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Nonconvolution transforms with oscillating kernels that map $\dot{B}_1^{0,1}$ into itself

by

G. SAMPSON (Auburn, Ala.)

Abstract. We consider operators of the form

$$(\Omega f)(y) = \int_{-\infty}^{\infty} \Omega(y, u) f(u)\, du$$

with $\Omega(y, u) = K(y, u) h(y - u)$, where K is a Calderón–Zygmund kernel and $h \in L^\infty$ (see (0.1) and (0.2)). We give necessary and sufficient conditions for such operators to map the Besov space $\dot{B}_1^{0,1}$ ($= B$) into itself. In particular, all operators with $h(y) = e^{i|y|^a}$, $a > 0$, $a \neq 1$, map B into itself.

0. Introduction. Form $\Omega(f)(y) = \int \Omega(y, u) f(u)\, du$ where

$$(0.1) \qquad \Omega(y, u) = \Omega_1(y, u)\psi(y - u) - \Omega_1(y + \tfrac{1}{2}, u)\psi(y + \tfrac{1}{2} - u)$$

with $\psi \in C^\infty$, $\psi(x) = 0$ if $|x| \leq 2$ and $\psi(x) = 1$ if $|x| \geq 3$. It follows from (0.1) that $\Omega(y, u) = 0$ if $|y - u| \leq \tfrac{3}{2}$. Here we take $\Omega_1(y, u) = K(y, u) h(y - u)$ where $h \in L^\infty$ and

$$(0.2) \qquad \begin{cases} \text{(a)} & |K(y, u)| \leq \dfrac{c}{|y - u|}, \\[2mm] \text{(b)} & |\nabla K| \leq \dfrac{c}{|y - u|^2}. \end{cases}$$

Our main result (Theorem 3.2) gives necessary and sufficient conditions on Ω in order that

$$(0.3) \qquad \|\Omega(f)\|_{\dot{B}_1^{0,1}} \leq c\|f\|_{\dot{B}_1^{0,1}}$$

where $\dot{B}_1^{0,1}$ ($= B$) is a Besov space which we discuss more fully in Section 1. In [20], we solved this problem for the case of convolutions.

In this paper, we work in one dimension and for the examples we have in mind we take $\Omega_1(y, u) = K(y, u)e^{i|y-u|^a}$, $a > 0$, $a \neq 1$. We show that

(0.3) holds for all such Ω's; this result appears in the Corollary in Section 3. The methods of this paper can be used to study the kernels $\Omega_1(y,u) = K(y,u)e^{ia(y-u)}$ for a large class of real-valued functions $a(y)$ (see [17] for the types of functions we have in mind). Also these results can be generalized to $\mathbb{R}^n$ (see [21] for the convolution case). By the definition (0.1) we get

$$(0.4) \qquad\qquad \int \Omega(y,u)\,dy = 0\,.$$

This condition (0.4), which is necessary in order to obtain our results, explains our definition (0.1) (see also [13] in this regard). Note that here we only focus on the problem at infinity ($\Omega(y,u)$ is supported in $|y-u| \geq \frac{3}{2}$), since the problem about the origin is considered in many works (for example see [5]).

We let c_i stand for a positive constant, and we use c generically. We define $1/p + 1/p' = 1$ and here we take $1 < p \leq 2$. When we write $\int f(x)\,dx$ we mean $\int_{-\infty}^{\infty} f(x)\,dx$ and when we write $\int_a^b f(x)\,dx$ we mean that $b \geq a$. We let $\chi_E(x)$ stand for the characteristic function of the set E.

Recently, D. Fan (*On oscillating kernels in* $\dot{B}_1^{0,1}$, preprint, 1993) has come up with a very interesting and very simple proof of these results in the convolution case.

We explained earlier our motivation for (0.1). However, when $0 < a < 1$, it follows that these kernels are in L^1. In the Corollary at the end of Section 3, we show these kernels satisfy (1.7)(c).

