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A multidimensional Lyapunov type theorem

by

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Abstract. Given functions $f_1, \dots, f_\nu \in \mathcal{L}^1(\mathbb{R}^n; \mathbb{R}^m)$, weights $p_1, \dots, p_\nu : \mathbb{R}^n \to [0,1]$ with $\sum p_i \equiv 1$, and any finite set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n \setminus \{0\}$, we prove the existence of a partition $\{A_1, \dots, A_\nu\}$ of $\mathbb{R}^n$ such that the two functions

$$f_p = \sum_{i=1}^{\nu} p_i f_i, \qquad f_A = \sum_{i=1}^{\nu} \chi_{A_i} f_i$$

have the same integral not only over $\mathbb{R}^n$, but also over every single line $x' + \mathbb{R}\mathbf{v}_j$, for each $j = 1, \dots, k$ and almost every x' in the orthogonal hyperplane $\mathbf{v}_j^\perp$. Equivalently, the Fourier transforms of f_p, f_A satisfy $\widehat{f}_p(y) = \widehat{f}_A(y)$ for every $y \in \bigcup \mathbf{v}_j^\perp$.

1. Introduction. Let $f_1, \dots, f_\nu$ be integrable functions from $\mathbb{R}^n$ into $\mathbb{R}^m$. If $p_1, \dots, p_\nu : \mathbb{R}^n \to [0,1]$ are measurable weights such that $\sum p_i(x) \equiv 1$, a well known version of Lyapunov's theorem [4, p. 453] states the existence of a measurable partition $\{A_1, \dots, A_\nu\}$ of $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \sum_{i=1}^{\nu} p_i(x) f_i(x)\, dx = \sum_{i=1}^{\nu} \int_{A_i} f_i(x)\, dx\,. \tag{1.1}$$

Now let v be any nonzero vector in $\mathbb{R}^n$ and denote by $v^\perp$ the hyperplane perpendicular to v. By Fubini's theorem, for almost every $x' \in v^\perp$, the maps $\lambda \mapsto f_i(x' + \lambda v)$, $i = 1, \dots, \nu$, are all integrable. Applying the above theorem to each single line $\{x' + \lambda v : \lambda \in \mathbb{R}\}$, one obtains the existence of a partition $\{A_1, \dots, A_\nu\}$ such that

$$\int_{-\infty}^{\infty} \sum_{i=1}^{\nu} p_i(x' + \lambda v) f_i(x' + \lambda v)\, d\lambda = \sum_{i=1}^{\nu} \int_{x' + \lambda v \in A_i} f_i(x' + \lambda v)\, d\lambda \quad \text{for a.e. } x' \in v^\perp\,. \tag{1.2}$$

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Of course, "almost everywhere" refers here to the $(n-1)$-dimensional Lebesgue measure on the hyperplane $v^\perp$.

In this paper, given any finite family $\mathcal{F}$ of nonzero vectors in $\mathbb{R}^n$, we prove the existence of a partition $\{A_1, \dots, A_\nu\}$ such that (1.2) holds *simultaneously* for every $v \in \mathcal{F}$.

THEOREM. *Let $f_1, \dots, f_\nu \in \mathcal{L}^1(\mathbb{R}^n; \mathbb{R}^m)$. Let $\overline{p}_1, \dots, \overline{p}_\nu : \mathbb{R}^n \to [0,1]$ be measurable functions such that $\sum \overline{p}_i(x) = 1$ for all $x \in \mathbb{R}^n$. Given any finite set $\{v_1, \dots, v_\mu\}$ of nonzero vectors in $\mathbb{R}^n$, there exists a measurable partition $\{A_1, \dots, A_\nu\}$ of $\mathbb{R}^n$ such that, for every $j = 1, \dots, \mu$, one has*

$$\int_{-\infty}^{\infty} \sum_{i=1}^{\nu} \overline{p}_i(x' + \lambda v_j) f_i(x' + \lambda v_j)\, d\lambda = \sum_{i=1}^{\nu} \int_{x'+\lambda v_j \in A_i} f_i(x' + \lambda v_j)\, d\lambda \tag{1.3}$$

for almost every x' in the orthogonal hyperplane $v_j^\perp$.

