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MATHEMATICS DEPARTMENT
AUBURN UNIVERSITY
AUBURN, ALABAMA 36849-5201
U.S.A.

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Ergodic properties of skew products with Lasota–Yorke type maps in the base

by

ZBIGNIEW S. KOWALSKI (Wrocław)

Abstract. We consider skew products $T(x,y) = (f(x), T_{e(x)}y)$ preserving a measure which is absolutely continuous with respect to the product measure. Here f is a 1-sided Markov shift with a finite set of states or a Lasota–Yorke type transformation and T_i, $i = 1, \ldots, \max e$, are nonsingular transformations of some probability space. We obtain the description of the set of eigenfunctions of the Frobenius–Perron operator for T and consequently we get the conditions ensuring the ergodicity, weak mixing and exactness of T. We apply these results to random perturbations.

0. Introduction. Let $\{T_i\}_{i=1}^s$ be a finite family of nonsingular transformations of a probability space $(Y, \mathcal{B}, p)$. Given a nonsingular transformation f of a probability space $(X, \mathcal{A}, \mu)$ and a mapping e from X to $\{1, \ldots, s\}$, we define the *skew product transformation*

$$T(x,y) = (f(x), T_{e(x)}y).$$

The purpose of this paper is the description of the ergodic properties of T. To this end we use our results on eigenfunctions of the Frobenius–Perron operator for T. The above problem was considered in [10] and [11] where the transformation f preserves the Bernoulli measure μ and the family of transformations may be infinite.

The paper consists of two parts. In the first part we assume that f is a 1-sided Markov shift preserving the measure μ with a finite set of states. In the second part we assume f to be a general Lasota–Yorke type transformation, i.e. f is piecewise C^1 and uniformly expanding.

PART I

1. Introduction. Let σ be the shift endomorphism in a space $X \subset \{1, \ldots, s\}^{\mathbb{N}}$ preserving μ. The measure μ is Markov and it is determined by

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a pair π, Q, where $Q = [q_{ij}]$ is a stochastic matrix and $\pi = (q_1, \ldots, q_s)$ a probabilistic vector with $\pi Q = \pi$. Let $B = \{(i,j) : q_{ij} > 0\}$. Let $\{T_{ij}\}_{(i,j)\in B}$ be a family of measurable negative nonsingular transformations of a probability space $(Y, \mathcal{B}, p)$. We define the skew product transformation

$$T(x,y) = (\sigma(x), T_{x(1)x(2)}y). \tag{1}$$

The Frobenius–Perron operator for T is given by the formula

$$P_T(g\otimes f)(x,y) = \sum_{i,j} \frac{q_i}{q_j} q_{ij} g(ix) 1_{A_j}(x)(P_{T_{ij}} f)(y),$$

where the summation is taken over $(i,j) \in B$. Here $(g\otimes f)(x,y) = g(x)f(y)$, where $g \in L_1(\mu)$, $f \in L_1(p)$, $(ix) = (i, x(1), x(2), \ldots)$, $A_i = \{x : x(1) = i\}$ and $P_{T_{ij}}$ denotes the Frobenius–Perron operator for T_{ij}.

In Section 2, under the above assumptions, we prove that if a function $F \in L_1(\mu\times p)$ satisfies $F\circ T = \lambda F$ for some $\lambda \in \mathbb{C}$, then there are functions $f_i \in L_1(p)$ such that

$$F(x,y) = \sum_{i=1}^{s} 1_{A_i}(x) f_i(y) \quad \mu\times p\text{-a.e.}$$

This is a generalization of Morita's result [11] for the Bernoulli case. From this we obtain conditions ensuring the weak mixing and exactness of absolutely continuous invariant measures (a.c.i.m.).

Ergodic properties of skew products with a Bernoulli shift in the base are considered, e.g., in [1], [3], [9], [13].

In Section 3 we apply the above results to perturbations of automorphisms.

2. Ergodic properties.

The following lemma provides the description of eigenfunctions for P_T.

LEMMA 1. *If $F \in L_1(\mu\times p)$ satisfies $\lambda P_T F = F$ for some $\lambda\in\mathbb{C}$, $|\lambda| = 1$, then there exist $f_i \in L_1(p)$, $i = 1, \ldots, s$, such that*

$$F(x,y) = \sum_{i=1}^{s} 1_{A_i}(x) f_i(y) \quad \mu\times p\text{-a.e.}$$

Proof. Let $A_{i_1\ldots i_n} = \{x : x(1) = i_1, \ldots, x(n) = i_n\}$ and let $f\in L_1(p)$. Then

$$P_T(1_{A_{i_1\ldots i_n}} \otimes f)(x,y) = \frac{q_{i_1}}{q_{i_2}} q_{i_1 i_2} 1_{A_{i_2\ldots i_n}}(x)(P_{T_{i_1 i_2}} f)(y).$$