1. General notions and preliminary estimates. In the papers [12, 17–20] we discussed the concept of regular kernels for convolution transforms. Here we generalize this notion to kernels of nonconvolution transforms. Now it is these kernels $\Omega_1(y,u)$ defined in the introduction which will be tested for regularity, rather than the kernels $\Omega(y,u)$. We say $\Omega_1(y,u)$ is *regular* if

$$(1.1) \qquad\qquad \Omega_1(y,u) = k(y,u)g(y-u)\,,$$

where $g : \mathbb{R} \to \mathbb{C}$, and for some $\varepsilon > 0$ $(0 < \varepsilon \leq 1)$,

(1.2)
$$\begin{cases} \text{(a)} & |k(y,u)-k(y',u)||g(y-u)| \leq \dfrac{c|y-y'|^\varepsilon}{|y-u|^{1+\varepsilon}} \quad \text{if } |y-u| \geq 2|y-y'|\,, \\[2ex] \text{(b)} & |k(y,u)-k(y,u')||g(y-u)| \leq \dfrac{c|u-u'|^\varepsilon}{|y-u|^{1+\varepsilon}} \quad \text{if } |y-u| \geq 2|u-u'|\,, \end{cases}$$

and

$$(1.3) \qquad \Omega_1 \in L_2^2\,, \quad \text{i.e.} \quad \|\Omega_1(f)\|_2 \leq c\|f\|_2\,, \quad \text{for ``nice'' } f \in L^2(\mathbb{R})\,.$$

If in addition Ω_1 satisfies the following properties we say that $\Omega_1 \in \mathfrak{R}$: for a given parameter $0 < a$ and for some $1 < p \leq 2$ (in case $a > 1$, we need even that $p' > a - 1$) we have

$$(1.4)(p) \qquad\qquad g \in L_p^p\,, \quad \text{i.e.} \quad \|g * f\|_p \leq c\|f\|_p\,,$$

$$(1.5)(p) \qquad\qquad |k(y,u)| \leq \frac{c}{|y-u|^{a/p'}}\,, \quad \text{and}$$

$$(1.6)(p) \qquad |g(y) - g(u)| \leq \frac{c|y-u|}{|u|^{2-a-a/p'}} \quad \text{if } |u| \geq \max(2|y-u|, 1)$$

plus

(1.7)
$$\begin{cases} \text{(a)} & |\Omega(y,u) - \Omega(y,u')| \leq \dfrac{c|u-u'|}{|y-u|^{2-a}} \\ & \qquad\qquad \text{if } |y-u| \geq \max(2|u-u'|, 1),\ 0 < a < 1\,, \\[2ex] \text{(b)} & |\Omega(y,u) - \Omega(y,u')| \leq c|u-u'| \\ & \qquad\qquad \text{if } \frac{3}{2} \leq |y-u| \leq 5,\ |u-u'| \leq 1 \text{ and } a > 1\,, \\[2ex] \text{(c)} & |\Omega(y,u)| \leq \dfrac{c}{|y-u|^{2-a}} \qquad \text{if } |y-u| \geq 2 \text{ and } 0 < a < 1\,. \end{cases}$$

Re m a r k. In the case when $p = 2$, we write (1.4), (1.5), and (1.6) respectively. In case $0 < a < 1$, we just need the standard decomposition (1.1) for Ω_1, i.e. k, g satisfies (1.4)–(1.6). However, in case $a > 1$, we assume that Ω_1 has two decompositions, one k, g (from (1.1)) satisfies (1.4)–(1.6) and the other k, g (of course this will be different from the first pair) will satisfy $(1.4)(p)$, $(1.5)(p)$, and $(1.6)(p)$ where $p' > a - 1$. This latter decomposition is confined to the proof of Proposition 2.5, in our estimates of I_1 there.

We stated in the introduction that $\Omega_1(y,u) = K(y,u)h(y-u)$ where $h \in L^\infty$. We are not suggesting that this is the decomposition we have in mind for (1.1), in fact for the most part it is not. See the Corollary in Section 3 for possible decompositions. Also, see [17].