The proof will be worked out in the next two sections. Applications of this theorem include multidimensional Aumann integrals and nonconvex optimal control problems for semilinear hyperbolic systems of partial differential equations in one space variable. These are considered in the forthcoming paper [3].

Several different approaches to the original theorem of Lyapunov [7] on the range of a vector measure can be found in [1, 5, 6, 8, 9].

2. Two geometric lemmas

LEMMA 1. *For any integer $m \geq 1$ and every finite set $\{v_1, \dots, v_\mu\} \subset \mathbb{R}^n \setminus \{0\}$, there exists a finite set $\mathcal{S} \subset \mathbb{R}^n$ such that*

$$\#(\mathcal{S}) > m \sum_{j=1}^{\mu} \#(\mathcal{S}_j)\,. \tag{2.1}$$

Here $\#(\mathcal{S})$ denotes the number of points in $\mathcal{S}$, while $\#(\mathcal{S}_j)$ counts the number of distinct lines parallel to v_j which intersect $\mathcal{S}$.

In the case where the vectors $v_1, \dots, v_\mu$ are linearly independent, the proof of the lemma is quite easy. Indeed, the finite lattice

$$\mathcal{S} \doteq \Big\{ \sum_{j=1}^{\mu} c_j v_j : c_j \in \{0, \dots, m\mu\} \Big\}$$

satisfies

$$\#(\mathcal{S}) = (1+m\mu)^\mu > m\mu(1+m\mu)^{\mu-1} = m \sum_{j=1}^{\mu} \#(\mathcal{S}_j)\,.$$

To cover the general case, let $\{\mathbf{e}_1, \dots, \mathbf{e}_\mu\}$ be the standard basis in $\mathbb{R}^\mu$ and consider the finite set

$$\mathcal{G} \doteq \Big\{ \sum_{j=1}^{\mu} c_j \mathbf{e}_j : c_j \in \mathbb{Z},\ 0 \leq c_j \leq m\mu,\ \forall j \Big\} \subset \mathbb{R}^\mu\,. \tag{2.2}$$

We now construct a linear map $\Lambda : \mathbb{R}^\mu \to \mathbb{R}^n$ with the properties:

(i) for each j, $\Lambda(\mathbf{e}_j)$ is parallel to v_j,
(ii) the restriction of Λ to $\mathcal{G}$ is one-to-one.

Let w be a vector such that $\langle w, v_j \rangle \neq 0$ for every j. Fix any transcendental irrational number $\xi \in \mathbb{R}$ and choose constants α_j so that

$$\langle w, \alpha_j v_j \rangle = \xi^j, \quad j = 1, \dots, \mu\,. \tag{2.3}$$

If $y = \sum b_j \mathbf{e}_j$, we now define

$$\Lambda(y) \doteq \sum_{j=1}^{\mu} \alpha_j b_j v_j\,. \tag{2.4}$$

The linear map Λ then satisfies both requirements. Indeed, by (2.4), $\Lambda(\mathbf{e}_j) = \alpha_j v_j$, hence (i) holds. If (ii) fails, then there exist integer coefficients $c_1, \dots, c_\mu$, not all zero, such that $\Lambda(\sum c_j \mathbf{e}_j) = 0$. From (2.3) and (2.4) it follows that

$$0 = \Big\langle w, \Lambda\Big(\sum_{j=1}^{\mu} c_j \mathbf{e}_j\Big)\Big\rangle = \Big\langle w, \sum_{j=1}^{\mu} \alpha_j c_j v_j \Big\rangle = \sum_{j=1}^{\mu} c_j \langle w, \alpha_j v_j \rangle = \sum_{j=1}^{\mu} c_j \xi^j\,. \tag{2.5}$$

According to (2.5), the transcendental number ξ is a root of a polynomial with integer coefficients. This is absurd, hence (ii) must hold.

