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Weighted Bergman projections and tangential area integrals

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Abstract. Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. In this paper we find sufficient conditions on a function f defined on Ω in order that the weighted Bergman projection $P_s f$ belong to the Hardy–Sobolev space $H_k^p(\partial\Omega)$. The conditions on f we consider are formulated in terms of tent spaces and complex tangential vector fields. If f is holomorphic then these conditions are necessary and sufficient in order that f belong to the Hardy–Sobolev space $H_k^p(\partial\Omega)$.

0. Introduction. Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^\infty$ boundary $\partial\Omega$. Thus, we suppose that there is a $\mathcal{C}^\infty$ defining function $r : \mathbb{C}^n \to \mathbb{R}$, a neighborhood $\mathcal{O}$ of $\overline{\Omega}$, the closure of Ω, and a constant $C > 0$ such that

$$\Omega = \{\zeta : r(\zeta) < 0\}, \quad \partial\Omega = \{\zeta : r(\zeta) = 0\},$$

$$|\nabla r| \neq 0 \quad \text{everywhere on } \partial\Omega,$$

and that the Levi form of r is positive definite on $\mathcal{O}$:

$$\sum_{j,k} \frac{\partial^2 r}{\partial\zeta_j \partial\overline{\zeta}_k}(\zeta) w_j \overline{w}_k \geq C|w|^2$$

for all $\zeta \in \mathcal{O}$ and all $w \in \mathbb{C}^n$.

If F is a function on Ω, we will say that F belongs to the *tent space* $T_2^p(\Omega)$ if the admissible area function

$$AF(\eta) = \left(\int_{\Gamma(\eta)} |F(z)|^2 \frac{dm(z)}{|r(z)|^{n+1}} \right)^{1/2} \tag{0.1}$$

defined for each $\eta \in \partial\Omega$ belongs to $L^p(d\sigma)$. Here, dm denotes Lebesgue measure on $\mathbb{C}^n$, $d\sigma$ denotes the "surface area measure" on $\partial\Omega$, and $\Gamma(\eta)$ is the admissible approach region we specify in Section 2 below. Tent spaces were defined and studied in the context of the upper half space $\mathbb{R}_+^{n+1}$ by

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Coifman, Meyer, and Stein in [CMSt]. Some of their results were generalized to contexts including the present one by Ahern and Nagel in [AN]. It can be shown that most of the results in [CMSt] have analogues in the present setting and we will comment on these when the need arises. It seems to be well known that tent spaces provide a convenient characterization for the Hardy classes H^p of holomorphic functions defined on Ω; see the discussions in [AB], [B] and [St2]. To state this characterization, for a smooth function f defined on Ω, let $\mathcal{D}_k f(z)$ denote the sum of the absolute values of all derivatives of f of order less than or equal to k evaluated at $z \in \Omega$.

THEOREM A. *A holomorphic function F defined on Ω belongs to the Hardy class H^p where $0 < p < \infty$ if and only if*

$$r\mathcal{D}_1 F \in T_2^p(\Omega).$$

More generally, we say that F belongs to the *Hardy–Sobolev space* $H_k^p(\Omega)$ if all derivatives of F of order less than or equal to k belong to $H^p(\Omega)$. It will be convenient to use multi-index notation: if $I = (k_1, \ldots, k_n)$ where each k_i is a nonnegative integer, then

$$D_I = \frac{\partial^{|I|}}{\partial\zeta_1^{k_1}\ldots\partial\zeta_n^{k_n}} \quad \text{and} \quad \overline{D}_I = \frac{\partial^{|I|}}{\partial\overline{\zeta}_1^{k_1}\ldots\partial\overline{\zeta}_n^{k_n}},$$

where $|I| = k_1 + \ldots + k_n$. The following tent space characterization of H_k^p is well known.

THEOREM B. *A holomorphic function F defined on Ω belongs to $H_k^p(\Omega)$ if and only if*

$$r^m\mathcal{D}_{k+m}F \in T_2^p(\Omega),$$

where m is any positive integer.

If $\zeta \in \mathcal{O}$ and $r(\zeta) = t$ let $\partial\Omega_t$ be the boundary of the domain $\{z : r(z) < t\}$. Let $\overline{D}r(\zeta)$ denote the normal to $\partial\Omega_t$,

$$\overline{D}r(\zeta) = \left(\frac{\partial r}{\partial\overline{\zeta}_1}(\zeta), \ldots, \frac{\partial r}{\partial\overline{\zeta}_n}(\zeta)\right).$$

We have the usual two vector fields associated with $\overline{D}r$,

$$N_1 = \sum_{j=1}^{n}\left(\frac{\partial r}{\partial\overline{\zeta}_j}\frac{\partial}{\partial\zeta_j} + \frac{\partial r}{\partial\zeta_j}\frac{\partial}{\partial\overline{\zeta}_j}\right)$$

and

$$N_2 = i\sum_{j=1}^{n}\left(\frac{\partial r}{\partial\overline{\zeta}_j}\frac{\partial}{\partial\zeta_j} - \frac{\partial r}{\partial\zeta_j}\frac{\partial}{\partial\overline{\zeta}_j}\right), \tag{0.2}$$

where N_1 corresponds to differentiation in the direction normal to $\partial\Omega_t$, while N_2 corresponds to differentiation in the direction tangential to $\partial\Omega_t$ determined by $i\overline{D}r$. Notice that if F is holomorphic then

$$N_1F(\zeta) = iN_2F(\zeta).$$

It can be shown that if $0 < p < \infty$ and if F is holomorphic on Ω then F is "controlled" by the behavior of its normal derivatives in the sense that $F \in H_k^p(\Omega)$ if and only if

$$rN_1^jF \in T_2^p(\Omega)$$

for $j = 0, \ldots, k+1$. This actually follows from a more general result of Beatrous [B] which holds for harmonic functions: if u is a harmonic function on a domain D in $\mathbb{R}^N$ with C^2 boundary and if X is a transverse vector field on D then u belongs to $H_k^p(D)$ if and only if the nontangential area function of $rX^{k+1}u$ belongs to $L^p(d\sigma)$.

We next recall what can be said about derivatives of holomorphic functions in directions that are tangential to $\partial\Omega_t$. It is well known that if $\mathbf{T}_\zeta(\partial\Omega_t)$ denotes the (complexified) tangent space of $\partial\Omega_t$ at ζ then

$$\mathbf{T}_\zeta(\partial\Omega_t) = \mathbb{C}^{n-1}(\zeta) \oplus \overline{\mathbb{C}^{n-1}}(\zeta) \oplus \mathbb{R}(i\overline{D}r(\zeta)),$$

where $\mathbb{C}^{n-1}(\zeta)$ denotes the orthogonal complement of the complex span of $\{\overline{D}r(\zeta)\}$ in $\mathbb{C}^n$ and $\mathbb{R}(i\overline{D}r(\zeta))$ denotes the real span of $i\overline{D}r(\zeta)$.