Therefore

$$P^n_T(1_{A_{i_1\ldots i_n}} \otimes f)(x,y) = \sum_{i=1}^{s} 1_{A_i}(x) g_i(y), \quad \text{where } g_i \in L_1(p),\ i = 1, \ldots, s.$$

Next we reason as in the proof of Theorem 3.1 of [11]. ∎

COROLLARY 1. *Any T-a.c.i.m. has the form $\sum_{i=1}^{s} \mu_{A_i} \times \overline{p}_i$, where $\mu_{A_i}(A) = \mu(A\cap A_i)$ and $\overline{p}_i$ is a p-a.c.m. for $i = 1, \ldots, s$.*

COROLLARY 2. *Assume that $\nu = \sum_{i=1}^{s}\mu_{A_i}\times\overline{p}_i$ is T-a.c.i.m. If $F\in L_1(\nu)$ satisfies $F\circ T = \lambda F$ for some $\lambda\in\mathbb{C}$, $|\lambda| = 1$, then there exist $f_i \in L_1(\overline{p}_i)$, $i = 1, \ldots, s$, such that $F(x,y) = \sum_{i=1}^{s} 1_{A_i}(x) f_i(y)$ ν-a.e. In particular, if A is a T-invariant set ($T^{-1}A = A$) then $A = \bigcup_{i=1}^{s} A_i \times B_i$ for some sets $B_i \in \mathcal{B}$.*

Proof. This is an easy consequence of Lemma 1 and the equality $hP_{T,\nu}G = P_T(Gh)$ for $G \in L_1(\nu)$. Here $P_{T,\nu}$ is the Frobenius–Perron operator for the measure ν and $h = d\nu/d(\mu\times p)$. ∎

We apply Corollary 2 to the description of weakly mixing skew products. To this end we introduce the *property* (R) of the family $\{T_{ij}\}_{(i,j)\in B}$ and a measure $\nu = \sum_{i=1}^{s}\mu_{A_i}\times\overline{p}_i$:

(R) There exists a pair $(i,j) \in B$ such that $\{A : T^{-1}_{sm}T^{-1}_{ms}A = T^{-1}_{sl}T^{-1}_{ls}A$ for every l, m such that $(s,m), (m,s), (l,s), (s,l) \in B\} = \{\emptyset, Y\}$ up to $\overline{p}_s$-null sets for $s = i, j$.

We say that a negative nonsingular transformation is nonsingular if it maps sets of measure zero to sets of measure zero.

THEOREM 1. *Let the measure μ be mixing and let ν be a T-a.c.i.m. If the transformations $\{T_{ij}\}_{(i,j)\in B}$ are nonsingular and have the property* (R), *then the endomorphism T is weakly mixing.*

Proof. It is sufficient to show that $T\times T$ is ergodic. By the definition of T,

$$(T\times T)((x,y),(u,v)) = ((\sigma\times\sigma)(x,u), (T_{x(1)x(2)} \times T_{u(1)u(2)})(y,v)).$$

Therefore $T\times T$ is a skew product with Markov base $\sigma\times\sigma$. Let A be a $T\times T$-invariant set. By Corollary 2, $A = \bigcup_{n,m=1}^{s} A_m\times A_n\times B_{mn}$. Let (i,j) be a pair given by the property (R). We get $(T^{-1}_{il}\times T^{-1}_{jm})B_{lm} = B_{ij}$. By nonsingularity of $\{T_{ij}\}_{(i,j)\in B}$ we have $B_{lm} \supset (T_{il}\times T_{jm})B_{ij}$. Therefore $B_{ij} \supset (T_{li}T_{il}\times T_{mj}T_{jm})B_{ij}$ and $(T^{-1}_{il}T^{-1}_{li}\times T^{-1}_{jm}T^{-1}_{mj})B_{ij} = B_{ij}$. Hence

$$I\times T_{lj}T_{jl}T^{-1}_{jm}T^{-1}_{mj}B_{ij} \subset B_{ij}, \quad T_{li}T_{il}T^{-1}_{im}T^{-1}_{mi}\times IB_{ij} \subset B_{ij}. \tag{2}$$

Let $B^y_{ij} = \{v : (y,v)\in B_{ij}\}$. By (2), $T^{-1}_{jm}T^{-1}_{mj}B^y_{ij} = T^{-1}_{jl}T^{-1}_{lj}B^y_{ij}$ for $\overline{p}_i$-a.e. y. Hence by (R) we get $B^y_{ij} = Y$ for $\overline{p}_i$-a.e. y. Consequently, $B_{ij} = E\times Y$ for some set E. By applying (2) to E, we get $E = Y$ and hence $B_{ij} = Y\times Y$. Therefore $A \supset A_i\times A_j\times Y\times Y$ and the ergodicity of $\sigma\times\sigma$ implies $A = X\times X\times Y\times Y$. ∎