We claim that the set $\mathcal{S} = \Lambda(\mathcal{G})$ satisfies (2.1). Indeed, if

$$\overline{x} = \sum_{j=1}^{\mu} \alpha_j \overline{c}_j v_j \in \mathcal{S}\,,$$

then the $m\mu + 1$ distinct points

$$\alpha_j c v_j + \sum_{l \neq j} \alpha_l \overline{c}_l v_l, \qquad c = 0, \dots, m\mu\,,$$

all lie on the same line through $\overline{x}$, parallel to v_j. This implies

$$\#(\mathcal{S}) \geq \#(\mathcal{S}_j) \cdot (1 + m\mu)$$

and hence (2.1), because

$$\#(\mathcal{S}) = \#(\mathcal{G}) = (1+m\mu)^\mu > m\mu(1+m\mu)^{\mu-1}$$
$$= m \sum_{j=1}^{\mu} \frac{\#(\mathcal{S})}{1+m\mu} \geq m \sum_{j=1}^{\mu} \#(\mathcal{S}_j)\,.$$

LEMMA 2. *Let $K \subset \mathbb{R}^n$ be a compact set with positive measure and let $\mathcal{S} = \{z_1, \ldots, z_N\}$ be any finite set. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ and a compact $K^* \subseteq K$ whose measure and diameter satisfy*

$$\text{meas}(K^*) > 0, \qquad \text{diam}(K^*) < \varepsilon\delta\,, \tag{2.6}$$

such that the sets $\delta z_i + K^* = \{x : x - \delta z_i \in K^*\}$, for $i = 1, \ldots, N$, are all contained in K.

Proof. For any vector $z \in \mathbb{R}^n$, writing $\chi_{\delta z+K}$ for the characteristic function of the set $\delta z + K$, one has

$$\lim_{\delta\to 0} \|\chi_K - \chi_{\delta z+K}\|_{\mathcal{L}^1} = 0\,. \tag{2.7}$$

Indeed, for every $\varepsilon > 0$ there exists a continuous function ϕ with compact support such that $\|\phi - \chi_K\|_{\mathcal{L}^1} < \varepsilon$. Defining $\phi_\delta(x) \doteq \phi(x - \delta z)$, we have

$$\begin{aligned}\limsup_{\delta\to 0} &\|\chi_K - \chi_{\delta z+K}\|_{\mathcal{L}^1}\\ &\leq \|\phi - \chi_K\|_{\mathcal{L}^1} + \|\phi_\delta - \chi_{\delta z+K}\|_{\mathcal{L}^1} + \limsup_{\delta\to 0} \|\phi - \phi_\delta\|_{\mathcal{L}^1} \leq 2\varepsilon\,.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this yields (2.7).

Using (2.7), we can now choose $\delta > 0$ sufficiently small so that

$$\sum_{i=1}^{N} \|\chi_K - \chi_{\delta z_i+K}\|_{\mathcal{L}^1} < \text{meas}(K)\,. \tag{2.8}$$

Consider the set

$$K' = \{x \in K : x + \delta z_i \notin K \text{ for some } i\}\,.$$

By (2.8), $\text{meas}(K') < \text{meas}(K)$, hence the set-theoretic difference $K \setminus K'$ has positive measure. Any compact set $K^* \subseteq K \setminus K'$ with positive measure and diameter smaller than $\varepsilon\delta$ clearly satisfies the conclusion of the lemma.

3. Proof of the theorem. Let $f_1, \ldots, f_\nu \in \mathcal{L}^1(\mathbb{R}^n; \mathbb{R}^m)$ be given, together with measurable weights $\overline{p}_1, \ldots, \overline{p}_\nu$. Define $\mathcal{W}$ to be the family of all weights $p = (p_1, \ldots, p_\nu) : \mathbb{R}^n \to [0,1]^\nu$ such that $\sum p_i(x) \equiv 1$ and, for every $j = 1, \ldots, \mu$, one has

$$\begin{aligned}\int_{-\infty}^{\infty} \sum_{i=1}^{\nu} p_i(x' + \lambda v_j) f_i(x' + \lambda v_j)\, d\lambda\\ = \int_{-\infty}^{\infty} \sum_{i=1}^{\nu} \overline{p}_i(x' + \lambda v_j) f_i(x' + \lambda v_j)\, d\lambda\end{aligned} \tag{3.1}$$

for a.e. $x' \in v_j^\perp$. The theorem will be proved by showing that $\mathcal{W}$ contains an extreme point $(p_1^*, \ldots, p_\nu^*)$, which in turn must satisfy

$$p_i^*(x) \in \{0,1\} \quad \forall i \in \{1, \ldots, \nu\}, \text{ for a.e. } x \in \mathbb{R}^n\,. \tag{3.2}$$