Suppose then that v is a C^∞ mapping from $\mathcal{O}$ to $\mathbb{C}^n$. Write

$$v(\zeta) = (v_1(\zeta), \ldots, v_n(\zeta)).$$

Then v determines the vector fields

$$T_v = \sum_{j=1}^{n} v_j(\zeta)\frac{\partial}{\partial\zeta_j} \quad \text{and} \quad \overline{T}_v = \sum_{j=1}^{n}\overline{v}_j(\zeta)\frac{\partial}{\partial\overline{\zeta}_j}.$$

Let $\langle\,,\rangle$ denote the Hermitian inner product on $\mathbb{C}^n$. If v is "complex tangential" to $\partial\Omega_t$, i.e.

$$\langle v(\zeta), \overline{D}r(\zeta)\rangle = 0$$

for all ζ, then it follows that

$$v(\zeta) \in \mathbb{C}^{n-1}(\zeta), \quad \overline{v}(\zeta) \in \overline{\mathbb{C}^{n-1}}(\zeta),$$

for all ζ on $\partial\Omega_t$, and the restrictions of both T_v and $\overline{T}_v$ to $\partial\Omega_t$ define vector fields on the manifold $\partial\Omega_t$. It is natural to say therefore that the vector fields T_v and $\overline{T}_v$ are *complex tangential* on Ω. Any such complex tangential vector field can be written as a linear combination (with C^∞ coefficients) of the vector fields

$$T_{i,j} = \frac{\partial r}{\partial\zeta_j}\frac{\partial}{\partial\zeta_i} - \frac{\partial r}{\partial\zeta_i}\frac{\partial}{\partial\zeta_j}$$

and

$$\overline{T}_{i,j} = \frac{\partial r}{\partial \overline{\zeta}_j}\frac{\partial}{\partial \overline{\zeta}_i} - \frac{\partial r}{\partial \overline{\zeta}_i}\frac{\partial}{\partial \overline{\zeta}_j},$$

where $1 \le i, j \le n$. The following characterization of H^p_k was given by Ahern and Bruna in [AB] for the case where Ω is the unit ball and by Grellier in [Gr] for the case where Ω is a strictly pseudoconvex domain. (Actually, Grellier obtains results for an even larger class of domains.)

THEOREM C. *Let* $\nabla^t_k F(z) = \sum |LF(z)|$ *where the sum is taken over all compositions* L *of* k *vector fields chosen from the collection of the vector fields* $T_{i,j}$. *Then a holomorphic function* F *belongs to* H^p_k *if and only if*

$$r^m \nabla^t_{2k+2m} F \in T^p_2(\Omega).$$

The statement of necessity in Theorem C illustrates the familiar principle that a holomorphic function behaves twice as nicely in the complex tangential directions as it does in the normal direction. The statement of sufficiency in the theorem is of interest because it only involves the vector fields $T_{i,j}$ and not their conjugates. Thus, the behavior of the functions LF, where L is as above, is enough to determine whether or not F is in the Hardy class H^p.

In this paper we consider an arbitrary (not necessarily holomorphic) function f defined on Ω. We want to find sufficient conditions on such a function that imply that certain weighted Bergman projections of f belong to the Hardy–Sobolev space H^p_k. It turns out that, in the spirit of Theorem C, we need only consider the behavior of the derivatives Lf where now we must let L be any composition of $2k + 2m$ or fewer vector fields $T_{i,j}$. If each function of the form $r^m Lf$ belongs to the tent space T^p_2 then this will be sufficient that a weighted Bergman projection of f belongs to H^p_k.

Our result differs from Theorem C in another way: we actually only work, locally at least, with a single complex tangential vector field $T = T_\nu$. Thus, there is some similarity between our results and those of Beatrous [B]. We point out, however, that we need to consider all the functions $f, Tf, T^2 f, \ldots, T^{2k+2m} f$ instead of just the ones where the operator has the highest order. Perhaps this is the price one must pay in order to get control in terms of a single vector field.

To make all this precise we recall the following definitions.

For $s > -1$ let $B_s(\zeta, z)$ be the Bergman kernel which gives the orthogonal projection of the space $L^2(dm_s)$ onto the space $H \cap L^2(dm_s)$. Here $dm_s = |r|^s\, dm$, $L^2(dm_s)$ is the space of functions f defined on Ω satisfying

$$\|f\|^2_{2,s} = \int_\Omega |f|^2\, dm_s < \infty,$$

and $H \cap L^2(dm_s)$ is the Bergman space of holomorphic functions in $L^2(dm_s)$. Thus, if $F \in H \cap L^2(dm_s)$ then

$$F(z) = \int_\Omega F(w) B_s(\zeta, z)\, dm_s(\zeta)$$

for all $z \in \Omega$. If $f \in L^1(dm_s)$ we define

$$P_s f(z) = \int_\Omega f(\zeta) B_s(\zeta, z)\, dm_s(\zeta). \tag{0.3}$$

Suppose that there is a finite covering of $\partial\Omega$ by open balls $\{\mathcal{O}_\nu\}$ and that for each ν there is a C^∞ mapping v^ν of $\mathbb{C}^n$ to $\mathbb{C}^n$ satisfying the following conditions:

$$|v^\nu(\zeta)| > 0 \quad \text{and} \quad \langle v^\nu(\zeta), \overline{D}r(\zeta)\rangle = 0 \quad \text{for all } \zeta \in \mathcal{O}_\nu.$$

Let each v^ν determine the complex tangential vector field $T_\nu = T_{v^\nu}$. Our main result is the following theorem.

THEOREM 1. *Suppose* $0 < p < \infty$, m *is a positive integer, and* s *is a positive real number satisfying the conditions* $(n + s - m + 1)p > n$ *and* $s - m > -1$. *Let* $\{\mathcal{O}_\nu\}$, $\{v^\nu\}$ *and* $\{T_\nu\}$ *be as above and suppose that* χ_ν *is the characteristic function of* $\mathcal{O}_\nu$. *Suppose that for each* ν, $r^m \chi_\nu T^j_\nu f \in T^p_2(\Omega)$ *for all* $j = 0, \ldots, 2k + 2m$. *Then* $P_s f \in H^p_k(\Omega)$.