Now, let p be a Borel measure on $[0,1]$ which is positive on open sets. Moreover, let $\{T_{ij}\}_{(i,j)\in B}$ be piecewise monotonic and continuous transformations of $[0,1]$ into itself so that there exists a partition $\beta_0 = \{I_1, I_2, \ldots\}$ of finite entropy with $I_i = (t_{i-1}, t_i)$, $0 = t_0 < t_1 < \ldots$, $\lim t_i = 1$, such that $T_{ij}|(t_l, t_{l+1})$ is continuous and strictly monotonic for all $(i,j) \in B$, $l = 0, 1, 2, \ldots$

THEOREM 2. *Suppose μ is mixing and the transformations T_{ij}, $(i,j) \in B$, are piecewise monotonic and continuous. Moreover, let ν be a T-invariant equivalent measure and assume that T_{ij}, $(i,j) \in B$, are 1-1 p-a.e. Then the property* (R) *implies: if $\nu = \mu \times \overline{p}_1$ for some measure $\overline{p}_1 \approx p$ and T_{ij} does not preserve the measure $\overline{p}_1$ for some (i,j), then T is an exact endomorphism.*

Proof. By Theorem 1 we get the ergodicity of T. Theorem 1 of [7] and the weak mixing property of T imply the exactness of T. ■

Remark 1. If μ is a Bernoulli measure, then we can replace the property (R) by $\{A : T_i^{-1}A = T_j^{-1}A,\ i,j = 1, \ldots, s\} = \{\emptyset, Y\}$.

3. Application to some class of generalized skew products. Let $\{T_\varepsilon\}_{\varepsilon\in(a,b)}$ be a one-parameter family of transformations of the interval $[0,1]$ into itself such that

$$T_\varepsilon^{-1}(y) = (1-\varepsilon)y + \varepsilon g(y)\,, \tag{3}$$

where $g \in C^2[0,1]$, $g(0) = 0$, $g(1) = 1$, and $a = (1 - \sup g')^{-1}$, $b = (1 - \inf g')^{-1}$. Moreover, assume that there exists exactly one point y_0 for which $g'(y_0) = 1$.

We take functions $\{T_{\varepsilon_{ij}}\}_{(i,j)\in B}$ such that $\sum_{i=1}^s q_i q_{ij} \varepsilon_{ij} = 0$ for $j = 1, \ldots, s$. Let T be an endomorphism of the Lebesgue space $([0,1], \mathcal{B}, m)$. The transformation

$$\overline{T}(x,y) = (\sigma(x), T_{\varepsilon_{x(1)x(2)}} T(y))$$

preserves the product measure $\mu \times m$.

THEOREM 3. *Let μ be a mixing measure. If T is an automorphism and there exists a pair $(i,j) \in B$ such that $\varepsilon_{im_1} \neq \varepsilon_{im_2} \neq \varepsilon_{im_3}$, $\varepsilon_{m_1 i} = \varepsilon_{m_2 i} = \varepsilon_{m_3 i} = 0$, $\varepsilon_{jl_1} \neq \varepsilon_{jl_2} \neq \varepsilon_{jl_3}$, $\varepsilon_{l_1 j} = \varepsilon_{l_2 j} = \varepsilon_{l_3 j} = 0$, for some m_i, l_i, $i = 1,2,3$, then the transformation $\overline{T}$ is weakly mixing.*

Proof. By Theorem 1, it is sufficient to check the property (R). Let $A \neq \emptyset$ and $s = i$ or j. If

$$T_{m_1 s} T T_{sm_1} TA = T_{m_2 s} T T_{sm_2} TA = T_{m_3 s} T T_{sm_3} TA$$

then by the assumptions we get

$$T_{sm_1} TA = T_{sm_2} TA = T_{sm_3} TA\,.$$

For $D = TA$ we have $T_{sm_1} D = T_{sm_2} D = T_{sm_3} D$. By Lemma 2 of [6] we obtain $D = [0,1]$ and consequently $A = [0,1]$. ■

Let T be an infinite interval exchange transformation of $[0,1]$ of the following type:

(i) there exists a partition $\beta_0 = \{I_1, I_2, \ldots\}$ given by $I_i = (t_{i-1}, t_i)$ with $0 = t_0 < t_1 < \ldots$, $\lim t_i = 1$, $H(\beta_0) < \infty$,

(ii) there exist real constants a_i so that $T(t) = t + a_i$ for $t \in I_i$,

(iii) the only accumulation point of $\{t_{i-1} + a_i\} \cup \{t_i + a_i\}$ is 1,

(iv) T is a 1-1 transformation.

COROLLARY 3. *If T is an infinite interval exchange transformation of $[0,1]$, then under the assumptions of Theorem 3, $\overline{T}$ is an exact endomorphism.*

In some particular cases we can obtain the exactness of $\overline{T}$ without the assumption of nullity of some parameters ε_{ij}. For example, the following statement is true.