1. Observe that $\mathcal{W}$ is a bounded, convex subset of $\mathcal{L}^\infty(\mathbb{R}^n; \mathbb{R}^\nu)$. To prove that it is also weak* closed, let $(p^l)_{l\geq 1}$ be a sequence in $\mathcal{W}$, converging to p^∞ in the weak* topology. For every $j \in \{1, \ldots, \mu\}$ and every measurable set A contained in the $(n-1)$-dimensional hyperplane $v_j^\perp$, we then have

$$\begin{aligned}\int_{v_j^\perp} \Big\{ \int_{-\infty}^{\infty} \sum_{i=1}^{\nu} p_i^\infty(x' + \lambda v_j) f_i(x' + \lambda v_j)\, d\lambda \Big\} \chi_A(x')\, dx'\\ = \lim_{l\to\infty} \int_{v_j^\perp} \int_{-\infty}^{\infty} \sum_{i=1}^{\nu} p_i^l(x' + \lambda v_j)[f_i(x' + \lambda v_j)\chi_A(x')]\, d\lambda\, dx'\\ = \int_{v_j^\perp} \Big\{ \int_{-\infty}^{\infty} \sum_{i=1}^{\nu} \overline{p}_i(x' + \lambda v_j) f_i(x' + \lambda v_j)\, d\lambda \Big\} \chi_A(x')\, dx'\,.\end{aligned} \tag{3.3}$$

Since A is arbitrary, from the above equalities we conclude

$$\int_{-\infty}^{\infty} \sum_{i=1}^{\nu} p_i^\infty(x' + \lambda v_j) f_i(x' + \lambda v_j)\, d\lambda = \int_{-\infty}^{\infty} \sum_{i=1}^{\nu} \overline{p}_i(x' + \lambda v_j) f_i(x' + \lambda v_j)\, d\lambda$$

for almost every $x' \in v_j^\perp$; therefore $p^\infty \in \mathcal{W}$. This proves that $\mathcal{W}$ is closed, hence weak* compact. By the theorem of Krein–Milman, $\mathcal{W}$ contains an extreme point, say $p^* = (p_1^*, \ldots, p_\nu^*)$.

2. We claim that the weights p_i^* take values in $\{0,1\}$ for almost every x. Indeed, if (3.2) fails, by Lusin's theorem there exists $\sigma > 0$, two distinct indices $h, k \in \{1, \ldots, \nu\}$ and a compact set $K \subset \mathbb{R}^n$ with positive measure such that

(i) the maps f_h, f_k, p_h^*, p_k^* are continuous when restricted to K,
(ii) $p_h^*(x), p_k^*(x) \in [\sigma, 1-\sigma]$ for every $x \in K$.

Let $\mathcal{S} = \{z_1, \ldots, z_N\}$ be the finite set considered in Lemma 1. Let ε be the minimum distance between any two distinct lines, both parallel to some v_j, intersecting $\mathcal{S}$. Otherwise stated:

$$\begin{aligned}\varepsilon \doteq \min\{|z - z' + \lambda v_j| :\ z, z' \in \mathcal{S},\ \lambda \in \mathbb{R},\ j = 1, \ldots, \mu,\\ z - z' \text{ is not parallel to } v_j\}\,.\end{aligned} \tag{3.4}$$

Since $\mathcal{S}$ is finite, $\varepsilon > 0$. Applying Lemma 2 we now obtain the existence of $\delta > 0$ and $K^* \subseteq K$ satisfying (2.6), such that the sets $K^* + \delta z_i$, $i = 1, \ldots, N$, are all contained in K. Observe that the choice of ε implies that these N sets are mutually disjoint. Indeed, if $\xi + \delta z_i = \xi' + \delta z_j$ for some $i \neq j$, then a contradiction is reached because

$$\varepsilon\delta \leq |\delta z_i - \delta z_j| = |\xi - \xi'| \leq \text{diam}(K^*) < \varepsilon\delta\,.$$