Remark. We have assumed that the vector fields T_ν are globally defined in order that it make sense that a function of the form $T_\nu f$ belong to the tent space $T^p_2(\Omega)$ of functions globally defined on Ω. However, the crucial property of T_ν is that it is nonvanishing and complex tangential in the ball $\mathcal{O}_\nu$ and it is the behavior of the iterates of $T_\nu f$ in that ball which is important. This is why we can allow the factor χ_ν in the hypothesis of Theorem 1.

Theorem 1 has the following corollary.

COROLLARY. *Suppose* $0 < p < \infty$ *and* m *is a positive integer. Let* $\{\mathcal{O}_\nu\}$, $\{v^\nu\}$ *and* $\{T_\nu\}$ *be as above. Suppose that* F *is holomorphic on* Ω *and that for each* ν, $r^m \chi_\nu T^j_\nu F \in T^p_2(\Omega)$ *for all* $j = 0, \ldots, 2k+2m$. *Then* $F \in H^p_k(\Omega)$.

The remainder of the paper is organized as follows. Section 1 concerns the weighted Bergman kernels. We will use a modification of the formula obtained by Ligocka [L] which expresses the most singular part of the kernel $B_s(\zeta, z)$ in terms of the defining function r of the region Ω. The modification is needed to achieve the property summarized in Lemma 1.1. We will also need a more detailed expansion of the kernel $B_s(\zeta, z)$ analogous to the representation achieved by Kerzman and Stein [KSt] for the Szegö kernel. We will need this expansion in order to prove that certain kernels obtained from the Bergman kernel $B_s(\zeta, z)$ map the tent spaces T^p_2 to themselves.

These kernels arise in the following fashion. As in [C] the idea of the proof of Theorem 1 is to integrate by parts to represent the projection $P_s f(z)$ in terms of integrals of the functions $T_v^j f$ where $j = 0, \ldots, 2k+2m$ multiplied by kernels which are less singular than the kernel B_s. Differentiating $k+1$ times in the z variable then represents the derivatives of $P_s f(z)$ as integrals over Ω against the measure dm_{s-m} of $r^m T_v^j f$ multiplied by kernels which then have the same singularity as the kernel B_{s-m+1}. In order to use Theorem B we then multiply by $|r(z)|$ and are left with the task of showing that certain kernels with the same singularity as the unweighted Bergman kernel B_0 map $T_2^p(\Omega)$ to itself. This is made precise in Section 2. We should mention that in order to prove the T_2^2 boundedness result, as in [C] we appeal to a theorem of M. G. Krein (see [C], Theorem B, for a statement and [GK] for a proof) to deduce L^2 boundedness from the boundedness of an operator and its adjoint on certain Lipschitz spaces Λ_α. In the present context, as opposed to the context of [C] the Lipschitz spaces we consider consist of functions defined on Ω which vanish to a certain order at the boundary and are therefore dense in the space $L^2(dm_{-1})$.

Before continuing we remark that the following notational conventions will be used throughout the paper. The term "smooth" will mean C^∞. We will use the letter C to stand for various positive constants which change their values from context to context while remaining independent of the important variables. Finally, the relation

$$A \doteq B$$

means that there are constants C_1 and C_2 such that $C_1 A \le B \le C_2 A$.

1. Weighted Bergman kernels. In order to work with the kernel $B_s(\zeta, z)$ we need to represent it in terms of the defining function r of the region Ω. We will rely essentially on the work of Ligocka [L] and Kerzman and Stein [KSt] but will need to modify the constructions of those authors as in [C] to suit our purposes. If $I = (k_1, \ldots, k_n)$ is a multi-index let

$$(\zeta - z)^I = (\zeta_1 - z_1)^{k_1} \ldots (\zeta_n - z_n)^{k_n} \quad \text{and} \quad I! = k_1! \ldots k_n!.$$

As in [C], for each integer $q \ge 2$ define

$$\widetilde{G}_q(\zeta, z) = \sum_{j=1}^n \widetilde{g}_j(\zeta, z)(\zeta_j - z_j),$$

where

$$\widetilde{g}_j(\zeta, z) = D_j r(\zeta) - \frac{1}{2}\sum_{k=1}^n D_k D_j r(\zeta)(\zeta_k - z_k)$$

$$+ \frac{1}{3}\sum_{|I|=2} \frac{1}{I!} D_I D_j r(\zeta)(\zeta - z)^I$$
$$- \frac{1}{4}\sum_{|I|=3} \frac{1}{I!} D_I D_j r(\zeta)(\zeta - z)^I + \ldots$$
$$+ \frac{(-1)^{q-1}}{m} \sum_{|I|=q-1} \frac{1}{I!} D_I D_j r(\zeta)(\zeta - z)^I .$$

Since

$$\widetilde{G}_q(\zeta, z) = \sum_{j=1}^n D_j r(\zeta)(\zeta_j - z_j) - \frac{1}{2}\sum_{j,k}^n D_k D_j r(\zeta)(\zeta_j - z_j)(\zeta_k - z_k) + e(\zeta, z),$$

where

$$|e(\zeta, z)| \le C|\zeta - z|^3,$$

it follows from the strict pseudoconvexity of Ω that (see [KSt] or [C])

$$\operatorname{Re}\widetilde{G}_q(\zeta, z) \ge \frac{r(\zeta) - r(z)}{2} + C|\zeta - z|^2,$$

for $\zeta \in \mathcal{O}$ and $z \in \Omega$ with $|z - \zeta| \le \delta$, where δ is a constant independent of z or ζ. Following Kerzman and Stein and Ligocka we patch in $|\zeta - z|^2$ and set

$$\widehat{G}_q(\zeta, z) = \sum_{j=1}^n g_j(\zeta, z)(\zeta_j - z_j),$$

where

$$g_j(\zeta, z) = \psi(|\zeta - z|)\widetilde{g}_j(\zeta, z) + (1 - \psi(|\zeta - z|))(\overline{\zeta}_j - \overline{z}_j).$$

Here, ψ is a smooth cut-off function defined for nonnegative reals, which is identically 1 near 0 and supported on a neighborhood of 0 so small that the inequality

$$-r(\zeta) + \operatorname{Re}\widehat{G}_q(\zeta, z) \ge C(-r(\zeta) - r(z) + |\zeta - z|^2)$$

holds for all $\zeta \in \Omega$ and all $z \in \Omega$. The constant C does not depend on z or ζ. Now we define a kernel $G_q(\zeta, z)$ for (ζ, z) in $\Omega \times \Omega$ which is nonvanishing on the diagonal by setting

$$G_q(\zeta, z) = -r(\zeta) + \widehat{G}_q(\zeta, z). \tag{1.1}$$

Notice that there exists a constant δ_0 such that for $z \in \mathcal{O}$, $\overline{\partial}_z G_q(\zeta, z) = 0$ provided $\zeta \in \mathcal{O}$ and $|z - \zeta| < \delta_0$. Furthermore, there is a neighborhood $\mathcal{U}$ of $\overline{\Omega}$ such that $G_q(\zeta, z)$ has positive real part for all ζ and z in $\mathcal{U}$ such that $|z - \zeta| > \delta_0/2$.