We need to discuss the Besov space $\dot{B}_1^{0,1}$ $(= B)$. We say that $f \in B$ if

(1.8)
$$\begin{cases} \text{(a)} & f(x) = \sum_{l=1}^{\infty} \lambda_l b_l(x), \text{ with} \\ \text{(b)} & b_l(x) = \frac{1}{h}(\chi_{[r,r+h]}(x) - \chi_{[r-h,r]}(x)),\ h > 0\,, \end{cases}$$

r is a real number, and

$$\|f\|_B = \inf \sum |\lambda_l|\,,$$

where the inf is taken over all possible representations (1.8)(a) of f (see [6]). An equivalent characterization of B is given in [9]. Throughout we let $\varphi \in C^\infty$, $\operatorname{supp} \varphi \subset \{|y| \leq 1\}$, $\int \varphi = 0$, $\int |\hat{\varphi}|^2 = 1$, and $\varphi_t(y) = (1/t)\varphi(y/t)$.

In [9], it was shown that $f \in B$ if and only if

$$\int_0^\infty \|\varphi_t * f\|_1 \frac{dt}{t} < \infty$$

and there are two positive constants c_1, c_2 so that

$$(1.9) \qquad c_1\|f\|_B \leq \int_0^\infty \|\varphi_t * f\|_1 \frac{dt}{t} \leq c_2\|f\|_B.$$

Throughout this paper we let $b(x)$ in (1.8)(b) represent the special atom centered at r with radius h.

Thus, by (1.8) and (1.9), proving that Ω maps B into itself is equivalent to showing that

$$(1.10) \qquad \int_0^\infty \|\varphi_t * \Omega(b)\|_1 \frac{dt}{t} \leq c,$$

where c is a positive constant independent of b. In Theorem 1.1 of [20], where we discussed this problem for convolution transforms, we obtained two conditions. Here we show that the nonconvolution kernels map B into B if and only if

$$(1.11) \qquad \int_0^h \frac{dt}{t} \int_{|x-r|\geq 6h} dx \left| \sum_{i=0}^1 (-1)^i \int dy\, \varphi_t(x-y) \right.$$
$$\left. \times \int k(x+\alpha_i, u) M b(u)\, du \right| \leq c,$$

$$(1.12) \qquad \int_h^\infty \frac{dt}{t} \int_{|x-r|\geq 6t} dx \left| \sum_{i=0}^1 (-1)^i \int dy\, \varphi_t(x-y) \right.$$
$$\left. \times k(y+\alpha_i, r) g\psi * b(y+\alpha_i) \right| \leq c,$$

where $\alpha_0 = 0$, $\alpha_1 = \frac{1}{2}$, and $M = g(y + \alpha_i - u)\psi(y + \alpha_i - u)$. A complete statement of this result appears in Theorem 3.2.

The next result which is needed here appeared in [20].

PROPOSITION 1.1. *Let $a > 0$ and $a \neq 1$. If Ω_1 is a regular kernel that satisfies* (1.4)–(1.6), *then*

$$(1.13) \qquad \int_0^h \frac{dt}{t} \int_{|x-r|\geq 6h} |k(x,r)| |\varphi_t * g * b(x)|\, dx \leq c$$

and

$$(1.14) \qquad \int_h^\infty \frac{dt}{t} \int_{|x-r|\geq 6t} |k(x,r)| |\varphi_t * g * b(x)|\, dx \leq c.$$

Proof. In case $0 < a < 1$, this result appears in the proof of Theorem 1.7 of [20] (it was called (1.9) and (1.10) in [20]). For the case $a > 1$, this result appears in the proof of Theorem 1.8 of [20]. ∎

We need this next series of results.

PROPOSITION 1.2. *If Ω satisfies* (1.3), *then*

$$\int_{3/16}^h \frac{dt}{t} \int_{|x-r|\leq 6t} |\varphi_t * \Omega(b)|\, dx \leq c.$$

Proof.

$$\int_{3/16}^h \frac{dt}{t} \int_{|x-r|\leq 6t} |\varphi_t * \Omega(b)|\, dx$$
$$\leq \int_{3/16}^h \frac{dt}{t} t^{1/2} \|\varphi_t * \Omega(b)\|_2 \leq \int_{3/16}^h \frac{dt}{t} t^{1/2} \|\Omega(b)\|_2 \leq c \int_{3/16}^h \frac{dt}{t} t^{1/2} \|b\|_2 \leq c. \; ∎$$