For each $\xi \in K^*$ we now consider a system of linear equations in N scalar variables $\theta_1, \dots, \theta_N$. Denote by $\pi_j : \mathbb{R}^n \to v_j^\perp$ the orthogonal projections. For each $x' \in v_j^\perp$, if the intersection of the line $\pi_j^{-1}(x')$ with the finite set $\xi + \delta\mathcal{S}$ is not empty, consider the equation

$$\sum_{\pi_j(\xi+\delta z_i)=x'} \theta_i [f_h(\xi + \delta z_i) - f_k(\xi + \delta z_i)] = 0 \in \mathbb{R}^m . \tag{3.5}$$

For each j, the number of nontrivial equations of the form (3.5) equals the number of distinct lines parallel to v_j which intersect $\xi + \delta\mathcal{S}$, i.e. $\#(\mathcal{S}_j)$. Observing that each vector equation in (3.5) is equivalent to m scalar equations, as j varies in $\{1, \dots, \mu\}$ we obtain a linear homogeneous system of $m \sum \#(\mathcal{S}_j)$ scalar equations in the $N = \#(\mathcal{S})$ real variables θ_i. Because of (2.1), this system has at least one nontrivial solution $(\theta_1, \dots, \theta_N)$. Since the coefficients in (3.5) are continuous functions of $\xi \in K^*$, the set of all solutions of this linear homogeneous system which satisfy the additional constraint

$$\max\{|\theta_1|, \dots, |\theta_N|\} = \sigma \tag{3.6}$$

is compact, nonempty, and depends on ξ in an upper semicontinuous way. By a standard selection theorem [2, p. 90], we can thus construct a solution $(\theta_1(\xi), \dots, \theta_N(\xi))$, satisfying (3.6), which depends measurably on the parameter $\xi \in K^*$.

Two new weight functions $p^+(\cdot)$, $p^-(\cdot)$ can now be defined by setting:

$$p_i^+(x) = p_i^-(x) = p_i^*(x) \quad \text{if } i \neq h, k \text{ or } x \notin K^* + \delta\mathcal{S}\,, \tag{3.7}$$

$$p_h^\pm(x) = p_h^*(x) \pm \theta_i(\xi), \quad p_k^\pm(x) = p_k^*(x) \mp \theta_i(\xi) \tag{3.8}$$
$$\text{if } x = \xi + \delta z_i \text{ for some } \xi \in K^*,\ i \in \{1, \dots, N\}\,.$$

Since the sets $K^* + \delta z_i$ are mutually disjoint, the definition (3.8) is meaningful. Clearly, $\sum p_i^+(x) = \sum p_i^-(x) = 1$ for every x. By (3.6), $p_h^\pm(x), p_k^\pm(x) \in [0,1]$. Moreover, p^+, p^- are both in $\mathcal{W}$. Indeed, this can be established by proving that

$$\int_{x'+\lambda v_j \in K^*+\delta\mathcal{S}} \theta(x' + \lambda v_j)[f_h(x' + \lambda v_j) - f_k(x' + \lambda v_j)]\, d\lambda = 0 \tag{3.9}$$

for every j and every $x' \in v_j^\perp$, where

$$\theta(x) = \begin{cases} \theta_i(\xi) & \text{if } x = \xi + \delta z_i \text{ for some } \xi \in K^*,\ i \in \{1, \dots, N\}, \\ 0 & \text{otherwise.} \end{cases}$$

If $(x' + \mathbb{R}v_j) \cap (K^* + \delta\mathcal{S}) = \emptyset$, the equality (3.9) is trivial. In the other case, choose any $\xi \in K^*$, $z \in \mathcal{S}$ such that $\pi_j(\xi + \delta z) = x'$ and set $x'' = \pi_j(\xi)$. Observe that x'' is uniquely defined. Indeed, if

$$\pi_j(\xi + \delta z) = \pi_j(\xi' + \delta z'), \qquad \pi_j(\xi) \neq \pi_j(\xi')$$

for some $\xi, \xi' \in K^*$, $z, z' \in \mathcal{S}$, then $\pi_j(\delta z) \neq \pi_j(\delta z')$. The choice of ε in (3.4) thus implies

$$|\pi_j(\xi - \xi')| = |\pi_j(\delta z) - \pi_j(\delta z')| \geq \varepsilon\delta\,,$$

contrary to the assumption $\operatorname{diam}(K^*) < \varepsilon\delta$.