In [L] Ligocka shows that if $q = 2$ then for each $s > -1$ there is a function $N_s(\zeta, z)$ which is C^∞ on $\mathbb{C}^n \times \mathbb{C}^n$ and for which the equality

$$F(z) = \int_\Omega F(\zeta) \frac{N_s(\zeta, z)}{G_q(\zeta, z)^{n+1+s}} \, dm_s(\zeta) \tag{1.2}$$

holds for all $F \in H \cap L^2(dm_s)$. In fact, her argument works with G_q in place of G_2 (of course the function N_s will also depend on q) and we will use this fact in the sequel. The reason that we prefer to work with the modified kernels G_q is contained in the next lemma which was stated in [C].

LEMMA 1.1. *For any $j = 1, \ldots, n$,*

$$\frac{\partial G_q(\zeta, z)}{\partial \zeta_j} = \sum_{|I|=q} (\zeta - z)^I e_I(\zeta, z),$$

where each e_I is a smooth function of z and ζ.

Remark. In Lemma 1.1 above $\partial/\partial\zeta_j$ denotes differentiation with respect to the first variable while the second is held fixed. We will follow a similar convention in the sequel when differentiating the function of two variables $G(\zeta, z)$; $\partial/\partial z_j$ will denote differentiation in the second variable while the first is held fixed. Similar remarks apply to the operators $\partial/\partial\overline{\zeta}_j$ and $\partial/\partial\overline{z}_j$.

Remark. At a certain point in our argument we will need to be working with the kernel G_q where we must choose q to be a large integer. Until that moment we will simplify the notation by suppressing the dependence on q and simply write G in place of G_q.

We now reason as in [KSt] (see also [L] and [Ra], Chapter VII) and relate the weighted Bergman kernel to the kernel in the denominator of (1.2). Let $\Psi_s(\zeta, z) = N_s/G_q^{n+1+s}$. The form $\overline{\partial}_z \Psi_s(\zeta, z)$ may be extended to be smooth in a neighborhood of $\overline{\Omega}$. By finding a solution to the $\overline{\partial}$ problem $\overline{\partial}_z u = \overline{\partial}_z \Psi(\zeta, z)$ we obtain a kernel $Q(\zeta, z)$ which is smooth on $\overline{\Omega} \times \overline{\Omega}$. Subtracting Q from the kernel of (1.2) yields a kernel $\widetilde{N}_s/G_q^{n+1+s}$ which is holomorphic in $z \in \Omega$ and still satisfies (1.2). For an arbitrary F in $L^1(dm_s)$ define the operator

$$\mathbf{G}F(z) = \int_\Omega F(\zeta) \frac{\widetilde{N}_s(\zeta, z)}{G(\zeta, z)^{n+1+s}} \, dm_s(\zeta).$$

Let $\mathbf{G}^*$ be the Hilbert space adjoint of $\mathbf{G}$ which is well defined on the space $L^2(dm_s)$. (See Corollary 2.2 below.) If $\mathbf{B} = \mathbf{G}^* - \mathbf{G}$ then it follows from the reproducing properties of P_s and $\mathbf{G}$ that

$$P_s = \mathbf{G} + P_s \mathbf{B}.$$

Iterate this last equality to get

$$P_s = \sum_{j=0}^{k} \mathbf{G}\mathbf{B}^j + \mathcal{R}_k, \tag{1.3}$$

where

$$\mathcal{R}_k = P_s \mathbf{B}^{k+1}.$$

From (1.3) we may use the methods in [KSt] (which rely on the "smoothing properties" of the operator $\mathbf{B}$) in order to analyze the remainder term $\mathcal{R}_k$ and get the following representation for the weighted Bergman kernel B_s. We remark that the terms arising from the "correction factor" Q introduced above are contained in the "remainder" term of the formula.

THEOREM D. *Let*

$$E(\zeta, z) = \frac{N_s(\zeta, z)}{G(\zeta, z)^{n+1+s}},$$

where $z, \zeta \in \Omega$, and set

$$A(\zeta, z) = \overline{E}(z, \zeta) - E(\zeta, z). \tag{1.4}$$

Then the weighted Bergman kernel of Ω is, for $\zeta \in \Omega$ and $z \in \Omega$,

$$B_s(\zeta, z) = E(\zeta, z) + \sum_{j=1}^{k} EA^j(\zeta, z) + R_{k+1}(\zeta, z),$$

where for each $\zeta \in \Omega$,

$$R_{k+1}(\zeta, z) \in C^{\alpha(k)}(\overline{\Omega})$$

in the variable z, and $\alpha(k) \to \infty$ as $k \to \infty$. The kernel composition EA^j means

$$EA^j(\zeta, z) = \int_{t_1 \in \Omega} \cdots \int_{t_j \in \Omega} E(t_1, z) A(t_2, t_1) \ldots A(t_j, t_{j-1}) A(\zeta, t_j) \, dm_s(t_j) \ldots dm_s(t_1).$$

2. Operators on tent spaces. In order to prove Theorem 1 we will need the following result.

THEOREM 2. *Suppose $0 < p < \infty$. Let $K(\zeta, z)$ be a kernel of the form*

$$K(\zeta, z) = \frac{r(z)^a r(\zeta)^b H(\zeta, z)}{G(\zeta, z)^{n+1+a+b+l}}$$

or

$$K(\zeta, z) = \frac{r(z)^a r(\zeta)^b H(\zeta, z)}{G(z, \zeta)^{n+1+a+b+l}},$$

where l is a nonnegative integer, $H(\zeta, z)$ a smooth function which satisfies

$$|H(\zeta,z)| \le C(|z-\zeta|^{2l} + |r(\zeta)|^l + |r(z)|^l),$$

and a and b are real numbers satisfying $a > 0$, $b > -1$, and $(n+b+1)p - n > 0$. Then the operator

$$\mathbf{K}f(z) = \int_{\Omega} f(\zeta)K(\zeta,z)\,dm(\zeta)$$

maps the space $T_2^p(\Omega)$ to itself.