PROPOSITION 1.3. *Suppose* (1.2)(a) *is satisfied. Then*

$$\int_0^s \frac{dt}{t} \int dx \left| \int dy\, \varphi_t(x-y) \int_{|x-u|\geq 3s} du\, (k(y,u) - k(x,u)) g(y-u) b(u) \right| \leq c.$$

Proof. We get

$$\int_0^s \frac{dt}{t} \int dx \left| \int dy\, \varphi_t(x-y) \int_{|x-u|\geq 3s} du\, (k(y,u) - k(x,u)) g(y-u) b(u) \right|$$
$$\leq \int_0^s \frac{dt}{t} \int dx \int dy\, |\varphi_t(x-y)| \int_{|x-u|\geq 3s} |k(y,u) - k(x,u)| |g(y-u)| |b(u)|$$
$$\leq c \int_0^s \frac{dt}{t} \int dx \int dy\, |\varphi_t(x-y)| \int_{|x-u|\geq 3s} \frac{|x-y|^\varepsilon}{|x-y|^{1+\varepsilon}} |b(u)|\, du$$

by (1.2)(a), since $|y - u| \geq |x - u| - |x - y| \geq 2s \geq 2|y - x|$

$$\leq c \int_0^s \frac{dt}{t} t^\varepsilon \int du\, |b(u)| \int_{|y-u|\geq 2s} \frac{dy}{|y-u|^{1+\varepsilon}} \int dx\, |\varphi_t(x-y)| \leq c. \; ∎$$

2. Preliminary estimates. The results of this section consist of mostly technical results. The parameter α we use throughout this paper just needs to be either 0 or 1/2, but our methods actually handle all the cases $0 \leq \alpha \leq \frac{1}{2}$, in a uniform way. The next result generalizes Proposition 1.2 of [20] to nonconvolution kernels.

PROPOSITION 2.1. *Let* $0 \leq \alpha \leq \frac{1}{2}$ *and* $a > 0$. *If* (1.2) *is satisfied,* $M_1 = k(y + \alpha, u) - k(x + \alpha, u)$, $M_2 = k(y + \alpha, u) - k(y + \alpha, r)$ *and* $N = g(y + \alpha - u)\psi(y + \alpha - u)$, *then*

$$I + II = \int_0^h \frac{dt}{t} \int_{|x-r| \geq 6h} dx \left| \int dy\, \varphi_t(x-y) \int du\, M_1 N b(u) \right|$$

$$+ \int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} dx \left| \int dy\, \varphi_t(x-y) \int du\, M_2 N b(u) \right| \leq c.$$

Proof. We begin by estimating I:

$$I \leq \int_0^h \frac{dt}{t} \int_{|x-r| \geq 6h} dx \left| \int du\, b(u) \int_{|y-u| \geq 3h} dy\, (k(y+\alpha, u) - k(x+\alpha, u)) \right.$$

$$\left. \times g(y + \alpha - u)\psi(y + \alpha - u)\varphi_t(x-y) \right|$$

$$+ \int_0^h \frac{dt}{t} \int_{|x-r| \geq 6h} dx \left| \int du\, b(u) \int_{|y-u| \leq 3h} dy\, (k(y+\alpha, u) - k(x+\alpha, u)) \right.$$

$$\left. \times g(y + \alpha - u)\psi(y + \alpha - u)\varphi_t(x-y) \right| = I_1 + I_2.$$

Now since $|y - u| \geq \frac{3}{2}$ (otherwise $I_1 = I_2 = 0$), we notice that $|y + \alpha - u| \geq |y - u| - |\alpha| \geq \frac{2}{3}|y - u|$. Since $|y - u| \geq 3h$, by Proposition 1.3 we get $I_1 \leq c$. For I_2, since $|y - u| \leq 3h$ we have $|x - u| \leq |x - y| + |y - u| \leq 3h + t \leq 4h$ and therefore $|x - r| \leq |x - u| + |u - r| \leq 4h + h = 5h$. But I_2 is restricted to x's with $|x - r| \geq 6h$, hence $I_2 = 0$. This completes our estimate for I.