THEOREM 4. *If $T = I$, where $I(y) = y$, and if there exists $(i,j) \in B$ such that $\varepsilon_{m_1 j} > \varepsilon_{jm_1}$, $0 > \varepsilon_{m_2 j} > \varepsilon_{jm_2}$, $0 < \varepsilon_{m_3 j} < \varepsilon_{jm_3}$ and $\varepsilon_{l_1 i} > \varepsilon_{il_1}$, $0 > \varepsilon_{l_2 i} > \varepsilon_{il_2}$, $0 < \varepsilon_{l_3 i} < \varepsilon_{il_3}$ for some numbers m_i, l_i, $i = 1,2,3$, then the transformation $\overline{T}$ is exact.*

We finish this section with an application of Theorem 1 to the class of random maps of interval which are considered in [12]. Let σ be a 1-sided (r,t)-Bernoulli shift and let T_1, T_2 be transformations of $[0,1]$ such that

(a) T_1 is C^2, $T_1(0) = 0$, $T_1'' \geq 0$ and $1/2 \leq T_1' < 1$, and

(b) T_2 is a Lasota–Yorke type map with partition $I_0, I_1, \ldots, I_n$; that is, $T_2'(y) \geq 2$ wherever defined, $T_2(0) = 0$ and $T_2''(y) \geq 0$ for $y \in I_0$, while $T_2(y) \leq y$ for $y \in I_1, \ldots, I_n$.

Let $T(x,y) = (\sigma(x), T_{x(0)} y)$. Then by Theorem 2 of [12], T has an invariant measure $\mu \times p$ such that $p \ll m$.

THEOREM 5. *If the transformation $T_1^{-1} T_2$ can be extended to a Lasota–Yorke map (on $[0,1]$) which is ergodic with respect to an invariant measure ν, $\nu \approx p$, then T is exact.*

Proof. By Remark 1 and Theorem 1, T is weakly mixing. Then the weak compactness of the iterations of the Frobenius–Perron operator $\mathcal{P}_T$ implies the exactness of T. ■

EXAMPLE. If $T_1(y) = y/2$, $T_2(y) = 2y \bmod 1$ and $1/2 < t < 2/3$, then T is exact.

PART II

4. Introduction. Let f be a Lasota–Yorke type transformation of $[0,1]$ into itself, i.e.

(a) there exists a partition $0 = a_0 < a_1 < \ldots < a_q = 1$ of $[0,1]$ such that the restriction f_i of f to (a_{i-1}, a_i) is C^1,

(b) $1/|f_i'|$ extends to a function of bounded variation on $[a_{i-1}, a_i]$ for $i = 1, \ldots, q$,

(c) $\inf |f'| > 1$.

Let P_f denote the Frobenius–Perron operator for f. Then for every function g of bounded variation (see [14])

$$\bigvee_0^1 P_f g \le 3\lambda^{-1} \bigvee_0^1 g + \frac{3\lambda^{-1}}{a} \int_0^1 |g|\, dm . \tag{4}$$

Here $\lambda = \inf |f'|$ and $a = \min_i (a_i - a_{i-1})$.

Let $\{T_i\}_{i=1}^q$ be a family of measurable transformations of a Lebesgue space $(Y, \mathcal{B}, p)$. We require that if $p(A) = 0$ then $p(T_i^{-1}A) = p(T_i A) = 0$ for $i = 1, \ldots, q$. Under this assumption T_i, $i = 1, \ldots, q$, is nonsingular and positively measurable transformation. We define the skew product transformation

$$T(x,y) = (f(x), T_{e(x)} y) .$$

Here $e : X \to \{1, \ldots, q\}$ is such that $e(x) = i$ for $x \in I_i$, $i = 1, \ldots, q$, where $I_i = [a_{i-1}, a_i)$ for $i = 1, \ldots, q-1$ and $I_q = [a_{q-1}, 1]$.

The Frobenius–Perron operator for T with respect to the measure $m \times p$ is given by the formula

$$P_T G(x,y) = \sum_{i=1}^q P_i G(f_i^{-1}(x), y) |(f_i^{-1})'(x)| 1_{f_i(I_i)}(x) .$$

Here $G \in L_1(m \times p)$ and P_i denotes the Frobenius–Perron operator for T_i.

For a function G from $[0,1] \times Y$ into $\mathbb{C}$, let $\mathbf{V}_y G$ denote the total variation of $G(\cdot, y)$, for every $y \in Y$. Moreover, for $G \in L_1(m \times p)$ we introduce the following definitions:

$$\mathbf{V} G = \inf \Big\{ \int \mathbf{V}_y F \, dp : F \text{ is any version of } G \Big\},$$

$$BV = \{ G \in L_1(m \times p) : \mathbf{V} G < \infty \} \quad \text{and}$$

$$\mathcal{D} = \{ G \in L_1(m \times p) : G \ge 0 ,\ \|G\|_1 = 1 \} ,$$

which are modifications of the analogous definitions from [2].