Our proof of Theorem 2 requires three steps. Roughly speaking, first, we appeal to the arguments of Ahern and Schneider to prove that $\mathbf{K}$ and its formal adjoint $\mathbf{K}^*$ are bounded operators on the Lipschitz spaces $\Lambda_{\alpha,0}(\Omega)$ of functions f in the usual Lipschitz space $\Lambda_\alpha(\Omega)$ (see [St1]) which vanish on $\partial\Omega$. (The adjoint is taken with respect to the inner product on the Hilbert space $L^2(dm_{-1})$.) Here, we need only consider $0 < \alpha < 1$. It is easy to see that $\Lambda_{\alpha,0}(\Omega)$ is dense in the Hilbert space $L^2(dm_{-1})$. Second, we invoke the theorem of Krein to conclude from this that $\mathbf{K}$ and $\mathbf{K}^*$ are bounded on $L^2(dm_{-1})$. Since $L^2(dm_{-1}) = T_2^2(\Omega)$ (see [CMSt]) this proves the $p = 2$ case of Theorem 2. Finally, we use the $p = 2$ case and the atomic decomposition of T_2^p to prove that $\mathbf{K}$ and $\mathbf{K}^*$ are bounded operators on T_2^p, for $0 < p \le 1$. The full result is then a consequence of interpolation and duality.

LEMMA 2.1. *Let $0 < \alpha < 1$ and let $\Lambda_{\alpha,0}$ be the closed subspace of $\Lambda_\alpha(\Omega)$ defined above. Let $K(\zeta,z)$ be a kernel of the form*

$$K(\zeta,z) = \frac{r(z)^a r(\zeta)^b H(\zeta,z)}{G(\zeta,z)^{n+1+a+b+l}}$$

or

$$K(\zeta,z) = \frac{r(z)^a r(\zeta)^b H(\zeta,z)}{G(z,\zeta)^{n+1+a+b+l}},$$

where H is as in Theorem 2, $a > \alpha$ and $b > -1$. Then the operator

$$\mathbf{K}f(z) = \int_{\Omega} f(\zeta)K(\zeta,z)\,dm(\zeta)$$

maps the space Λ_α to itself and the space $\Lambda_{\alpha,0}$ to itself.

Proof. The proof of the Lipschitz continuity of $\mathbf{K}f$ proceeds along the same lines as the proof of the main result of [AS2] which concerns a similar result for the Szegö kernel. It is based on the construction of the kernel G which shows that there is a finite collection of open balls $\{\mathcal{O}_\nu\}$ such that $\partial\Omega \subseteq \bigcup \mathcal{O}_\nu$ such that the following conditions hold:

$$G(\zeta,\zeta) = -r(\zeta), \quad \zeta \in \bigcup \mathcal{O}_\nu; \tag{2.1}$$

$$\frac{\partial G}{\partial \zeta_j}(\zeta,\zeta) = 0, \quad \frac{\partial G}{\partial \overline{\zeta}_j}(\zeta,\zeta) = -\overline{D}_j r(\zeta), \quad j = 1,\ldots,n, \ \zeta \in \bigcup \mathcal{O}_\nu; \tag{2.2}$$

$$\frac{\partial G}{\partial \overline{z}_j}(z,z) = 0, \quad \frac{\partial G}{\partial z_j}(z,z) = -D_j r(z), \quad j = 1,\ldots,n, \ z \in \bigcup \mathcal{O}_\nu; \tag{2.3}$$

$$|G(\zeta,z)| \ge C(|r(\zeta)| + |r(z)| + |\zeta - z|^2), \quad z,\zeta \in \overline{\Omega} \cap \mathcal{O}_\nu. \tag{2.4}$$

To prove the first part of the lemma it is sufficient to establish the estimate

$$|D\mathbf{K}f(z)| \le C|r(z)|^{\alpha-1},$$

where D denotes a first order derivative. This in turn will follow if for ε sufficiently small and positive we set $\Omega_\varepsilon = \{\zeta : -\varepsilon < r(\zeta) < 0\}$, define

$$\mathbf{K}_\varepsilon f(z) = \int_{\Omega_\varepsilon} f(\zeta)K(\zeta,z)\,dm$$

and prove that

$$|D\mathbf{K}_\varepsilon f(z)| \le C|r(z)|^{\alpha-1}.$$

Now the open set Ω_ε may be written as the union of finitely many open sets $\mathcal{N}$ on which there exists a smooth projection P from $\mathcal{N}$ to $\partial\Omega$. Such a projection exists since ∇r is nonvanishing on $\partial\Omega$, and can be chosen so that the mapping $z \to (P(z), r(z))$ is a C^∞ diffeomorphism of $\mathcal{N}$ onto $(\mathcal{N} \cap \partial\Omega) \times (-\varepsilon_0, 0)$, for some $\varepsilon_0 > 0$. We may assume that $\varepsilon < \varepsilon_0$ and by partitioning unity we may also assume that the function f is supported on one of the sets $\mathcal{N}$. It follows then that we may write

$$\mathbf{K}_\varepsilon f(z) = \int_0^\varepsilon \int_{\partial\Omega_t} f(\zeta)K(\zeta,z)J(\zeta,t)\,d\sigma_t(\zeta)\,dt,$$

where $J(\zeta,t)$ is positive, smooth and nonvanishing and $d\sigma_t$ is surface area on the manifold $\partial\Omega_t$. If we apply a first order derivative D to $\mathbf{K}_\varepsilon f$ we may use properties (2.1)–(2.4) above and the method of [AS1] or [AS2] on the resulting inside integral to estimate that $|D\mathbf{K}_\varepsilon f(z)|$ is bounded by a constant times the sum of the terms

$$\int_0^\varepsilon \int_{\partial\Omega_t} |G(\zeta,z)|^{-n-2-b+\alpha}\,d\sigma_t(\zeta)\,t^b\,dt \tag{2.5}$$

and

$$|r(z)|^{a-1} \int_0^\varepsilon \int_{\partial\Omega_t} |G(\zeta,z)|^{-n-1-a-b+\alpha}\,d\sigma_t(\zeta)\,t^b\,dt \tag{2.6}$$

plus a term which is bounded independent of z. Note that $a + b > -1$. We may use the integral estimates of [AS1] to show that the expression in (2.5) is less than a constant times

$$\int_0^\varepsilon \frac{t^b}{(|r(z)|+t)^{b+2-\alpha}}\,dt$$

and the expression in (2.6) is less than a constant times

$$|r(z)|^{a-1}\int_0^\varepsilon \frac{t^b}{(|r(z)|+t)^{a+b+1-\alpha}}\,dt.$$

The desired estimates now follow easily.