Now we estimate the term II:

$$II \leq \int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} dx \left| \int dy\, \varphi_t(x-y) \right.$$

$$\times \int_{|y-u| \leq 3|u-r|} du\, (k(y+\alpha, u) - k(y+\alpha, r))$$

$$\left. \times g(y + \alpha - u)\psi(y + \alpha - u)b(u) \right|$$

$$+ \int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} dx \left| \int dy\, \varphi_t(x-y) \right.$$

$$\times \int_{|y-u| \geq 3|u-r|} du\, (k(y+\alpha, u) - k(y+\alpha, r))$$

$$\left. \times g(y + \alpha - u)\psi(y + \alpha - u)b(u) \right| = II_1 + II_2.$$

As above $|y + \alpha - u| \geq \frac{2}{3}|y - u| \geq 2|u - r|$ and so by (1.2)(b) we get

$$II_2 \leq \int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} dx \int dy\, |\varphi_t(x-y)|$$

$$\times \int_{|y-u| \geq 3|u-r|} du\, \frac{|u-r|^\varepsilon}{|y-u|^{1+\varepsilon}} |b(u)|,$$

but here $\frac{4}{3}|y - u| \geq |y - r| \geq \frac{2}{3}|y - u|$ and $\frac{7}{6}|x - r| \geq |y - r| \geq \frac{5}{6}|x - r|$ hence

$$II_2 \leq ch^\varepsilon \int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} \frac{dx}{|x-r|^{1+\varepsilon}} \int dy\, |\varphi_t(x-y)| \int du\, |b(u)| \leq c.$$

For the term II_1, $|x - r| \leq |x - y| + 4|u - r| \leq 5t$; but II_1 is restricted to x's with $|x - r| \geq 6t$ and so $II_1 = 0$. This completes the proof of the proposition. ∎

PROPOSITION 2.2. *Let* $h > 0$ *and* $0 \leq \alpha \leq \frac{1}{2}$. *Suppose that* (0.1), (1.1) *and* (1.2)(a) *hold. Then if*

$$(2.1) \quad \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \leq 6h} dx \left| \int dy\, \varphi_t(x-y) \int du\, k(x+\alpha, u) M b(u) \right| \leq c,$$

where $M = g(y + \alpha - u)\psi(y + \alpha - u)$, *then*

$$(2.2) \quad \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \leq 6h} |\varphi_t * \Omega(b)|\, dx \leq c.$$

Proof. Because of (0.1) and (1.1) it suffices to show that

$$I + II$$

$$= \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \leq 6h} dx \left| \int dy\, \varphi_t(x-y) \right.$$

$$\left. \times \int du\, (k(y+\alpha, u) - k(x+\alpha, u))\psi(y + \alpha - u)g(y + \alpha - u)b(u) \right|$$

$$+ \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \leq 6h} dx$$

$$\times \left| \int dy\, \varphi_t(x-y) \int k(x+\alpha, u)g(y + \alpha - u)\psi(y + \alpha - u)b(u)\, du \right| \leq c.$$

Because of (2.1) we need to estimate I. Since $|y + \alpha - u| \geq \frac{2}{3}|y - u| \geq 1 \geq 2t$, by (1.2)(a) we get $I \leq c$. This completes our result. ∎

As we have seen so far, we have made use of the decomposition (1.1). For the next result we make a further decomposition, namely,

$$\Omega_1(y,u) = k(y,u)|g(y-u)|r(y-u)$$

where $r(u) = g(u)/|g(u)|$ and does not vanish except on sets of measure zero. Furthermore, using the notation of the introduction, we get $K(y,u) = k(y,u)|g(y-u)|$. Set $K_\lambda(y,u) = k(y,u)|g(y-u-\lambda)|$.