In Section 5 we prove that if for every function $G \in L_1(m \times p)$ the limit $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_T^k G$ exists in L_1 then for every bounded function F satisfying $F(T) = \alpha F$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$, and each $H \in \mathcal{D}$ such that $P_T H = H$ we have $H \in BV$ and $FH \in BV$. In Section 6 we obtain the description of T-invariant sets. In the case of a markovian partition, i.e. when

$$m(f(I_i) \cap I_j) > 0 \quad \text{implies} \quad I_j \subset f(I_i) \quad \text{for } i,j = 1, \ldots, q ,$$

we prove that if A is a T-invariant set and $G \in \mathcal{D}$, $P_T G = G$, then

$$A \cap \{G > 0\} = \bigcup_{i=1}^q I_i \times B_i .$$

Otherwise, if $\inf |f'| > 2$ then $A \cap \{G > 0\} \supset I_i \times B$ for some i, where $p(B) > 0$.

In the same manner as in part I we apply the above results to describe the ergodic properties of T.

5. Regularity of the eigenfunctions of the Frobenius–Perron operator

LEMMA 2. *If $G \in BV$ then $\mathbf{V} P_i G \le \mathbf{V} G$ for $i = 1, \ldots, q$.*

Proof. Let F be such that $F = G$ a.e. and $\int \mathbf{V}_y F \, dp < \infty$. Then

$$\begin{aligned}
\int \mathbf{V}_y F\, dp &= \int P_i \mathbf{V}_y F \, dp \\
&= \int P_i \Big(\sup \sum_k |F(x_k, y) - F(x_{k+1}, y)| \Big) dp \\
&\ge \int \sup P_i \Big(\sum_k |F(x_k, y) - F(x_{k+1}, y)| \Big) dp \\
&\ge \int \sup \Big(\sum_k |P_i F(x_k, y) - P_i F(x_{k+1}, y)| \Big) dp \\
&= \int \mathbf{V}_y P_i F \, dp . \ \blacksquare
\end{aligned}$$

LEMMA 3. *If $G \in BV$ then*

$$\mathbf{V} P_T G \le 3\lambda^{-1} \mathbf{V} G + \frac{3\lambda^{-1}}{a} \|G\|_1 .$$

Proof. Let $F = G$ a.e. and $\int \mathbf{V}_y F \, dp < \infty$. Then

$$\begin{aligned}
&\int \mathbf{V}_y P_T F \, dp \\
&\quad = \int \mathbf{V}_y \sum_{i=1}^q P_i F(f_i^{-1}(x), y) |(f_i^{-1})'(x)| 1_{f(I_i)}(x)\, dp \\
&\quad \le \sum_{i=1}^q \mathbf{V}_y P_i F(f_i^{-1}(x), y) |(f_i^{-1})'(x)| 1_{f(I_i)}(x)\, dp
\end{aligned}$$

$$\le \int \sum_{i=1}^{q} \mathop{\mathbf{V}}_{y} F(f_i^{-1}(x),y)|(f_i^{-1})'(x)|1_{f(I_i)}(x)\,dp \quad \text{(by Lemma 2)}$$

$$\le 3\lambda^{-1}\int \mathop{\mathbf{V}}_{y} F\,dp + \frac{3\lambda^{-1}}{a}\int |G|\,d(m\times p).$$

The last inequality is a consequence of (4). ■

PROPERTY 1 ([2]). *If* $\lim_{n\to\infty} F_n = F$ *in* L_1 *norm then*

$$\mathbf{V}F \le \limsup_{n\to\infty} \mathbf{V}F_n .$$

As a corollary we obtain

THEOREM 6. *If for every* $G \in BV$ *the limit*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P_T^k G = Q_T G \quad \textit{exists in } L_1 ,$$

then $\mathbf{V}Q_T G \le c\|G\|_1$, *where the constant* c *does not depend on* G.

Proof. This follows immediately from Lemma 3, Property 1 and the continuity of the operator Q_T in L_1. ■

Remark 2. The assumption of Theorem 6 is related to the condition assuring weak sequential compactness in L_1-space, and to the Kakutani–Yosida ergodic theorem. It is satisfied, for instance, if:

(i) the transformations f, T_i, $i = 1,\ldots,q$, are piecewise C^2 Lasota–Yorke type maps (Theorem 1 of [2]),

(ii) there exists an equivalent T-invariant measure (Hopf's theorem [8]).

For the rest of this paper we will assume that the assumptions of Theorem 6 are satisfied. We denote by μ_G a T-a.c.i.m. such that $d\mu_G/d(m\times p) = G$.

LEMMA 4. *If* $F \in L_\infty(\mu_G)$ *satisfies* $P_T(FG) = \alpha FG$ *for some* $\alpha \in \mathbb{C}$, $|\alpha| = 1$, *then* $FG \in BV$.