The left hand side of (3.9) can now be rewritten as

$$\int_{x''+\lambda v_j \in K^*} \sum_{i \in I(x')} \theta_i(x'' + \lambda v_j)[f_h(x'' + \lambda v_j + \delta z_i) - f_k(x'' + \lambda v_j + \delta z_i)]\, d\lambda\,, \tag{3.10}$$

where the set $I(x')$ of indices is defined as

$$I(x') = \{i \in \{1, \dots, N\} : \pi_j(x'' + \delta z_i) = x'\}\,.$$

By construction, the functions θ_i satisfy (3.5), hence the integrand in (3.10) vanishes identically and (3.9) holds. This proves that $p^+, p^- \in \mathcal{W}$.

The definitions (3.7), (3.8) imply $p^* = (p^+ + p^-)/2$, in contradiction with the extremality of p^*. Therefore, (3.2) must hold. The partition $\{A_1, \dots, A_\nu\}$ defined by

$$A_i = \{x \in \mathbb{R}^n : p_i^*(x) = 1\}$$

clearly satisfies the conclusions of the theorem.

4. An example. On the unit square $Q = [0,1] \times [0,1]$, define the functions

$$f_1(x,y) \equiv 1, \qquad f_2(x,y) \equiv 0$$

and the weights

$$p_1(x,y) = x, \qquad p_2(x,y) = 1 - x\,.$$

An application of our theorem with $v_1 = (1,0)$, $v_2 = (0,1)$ yields the existence of a subset $A_1 \subset Q$ whose horizontal and vertical sections satisfy

$$\operatorname{meas}\{y : (x,y) \in A_1\} = x \quad \text{for a.e. } x \in [0,1]\,,$$

$$\operatorname{meas}\{x : (x,y) \in A_1\} = \frac{1}{2} \quad \text{for a.e. } y \in [0,1]\,.$$

This can be accomplished, for example, by choosing

$$A_1 = \{(x,y) : x \leq y \leq 2x \text{ or } y \leq 2x - 1\} \cap Q\,.$$

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Generalized inverses in C^*-algebras II

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Abstract. Commutativity and continuity conditions for the Moore–Penrose inverse and the "conorm" are established in a C^*-algebra; moreover, spectral permanence and B^*-properties for the conorm are proved.

Suppose A is a ring, with identity 1 and invertible group A^{-1} (more generally, an "additive category"); then an element $a \in A$ will be called *regular* if it has a *generalized inverse* in A, $b \in A$ for which

$$a = aba\,. \tag{0.1}$$

It is clear that both products

$$ba = (ba)^2 \quad \text{and} \quad ab = (ab)^2 \tag{0.2}$$

are *idempotents* of A; in the presence of an *involution* $* : A \to A$ we can also enquire whether or not they are *self-adjoint*: when $(ba)^* = ba$ and $(ab)^* = ab$ then (provided also $b = bab$) the generalized inverse is called a *Moore–Penrose inverse* for A. If this exists then ([7], Theorem 5) it is uniquely determined, and lies in the *double commutant* of a and a^*; when A is a C^*-algebra then ([7], Theorem 6) every regular element has a Moore–Penrose inverse. We write a^+ for the Moore–Penrose inverse of $a \in A$; thus

$$a = aa^+a;\quad a^+ = a^+aa^+;\quad (a^+a)^* = a^+a;\quad (aa^+)^* = aa^+\,. \tag{0.3}$$

By the uniqueness it is clear that

$$(a^*)^+ = (a^+)^*\,. \tag{0.4}$$

We recall also that, in a C^*-algebra A, necessary and sufficient for an element $a \in A$ to be regular (and hence have a Moore–Penrose inverse) is ([7], Theorems 2 and 8) that the range ideal be closed:

$$aA = \mathrm{cl}\, aA\,. \tag{0.5}$$

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