LEMMA 2.3. *Let $h > 0$, $0 \le \alpha \le \frac{1}{2}$ and $\beta = 1/(1-2a)$ and $a > 0$. Suppose that (0.1), (1.1), $(1.2)(a)$, (1.5) and (1.6) hold. Then if*

$$(2.3) \qquad \left| \int_c^d r(x+\alpha-u-\lambda)\chi(|x-u| \ge 5t^\beta)\, du \right| \le ct^{(1-a)\beta}$$

if $|\lambda| \le t \le 1$ and $a > 1$, and

$(2.4) \qquad K_\lambda(y,u)$ *satisfies (0.2)*   *if $|y-u| \ge \max(2|\lambda|, 1)$,*

then (2.2) holds.

P r o o f. By Proposition 2.2 it suffices to show (2.1). Now,

$$\int_0^{3/16} \frac{dt}{t} \int_{|x-r| \ge 6h} dx \left| \int du\, b(u)k(x+\alpha,u) \right.$$

$$\left. \times \int dy\, \varphi_t(x-y)g(y+\alpha-u)\psi(y+\alpha-u) \right|$$

$$\le \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \le 6h} dx \left| \int_{|x-u| \le 5} du\, b(u)k(x+\alpha,u) \right.$$

$$\left. \times \int dy\, \varphi_t(x-y)\psi(y+\alpha-u)g(y+\alpha-u) \right|$$

$$+ \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \le 6h} dx \left| \int_{|x-u| \ge 5} du\, b(u)k(x+\alpha,u) \right.$$

$$\left. \times \int dy\, \varphi_t(x-y)\psi(y+\alpha-u)g(y+\alpha-u) \right|$$

$$= I + II.$$

We have

$$I \le \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \le 6h} dx \left| \int_{|x-u| \le 5} du\, b(u)k(x+\alpha,u) \right.$$

$$\left. \times \int dy\, \varphi_t(x-y)(\psi(y+\alpha-u) - \psi(x+\alpha-u))g(y+\alpha-u) \right|$$

$$+ \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \le 6h} dx \left| \int_{|x-u| \le 5} du\, b(u)k(x+\alpha,u)\psi(x+\alpha-u) \right.$$

$$\left. \times \int dy\, \varphi_t(x-y)g(y+\alpha-u) \right|$$

$$= I_1 + I_2.$$

For the term I_1, $|x+\alpha-u| \ge 1$ since otherwise $|y+\alpha-u| \le |x+\alpha-u| + |x-y| \le 1 + t \le 2$ and so $I_1 = 0$. Thus

$$I_1 \le \frac{c}{h} \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \le 6h} dx \int_{|x-u| \le 5} du$$

$$\times \int dy\, |\varphi_t(x-y)||x-y||k(x+\alpha,u)g(y+\alpha-u)|$$

and our result follows by $(2.4)(a)$ (i.e. K_λ satisfies $(0.2)(a)$) for then

$$(2.5) \qquad |k(x+\alpha,u)g(x+\alpha-u+y-x)| \le c \quad \text{if}$$

$$(2.6) \qquad \begin{aligned} &|x+\alpha-u| \ge 2|y-x| \quad \text{and} \\ &|x+\alpha-u| \text{ is uniformly bounded away from zero.} \end{aligned}$$

But in fact we get $|x+\alpha-u| \ge 1 \ge 2t$, hence $I_1 \le c$.

Now we estimate I_2:

$$(2.7) \quad I_2 = \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \le 6h} dx \left| \int_{|x-u| \le 5} du\, b(u)k(x+\alpha,u)\psi(x+\alpha-u) \right.$$

$$\left. \times \int dy\, \varphi_t(x-y)(g(y+\alpha-u) - g(x+\alpha-u)) \right|$$

$$\le c \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \le 6h} dx \int_{|x-u| \le 5} |b(u)||x+\alpha-u|^{a-2} t \int dy\, |\varphi_t(x-y)|$$

by (1.5) and (1.6), since $|x+\alpha-u| \ge 2 \ge 2t \ge 2|x-y|$, and since $|x+\alpha-u| \le 5\frac{1}{2}$ it follows that $I_2 \le c$.