To prove the second part of the lemma, observe that if $f\in\Lambda_{\alpha,0}$ then $|f(z)|\le C|r(z)|^\alpha$ and we have the estimate

$$\begin{aligned}|\mathbf{K}_\varepsilon f(z)| &\le C\int_\Omega |r(\zeta)|^\alpha|K(\zeta,z)|\,dm(\zeta)\\ &\le \int_\Omega \frac{|r(z)|^a|r(\zeta)|^{b+\alpha}}{|G(\zeta,z)|^{n+1+a+b}}\,dm(\zeta)\\ &\le C|r(z)|^a\int_0^\varepsilon \frac{t^{b+\alpha}}{(|r(z)|+t)^{1+a+b}}\,dt\\ &\le C|r(z)|^\alpha,\end{aligned}$$

since $a>\alpha$. This finishes the proof.

Remark. The simple method used to obtain the last estimate can also be used to obtain bounds on any first order derivative of $\mathbf{K}f$ if $f\in\Lambda_{\alpha,0}$ and therefore gives an easier proof of the second statement of the lemma. In fact, since the kernel G has positive real part, if $K(\zeta,z)$ is one of the kernels of Lemma 2.1 with the function G replaced by $|G|$, then the proof shows that the operator obtained from this kernel maps $\Lambda_{\alpha,0}$ to itself. This is reminiscent of the Forelli–Rudin result (see [Ru], Chapter 7) that shows that the absolute value of certain weighted Bergman kernels on the unit ball give rise to bounded operators on weighted L^2 spaces.

COROLLARY 2.2. *Let* $\mathbf{K}$ *be an operator of the form considered in Theorem 2 above. Then* $\mathbf{K}$ *is a bounded operator on* $L^2(dm_{-1})$.

Proof. Let $\alpha>0$ and suppose that f and g belong to $\Lambda_{\alpha,0}$. If

$$K(\zeta,z)=\frac{r(z)^a r(\zeta)^b H(\zeta,z)}{G(\zeta,z)^{n+1+a+b+l}},$$

and

$$\langle f,g\rangle=\int_\Omega f\,\overline{g}\,dm_{-1},$$

then it is easy to see that $\langle\mathbf{K}f,g\rangle=\langle f,\mathbf{K}^*g\rangle$, where

$$\mathbf{K}^*g(z)=\int_\Omega g(\zeta)K^*(\zeta,z)\,dm(\zeta)$$

and

$$K^*(\zeta,z)=\frac{r(z)^{b+1}r(\zeta)^{a-1}\,\overline{H}(\zeta,z)}{\overline{G}(z,\zeta)^{n+1+a+b+l}}.$$

If we take α sufficiently small it is not hard to see that $\mathbf{K}$ and $\mathbf{K}^*$ are bounded on $\Lambda_{\alpha,0}$ by Lemma 2.1. The conclusion of Theorem 2 for the first form of the kernel K therefore follows from the theorem of Krein. The same argument works for the second form of K and the proof is complete.

In order to use the last two results to complete the proof of Theorem 2 we must recall some facts about the geometry of Ω and the atomic decomposition of the tent space T_2^p.

Let ϱ be the pseudometric defined on $\partial\Omega$ by

$$\varrho(\zeta,\eta)=|\langle\zeta-\eta,\overline{D}r(\zeta)\rangle|+|\langle\zeta-\eta,\overline{D}r(\eta)\rangle|+|\zeta-\eta|^2$$

and for $\zeta\in\partial\Omega$ and $\delta>0$ define the nonisotropic ball

$$Q(\zeta,\delta)=\{\eta:\eta\in\partial\Omega \text{ and } \varrho(\zeta,\eta)<\delta\}.$$

See [St3, Chapter II] for a discussion of the properties of ϱ and the collection of balls Q.

With the pseudometric ϱ defined above we may work locally on sets $\mathcal{N}$ on which there is a smooth projection P from $\mathcal{N}$ to $\partial\Omega$ and a diffeomorphism of $\mathcal{N}$ onto $(\mathcal{N}\cap\partial\Omega)\times(-\varepsilon_0,\varepsilon_0)$. Thus each point ζ in $\mathcal{N}$ is identified with the pair $(P(\zeta),r(\zeta))$. The following lemma can be proved by applying Lemma 5.3 of [C] twice.

LEMMA 2.3. *If* ζ *and* z *belong to* $\mathcal{N}\cap\Omega$ *then*

$$|r(\zeta)|+|r(z)|+\varrho(P(\zeta),P(z))\doteq|G(\zeta,z)|.$$

For each $\eta\in\mathcal{N}\cap\partial\Omega$ define the approach region

$$\Gamma_t(\eta)=\{\zeta\in\mathcal{N}:|\varrho(P(\zeta),\eta)|\le t|r(\zeta)|\},$$

where $t>0$. In what follows we will assume that a single t has been chosen and suppress the dependence of the approach region on the parameter t. If E is a subset of $\mathcal{N}\cap\partial\Omega$ may define the "tent" over E as

$$\widehat{E}=\{\zeta\in\mathcal{N}:Q(P(\zeta),|r(\zeta)|)\subset E\}.$$

If $0<p\le 1$ a function β defined on $\mathcal{N}$ is called a T_2^p *atom* if there is a ball $Q=Q(\eta,\delta)$ such that β is supported in $\widehat{Q}$ and

$$\int_\Omega|\beta|^2\frac{dm}{|r|}\le\delta^{n(1-2/p)}.$$

Take a finite cover of $\partial\Omega$ by open sets $\mathcal{N}_j$ where each $\mathcal{N}_j$ is as $\mathcal{N}$ above and call a function defined on Ω a T_2^p *atom* if it is a T_2^p atom arising from one of the sets $\mathcal{N}_j$. The proof of the atomic decomposition for $T_2^p(\mathbb{R}_+^{n+1})$ given in [CMSt] may be modified to show that there is an open subset $\mathcal{W} \subset \bigcup \mathcal{N}_j$ containing $\partial\Omega$ such that any function f belonging to $T_2^p(\Omega)$ which is supported on $\mathcal{W}$ has an "atomic decomposition"

$$f = \sum_{j=1}^{\infty} \lambda_j \beta_j ,$$

where $\sum |\lambda_j|^p < \infty$ and β_j is a T_2^p atom. We remark that in modifying the proof, in place of the Whitney decomposition used in [CMSt] we may use a Whitney decomposition of the type described in [AN], on page 364. This leads to a formula analogous to the one in [CMSt] on page 315 of the form $\widehat{O}_k^* - \widehat{O}_{k+1}^* = \bigcup \Delta_j^k$. Our sets Δ_j^k, however, are not disjoint. But if we use $\widetilde{\Delta}_j^k = \Delta_j^k - \bigcup_{l=1}^{j-1} \Delta_l^k$ in place of Δ_j^k, then the desired atomic decomposition can be achieved.