Proof. Let $\widetilde{P}_T H = \overline{\alpha} G^{-1} P_T(HG)$. Then $\widetilde{P}_T F = F$. By Lemma 3,

$$\mathbf{V}G\widetilde{P}_T H \le 3\lambda^{-1}\,\mathbf{V}HG + \frac{3\lambda^{-1}}{a}\|HG\|_1 , \quad \text{for } HG \in BV$$

and by definition $G\widetilde{P}_T^n H = \overline{\alpha}^n P_T^n(HG)$. Therefore, it is enough to show that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\widetilde{P}_T^k H \quad \text{exists in } L_1 \text{ for every } H \in L_1(\mu_G). \tag{5}$$

The operator $\widetilde{P}_T$ has the following properties:

(i) $\|\widetilde{P}_T H\|_\infty \le \|H\|_\infty$ for $H \in L_\infty(\mu_G)$,

(ii) $\int |\widetilde{P}_T H|\,d\mu_G \le \int |H|\,d\mu_G$ for $H \in L_1(\mu_G)$.

Hence $\widetilde{P}_T$ is a linear L_1-L_∞-contraction and by the Dunford–Schwartz theorem we obtain (5). ■

COROLLARY 4. *If* $G \in \mathcal{D}$ *is a function such that* $P_T G = G$, A *is* T*-invariant set and* F *is bounded function such that* $F(T) = \alpha F$ *for some* $\alpha \in \mathbb{C}$, $|\alpha| = 1$, *then* $\mathbf{V}_y G < \infty$, $\mathbf{V}_y 1_A G < \infty$ *and* $\mathbf{V}_y FG < \infty$ *a.e.*

Here $\mathbf{V}_y H < \infty$ a.e. means that for a.e. y there exists a set $A_y \subset [0,1]$, $m(A_y) = 0$, such that $\mathbf{V}_{[0,1]\setminus A_y} H(\cdot,y) < \infty$.

6. Ergodic properties. Let $D_G = \{(x,y) : G(x,y) > 0\}$ for $G \in \mathcal{D}$ with $P_T G = G$. Then $TD_G = D_G$ with respect to $m\times p$. Fixing a density G we write $\mu = \mu_G$ and $D = D_G$. Moreover, let β denote the partition $\{I_1,\ldots,I_q\}$.

LEMMA 5. *Let* A *be a* T*-invariant set such that* $\mu(A) > 0$. *Then there exists a set* $B \in \mathcal{B}$, $p(B) > 0$, *such that*

$$\bigcup_{y\in B} I_y \times \{y\} \subset A \cap D$$

for some nonempty open intervals I_y.

Proof. Let

$$B = \{y : \mathop{\mathbf{V}}_{y} G < \infty\} \cap \{y : \mathop{\mathbf{V}}_{y} 1_A G < \infty\} \cap \{y : p((A\cap D)_y) > 0\}.$$

Here $(A\cap D)_y = \{x : (x,y) \in A\cap D\}$. By Corollary 4, $p(B) > 0$. Let $y \in B$. From the definition of B it is easy to see that $(A\cap D)_y$ contains a nonempty interval. ■

LEMMA 6. *Let* A *be a* T*-invariant set.*

(i) *If the partition* β *is markovian then* $A\cap D = \bigcup_{i=1}^q I_i \times B_i$.

(ii) *If* $\inf|f'| > 2$ *then* $A\cap D \supset I_i \times E$ *for some* i *and* $E \in \mathcal{B}$, $p(E) > 0$.

Proof. The property $\inf|f'| > 2$ implies that for every nonempty interval I, either $I \supset I_i$ or $m(f(I_i\cap I)) \ge (\lambda/2)m(I)$ for some i. On the other hand, by the markovian property, for every nonempty interval I such that $I \subset I_i$ for some i, there exists j such that $f(I) \supset I_j$ or $f(I) \subset I_j$ and $m(f(I)) \ge \lambda m(I)$. Hence for every interval I there exists a sequence $i_1,\ldots,i_k$ and $\widetilde{I} \subset I$ such that $f_{i_k}\circ\ldots\circ f_{i_1}(\widetilde{I}) = I_i$ for some i. Let B be the set given by Lemma 5. Then there exist i, $B_1 \subset B$ with $p(B_1) > 0$, and a sequence $i_1,\ldots,i_k$ such that $f_{i_k}\circ\ldots\circ f_{i_1}(\widetilde{I}_y) = I_i$ for every $y \in B_1$ and for some interval $\widetilde{I}_y \subset I_y$. By invariance of A we get

$$A \cap D = T^k(A \cap D) \supset \bigcup_{y \in B_1} (f_{i_k} \circ \ldots \circ f_{i_1}(\widetilde{I}_y) \times T_{i_k} \circ \ldots \circ T_{i_1}(y)) = I_i \times T_{i_k} \circ \ldots \circ T_{i_1}(B_1) = I_i \times E_1 .$$

The nonsingularity of $\{T_i\}_{i=1}^q$ implies (ii). Let the partition β be markovian and

$$A_1 = \bigcup_{i=1}^q I_i \times B_i , \quad \text{where} \quad B_i = \{y : p((A \cap D)_y - I_i) = 0\} .$$

