functions satisfying a vanishing mean oscillation property with a lag mapping on a space of homogeneous type.

1. Main results. Let X be a set. A symmetric function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is a *quasi-distance* on X if $d(x,y) = 0$ iff $x = y$ and there exists a constant K such that $d(x,z) \le K[d(x,y) + d(y,z)]$ for $x, y, z \in X$. The ball with center $x \in X$ and radius $r > 0$ is the set $B(x,r) = \{y \in X : d(x,y) < r\}$. We shall say that a positive measure μ defined on a σ-algebra contain-

1991 *Mathematics Subject Classification*: Primary 42B99.