LEMMA 2.4. *Let $0 < p \le 1$ and let β be a T_2^p atom. Suppose that* $\mathbf{K}$ *is the kernel described in Theorem* 1. *Then there is an absolute constant C independent of β such that*

$$\int_{\partial\Omega} |A(\mathbf{K}\beta)|^p d\sigma \le C .$$

Proof. Suppose that β is supported on $\widehat{Q}$ where $Q = Q(e,\delta)$, and $e \in \partial\Omega$. We first use Hölder's inequality to estimate that if $mQ = Q(e, m\delta)$ then

$$\int_{mQ} |A(\mathbf{K}\beta)|^p \, d\sigma \le \Big(\int_{\partial\Omega} |A(\mathbf{K}\beta)|^2 \, d\sigma \Big)^{p/2} \sigma(mQ)^{1-p/2} .$$

Using the fact that $T_2^2 = L^2(dm_{-1})$ it follows that

$$\int_{\partial\Omega} |A(\mathbf{K}\beta)|^2 \, d\sigma = C \int_{\Omega} |\mathbf{K}\beta|^2 \frac{dm}{|r|}$$

and this last integral is dominated by a constant times

$$\int_{\Omega} |\beta|^2 \frac{dm}{|r|} < \delta^{n(1-2/p)} ,$$

where we have used Lemma 2.2. Since $\sigma(mQ) \le C(m\delta)^n$, it follows that

$$\int_{mQ} |A(\mathbf{K}\beta)|^p \, d\sigma \le C ,$$

where C is a constant depending only on m.

Next, let $\eta \in \partial\Omega$ and suppose that $\varrho(\eta, e) > m\delta$. Let $d = \varrho(\eta, e)$. If m is sufficiently large we can use Lemma 2.3 to conclude that there exists a constant C such that the estimate

$$|G(\tau, z)| \ge C(d + |r(z)|)$$

holds whenever $z \in \Gamma(\eta)$ and $\tau \in \widehat{Q}$. Use the formula for K to estimate then that

$$|K(\tau, z)| \le C \frac{|r(z)|^a |r(\tau)|^b}{|G(\tau, z)|^{n+1+a+b}} \le C \frac{|r(z)|^a |r(\tau)|^b}{(d + |r(z)|)^{n+1+a+b}} ,$$

and therefore, if $z \in \Gamma(\eta)$,

$$\begin{aligned} |\mathbf{K}\beta(z)| &\le C \frac{|r(z)|^a}{(d + |r(z)|)^{n+1+a+b}} \Big(\int_{\widehat{Q}} |\beta|^2 \frac{dm}{|r|} \Big)^{1/2} \Big(\int_{\widehat{Q}} |r|^{2b+1} \, dm \Big)^{1/2} \\ &\le C \frac{|r(z)|^a}{(d + |r(z)|)^{n+1+a+b}} \delta^{n(1-2/p)/2} \delta^{(2b+2+n)/2} \\ &= C \frac{\delta^{n+b+1-n/p} |r(z)|^a}{(d + |r(z)|)^{n+1+a+b}} . \end{aligned}$$

We now integrate over $\Gamma(\eta)$ using the local product structure and get the estimate

$$\begin{aligned} A(\mathbf{K}\beta)(\eta)^2 &= \int_{\Gamma(\eta)} |\mathbf{K}\beta(z)|^2 \frac{dm(z)}{|r(z)|^{n+1}} \\ &\le C\delta^{2(n+b+1-n/p)} \int_{\Gamma(\eta)} \frac{|r(z)|^{2a-n-1}}{(d + |r(z)|)^{2(n+1+a+b)}} \, dm(z) \\ &\le C\delta^{2(n+b+1-n/p)} \int_0^{\varepsilon} \int_{Q(\eta, Ct)} \frac{t^{2a-n-1}}{(d+t)^{2(n+1+a+b)}} \, d\sigma \, dt \\ &\le C \frac{\delta^{2(n+b+1-n/p)}}{d^{2(n+b+1)}} . \end{aligned}$$

Therefore

$$\int_{\partial\Omega - mQ} |A(\mathbf{K}\beta)|^p \, d\sigma \le C\delta^{p(n+b+1-n/p)} \int_{\partial\Omega - mQ} \frac{dm(\eta)}{|\varrho(\eta, e)|^{p(n+b+1)}} \le C,$$

provided $(n+b+1)p > n$. Together with the previous estimate this completes the proof.

We now can complete the proof of Theorem 2. It is easy to see that it is enough to show that $\mathbf{K}f \in T_2^p$ whenever f is in T_2^p and is supported on the set $\mathcal{W}$ above. For $0 < p \le 1$ and $p = 2$ this is a consequence of Corollary 2.2 and Lemma 2.4. We may then use the interpolation results of [CMSt] (which can be generalized to the present context) to get the theorem for $1 < p < 2$. If we then use the duality (proved in [CMSt] and also true in this context) which is achieved by the pairing $\langle f, g\rangle = \int_\Omega f\,\overline{g}\,dm_{-1}$ between T_2^p and T_2^q where q is conjugate to p plus the fact, and this is why we need two forms of the kernel K, that the adjoint of $\mathbf{K}$ with respect to this pairing is bounded on T_2^q for $1 < q < 2$, the argument is complete.

With Theorem 2 established we are ready to give the proof of Theorem 1. The argument is quite similar to the proof of Theorem 3 in [C] and we therefore sketch the details.

Proof of Theorem 1. The hypotheses on f imply that $r^m f \in L^1(dm_{s-m})$ so the projection $P_s f$ is well defined. Write

$$P_s f(z) = \int_\Omega f(\zeta) B_s(\zeta, z)\,dm_s(\zeta).$$

We now use the expansion given by Theorem D to write that $P_s f$ is a sum of expressions

$$U_j(z) = \int_\Omega f(\zeta) E A^i(\zeta, z)\,dm_s(\zeta)$$

and

$$U_{M+1}(z) = \int_\Omega f(\zeta) \mathcal{R}_{M+1}(\zeta, z)\,dm_s(\zeta),$$

where $j = 0, \ldots, M$ and where M will depend only on p.