Since $TA_1 \subset A_1 \subset A$, the assumption that $\mu(A - A_1) > 0$ implies the existence of a set A_3 such that $(A - A_1) \cap D = A_3 \cap D$ and $T^{-1}A_3 = A_3$. By the first part of the proof we get $I_j \times E_j \subset A_3 \cap D$ for some j and $E_j \in \mathcal{B}$, $p(E_j) > 0$, which contradicts the definition of A_1. ■

COROLLARY 5. (i) *If the partition β is markovian, then*

$$D = \bigcup_{i=1}^q I_i \times E_i .$$

(ii) *If* $\inf|f'| > 2$, *then $D \supset I_i \times E$ for some i and $E \in \mathcal{B}$, $p(E) > 0$.*

From now on let ν denote an f-a.c.i.m. (for the existence see [14]) and assume that $D \subseteq \operatorname{supp} \nu \times Y$. Moreover, let

$$Z_i = \{(i_1, \ldots, i_s) : I_i \subset \operatorname{supp} \nu , \ f_{i_s} \circ \ldots \circ f_{i_1} I_i \supset I_i \text{ and } m(I_i - f_{i_k} \circ \ldots \circ f_{i_1} I_i) > 0 \text{ for } 1 \le k < s\} .$$

THEOREM 7. *Let f be ergodic. If the partition β is markovian and there exists $(i_1, \ldots, i_s) \in Z_i$ for some i such that*

$$\forall B \in \mathcal{B}, \quad T_{i_s} \circ \ldots \circ T_{i_1} B \subset B \Rightarrow p(B) \in \{0, 1\} ,$$

then T is ergodic.

Proof. Let A be a T-invariant set of positive measure. By Lemma 6 we obtain $A \cap D = \bigcup_{j=1}^q I_j \times E_j$. First we show that if $I_j \subset \operatorname{supp} \nu$ then $p(E_j) > 0$. Let j_0 be such that $p(E_{j_0}) > 0$. Then for every $I_j \subset \operatorname{supp} \nu$ there exists a sequence $j_1, \ldots, j_s$ and the set $\widetilde{I} \subset I_{j_0}$ with $f_{j_s} \circ \ldots \circ f_{j_1}(\widetilde{I}) = I_j$. Therefore

$$T^s(\widetilde{I} \times E_{j_0}) = I_j \times T_{j_s} \circ \ldots \circ T_{j_1} E_{j_0} \subset I_j \times E_j .$$

The above implies that $p(E_j) \ge p(T_{j_s} \circ \ldots \circ T_{j_1} E_{j_0}) > 0$. From the assumptions of our theorem and by $p(E_i) > 0$ we get $p(E_i) = 1$ and consequently $A \cap D \supset I_i \times Y$. The ergodicity of f implies $A \cap D \supset \bigcup_j T^j(I_i \times Y) = \bigcup_j f^j(I_i) \times Y = \operatorname{supp} \nu \times Y$, which finishes our proof. ■

COROLLARY 6. *If f is ergodic,* $\inf|f'| > 2$ *and for every i there exists $(i_1, \ldots, i_s) \in Z_i$ such that*

$$\forall B \in \mathcal{B}, \quad T_{i_s} \circ \ldots \circ T_{i_1} B \subset B \Rightarrow p(B) \in \{0, 1\} ,$$

then T is ergodic.

Now we proceed to consider the problem of weak mixing of T.

LEMMA 7. *If f is mixing and T is not weakly mixing, then there exists a set A which is $T \times T$-invariant, $0 < (\mu \times \mu)(A) < 1$, and*

(i) *if the partition β is markovian then*

$$A \cap D \times D = \bigcup_{i,j=1}^q I_i \times I_j \times B_{ij} ,$$

(ii) *if* $\inf|f'| > 2$ *then $A \cap D \times D \supset I_i \times I_i \times B$ for some i, where $B \in \mathcal{B} \times \mathcal{B}$ and $(p \times p)(B) > 0$.*

Proof. By Lemma 6 we may assume that T is ergodic. Hence, if it is not weakly mixing then there exists a measurable nonconstant function F such that $|F| = 1$ and $F(T) = \alpha F$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Therefore

$$F(x, y) = \cos \varphi(x, y) + i \sin \varphi(x, y) ,$$

where $\varphi(x, y) = \arg F(x, y)$ $(-\pi \le \varphi(x, y) < \pi)$.

By Corollary 4 we have $\mathbf{V}_y FG < \infty$ and $\mathbf{V}_y G < \infty$ for a.e. y, which implies that for a.e. y there exists a set A_y, $m(A_y) = 0$, such that $\varphi(\cdot, y) 1_{D \times D}(\cdot, y) \mid [0, 1] - A_y$ is continuous. For $\delta, \gamma \in [-\pi, \pi)$, we define

$$A_{\delta\gamma} = \{(x, y, z, v) : \gamma \le \arg(F(x, y)\overline{F}(z, v)) < \delta\} .$$

We can find $\gamma < \delta$ such that $(\mu \times \mu)(A_{\delta\gamma}) < 1$ and $(\mu \times \mu)(A_{\delta\gamma} \cap I_i \times I_i \times E \times E) > 0$ where $I_i \times E$ is the set given by Corollary 5. The set $A_{\delta\gamma}$ is $T \times T$-invariant. Let B be the set of pairs $(y_0, v_0) \in E \times E$ for which there exists a pair $(x_0, z_0) \in I_i \times I_i$ such that $(x_0, y_0, z_0, v_0) \in A_{\delta\gamma}$ and moreover $\varphi(x) = \varphi(x, y_0) \mid [0, 1] - A_{y_0}$ is continuous at x_0 and $\psi(z) = \varphi(z, v_0) \mid [0, 1] - A_{v_0}$ is continuous at z_0. Then $(p \times p)(B) > 0$.

Let $(y_0, v_0) \in B$. Then $\gamma < \varphi(x_0) - \psi(z_0) + 2k\pi < \delta$, for some $k \in \{-1, 0, 1\}$. Since $\varphi(x) - \psi(z)$ is continuous at (x_0, z_0), there exist intervals I_{y_0} and J_{v_0} such that $I_{y_0} \times J_{v_0} \subset I_i \times I_i$ and $\gamma < \varphi(x) - \psi(z) + 2k\pi < \delta$ for a.e. $(x, z) \in I_{y_0} \times J_{v_0}$. We get

$$\bigcup_{(y,v) \in B} I_y \times J_v \times \{(y, v)\} \subset A_{\delta\gamma} \cap D \times D .$$

The same reasoning as in the proof of Lemma 6 applies to the case of markovian partition ((i)).

For the second case it is sufficient to show that for every I_y, J_v there exist sequences $i_1,\dots,i_s$, $j_1,\dots,j_s$ such that

$$m(f_{i_s}\circ\dots\circ f_{i_1}(I_y)\cap f_{j_s}\circ\dots\circ f_{j_1}(J_v))>0\,.$$

Let I, J be two intervals such that $I\cup J\subseteq\operatorname{supp}\nu$. By Theorem 1 of [4] there exists a positive integer $k(I)$ such that $f^{k(I)}(I)=\operatorname{supp}\nu$. Let $i_1,\dots,i_{k(I)}$ be a sequence such that $m(f_{i_{k(I)}}\circ\dots\circ f_{i_1}(J))>0$. Then we can find another sequence $j_1,\dots,j_{k(I)}$ such that

$$m(f_{i_{k(I)}}\circ\dots\circ f_{i_1}(J)\cap f_{j_{k(I)}}\circ\dots\circ f_{j_1}(I))>0\,.$$

The rest of the proof runs in the same manner as the proof of Lemma 6. ∎

Let $B=\{(i,j):f(I_i)\supset I_j\}$ and let $\mu\approx m\times p$. The property (R) of the family $\{T_i\}_{i=1}^q$ and the measure μ may be formulated in the following form:

There exists a pair $(i,j)\in B$ for which $\{A:T_s^{-1}T_m^{-1}A=T_s^{-1}T_l^{-1}A$ for every l, m such that (s,m), (m,s), (l,s), $(s,l)\in B\}=\{\emptyset,Y\}$ with respect to p, for $s=i,j$.

THEOREM 8. *If f is mixing and the partition β is markovian, then the property* (R) *implies weak mixing of T.*

THEOREM 9. *If f is mixing, $\inf|f'|>2$ and if for every i there exist $(i_1,\dots,i_s),(j_1,\dots,j_s)\in Z_i$ such that*

$$\forall B\in\mathcal{B}\times\mathcal{B}\,,\quad (T_{i_s}\times T_{j_s})\circ\dots\circ(T_{i_1}\times T_{j_1})B\subset B\ \Rightarrow\ (p\times p)(B)\in\{0,1\}\,,$$

then T is weakly mixing.

Remark 3. If T satisfies the assumptions of Theorem 8 or 9, and the transformations T_i are piecewise monotonic, continuous and 1-1 p-a.e., $\mu=\nu\times\overline{p}_1$ for some measure $\overline{p}_1\approx p$ and T_i does not preserve the measure $\overline{p}_1$ for some i, then T is exact.

Remark 4. If T satisfies the assumptions of Theorem 8 or 9 and the transformations f, T_i are piecewise C^2 Lasota-Yorke type maps, then T is exact.

In our considerations we may admit the situation when the family $\{T_i\}$ is infinite and f is a Lasota–Yorke map with countably many intervals of monotonicity. More precisely, we assume that $\{e^{-1}(i)\}_{i<\infty}=\{I_i\}_{i<\infty}$ and

$$\sum_{i=1}^{\infty}(\sup 1/|f_i'|+\mathbf{V}\,1/|f_i'|)<\infty\,.$$

It is not difficult to see that the results of this paper remain true in the above case.

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INSTITUTE OF MATHEMATICS
WROCŁAW TECHNICAL UNIVERSITY
WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW, POLAND

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