Our strategy now is to apply a differential operator $\mathcal{D}$ of order $k+1$ (in the variable z) to each term above, multiply by $|r(z)|$ and show that the result is in the tent space T_2^p. We can easily handle the "remainder term" U_{M+1} since it follows from Theorem D that we may choose M so that the kernel $\mathcal{R}_{M+1}$ is as smooth as we like. Therefore for M large the expression $r(z)\mathcal{D}U_{M+1}(z)$ will lie in T_2^p.

We next consider the term U_0. There is no loss of generality in supposing that f is supported on one of the sets $\mathcal{O}_\nu$ described in the hypothesis or that each of the sets $\mathcal{O}_\nu$ is contained in one of the sets $\mathcal{N}_j$ discussed previously, since a partition of unity argument reduces the general case to this one. Using the explicit expression for the kernel E given in Theorem D apply Lemmas 2.2 and 2.3 of [C] and the product structure on $\mathcal{O}_\nu$ to integrate by parts $2k + 2m$ times and express $U_0(z)$ as a sum of terms of two types. The first type arises if during the integration by parts an operator T_ν was applied to the factor G. Recall that $G = G_q$ and we have Lemma 1.1 at our disposal. In this case the resulting term has the form

$$\text{(2.7)} \qquad \int_\Omega r^m(\zeta) g(\zeta) \frac{H(\zeta, z)}{G(\zeta, z)^{n+s+1+2k+2m}}\,dm_{s-m}(\zeta),$$

where g is a linear combination of the functions $T_\nu^l f$, $l = 0, \ldots, 2k+2m$, and H is C^∞ on $\mathbb{C}^n \times \mathbb{C}^n$ and satisfies the condition

$$|H(\zeta, z)| \le C|\zeta - z|^{q-2k-2m}.$$

Applying the operator $\mathcal{D}$ to (2.7) and multiplying by $|r(z)|$ yields a term of the form

$$\text{(2.8)} \qquad |r(z)| \int_\Omega u(\zeta) \frac{H(\zeta, z)}{G(\zeta, z)^{n+s+2+3k+2m}}\,dm_{s-m}(\zeta),$$

where $u \in T_2^p$ and H satisfies the condition

$$|H(\zeta, z)| \le C|\zeta - z|^{q-3k-2m-1}.$$

By specifying at the outset that q be very large we can then apply Theorem 2 to deduce that (2.8) is in T_2^p.

Continuing to the other terms where the kernel G has not been differentiated, it can be shown that if $U(z)$ represents such a term then

$$\text{(2.9)} \qquad |r(z)|\mathcal{D}U(z) = \int_\Omega u(\zeta) \frac{|r(z)| H(\zeta, z)}{G(\zeta, z)^{n+s+k+2}}\,dm_{s-m}(\zeta),$$

where u is in T_2^p and

$$|H(\zeta, z)| \le C(|\zeta - z|^{2k+2m} + |r(\zeta)|^{k+m} + |r(z)|^{k+m}).$$

We may now apply Theorem 2 to conclude that the right hand side of (2.9) is in the tent space T_2^p.

We finally discuss the terms U_j where $j = 1, \ldots, M$. It follows from Theorem D that

$$U_j(z) = \int_\Omega \mathbf{A}^j f(t) E(t, z)\,dm_s(t),$$

where $\mathbf{A}$ is given by (1.4). Thus

$$\begin{aligned}\mathbf{A}f(t) &= \int_\Omega f(\zeta)\overline{E}(t, \zeta)\,dm_s(\zeta) - \int_\Omega f(\zeta) E(\zeta, t)\,dm_s(\zeta)\\ &= \overline{E}f(t) - Ef(t).\end{aligned}$$

The argument above shows that $r\mathcal{D}Ef \in T_2^p$ where $\mathcal{D}$ is any differential operator of order $k+1$. The same argument applies to $\overline{E}f$ where in place of Lemma 1.1 we use property (4.1) of [C]:

$$\frac{\partial G}{\partial \overline{z}_j}(\zeta, z) = 0, \quad j = 1, \ldots, n, \ \zeta, z \in \mathcal{O}_\nu.$$

Integrating by parts (using the vector field N_2 given by (0.2)) as was done during the proof of Lemma 4.1 in [C] allows us to use Theorem 2 to deduce that the property that $r\mathcal{D}f \in T_2^p$ where $\mathcal{D}$ is any differential operator of order $k+1$ is preserved by the operators E and $\overline{E}$. Therefore the function $r\mathcal{D}\mathbf{A}^j f$ also lies in T_2^p. The same argument then shows that $r\mathcal{D}U_j$ is in T_2^p and the proof is complete.

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Pointwise inequalities for Sobolev functions and some applications

by

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Abstract. We get a class of pointwise inequalities for Sobolev functions. As a corollary we obtain a short proof of Michael–Ziemer's theorem which states that Sobolev functions can be approximated by C^m functions both in norm and capacity.

1. Introduction. In this paper, we prove some pointwise inequalities for Sobolev functions, i.e. functions in the Sobolev classes $W^{m,p}(\Omega)$, where m is an integer, $p \geq 1$, and Ω is an open subset of $\mathbb{R}^n$. For simplicity we restrict the discussion to the case $\Omega = \mathbb{R}^n$ and $mp < n$. The generalized derivatives $D^\alpha f$, $|\alpha| \leq m$, are defined as equivalence classes of measurable functions. For our pointwise estimates, presented in a form valid for each point of the domain Ω, it is essential to select a representative in each class which is a Borel function, i.e. a function well defined at each point of its domain, essentially by an everywhere convergent limiting process of sequences of continuous or continuously differentiable real-valued functions. This is best illustrated by the well known procedure of selecting a Borel function $\widetilde{f}(x)$ for the class of real-valued Lebesgue spaces $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ using the formula

$$\widetilde{f}(x) = \limsup_{r\to 0} \fint_{B(x,r)} f(y)\,dy = \limsup_{r\to 0} f_r(x), \qquad r > 0,$$

where $f_r(x)$ are the Steklov means of the Lebesgue function f. Note that the above limiting process is rather delicate and should be applied with extreme care; in particular, it is not additive, and in general $\widetilde{f}(x) \neq -(\widetilde{-f})(x)$.

An important remark is that our main pointwise inequalities for the Borel function $\widetilde{f}(x)$ may be formulated in terms of the averaged Steklov type

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