Now for $0 < a < 1$ we get

$$II = \int_0^{3/16} \frac{dt}{t} \int_{|x-r| \le 6h} dx \left| \int_{|x-u| \ge 5} du\, b(u)k(x+\alpha,u) \right.$$

$$\left. \times \int dy\, \varphi_t(x-y)g(y+\alpha-u) \right|$$

since $|y+\alpha-u| \ge |x-u| - \frac{1}{2} - t \ge 3$ and so $\psi(y+\alpha-u) = 1$. Now arguing

as in (2.7) we get

$$II \leq c \int_0^{3/16} \frac{dt}{t} \int_{|x-r|\leq 6h} dx \int_{|x-u|\geq 5} du\, |b(u)||x+\alpha-u|^{a-2}t \leq c,$$

and this completes our result in case $0 < a < 1$.

To obtain (2.1) in case $a > 1$, we notice that

$$II \leq \int_0^{3/16} \frac{dt}{t} \int_{|x-r|\leq 6h} dx \left| \int_{5\leq|x-u|\leq 5t^\beta} du\, k(x+\alpha,u)b(u)\varphi_t * g(x+\alpha-u) \right|$$

$$+ \int_0^{3/16} \frac{dt}{t} \int_{|x-r|\leq 6h} dx \left| \int_{|x-u|\geq 5t^\beta} du\, b(u)k(x+\alpha,u)\varphi_t * g(x+\alpha-u) \right|$$

$$= II_1 + II_2\,.$$

As in (2.7),

$$II_1 \leq c \int_0^{3/16} dt \int_{|x-r|\leq 6h} dx \int_{5\leq|x-u|\leq 5t^\beta} du\, |b(u)||x-u|^{a-2} \int dy\, |\varphi_t(x-y)|$$

$$\leq c \int_0^{3/16} dt\, t^{-(a-1)/(2a-1)} \leq c \quad \text{since } a > 1\,.$$

We are left with estimating the term II_2. We consider the integrand of II_2 which becomes

$$A = \int_{-1}^1 ds\, \varphi(s) \int_{|x-u|\geq 5t^\beta} du\, b(u)k(x+\alpha,u)$$

$$\times |g(x+\alpha-u-st)|r(x+\alpha-u-st)\,.$$

Thus, $b(u)\chi(|x-u| \geq 5t^\beta)$ is supported in a fixed finite number of u-intervals (for each x) whose endpoints we shall denote by c and d, and so after integration by parts a typical term becomes

$$(2.8) \qquad A = k(x+\alpha,d)|g(x+\alpha-d-st)|$$

$$\times \int_c^d r(x+\alpha-u-st)\chi(|x-u| \geq 5t^\beta)\, du$$

$$- \frac{1}{h} \int_c^d du\, \frac{\partial}{\partial u}(k(x+\alpha,u)|g(x+\alpha-u-st)|)$$

$$\times \int_c^u dv\, r(x+\alpha-v-st)\chi(|x-v| \geq 5t^\beta)\,.$$

By (2.4) and (2.8) we conclude that

$$|A| \leq \frac{c}{h|x-d|} \left| \int_c^d r(x+\alpha-u-st)\chi(|x-u| \geq 5t^\beta)\, du \right|$$

$$+ \frac{c}{h} \int_c^d \frac{du}{|x-u|^2} \left| \int_c^u dv\, r(x+\alpha-v-st)\chi(|x-v| \geq 5t^\beta) \right|$$

$$\leq \frac{c}{h} t^{a/(2a-1)}$$

by (2.3) and hence it follows that $II_2 \leq c$. This completes the proof of the lemma. ∎

(2.9)   Let $\psi_0, \psi \in C^\infty$ so that $0 \leq \psi \leq 1, \psi_0$ is supported in $|u| \leq 1, \psi$ is supported in $\frac{1}{4} \leq |u| \leq 2$ and $\psi_l(u) = \psi(u/2^l)$ with $\sum_{l=0}^\infty \psi_l(u) = 1$.

Next let $\Omega_l(f)(u) = \int \Omega(y,u)\psi_l(y-u)f(y)\, dy$ and

$$\Omega_l(f)(y) = \int \Omega(y,u)\psi_l(y-u)f(u)\, du.$$

We need the next condition for either representation of Ω_l:

$$(2.10) \qquad\qquad \|\Omega_l(f)\|_2 \leq c_l \|f\|_2\,.$$

PROPOSITION 2.4. *Let* $a > 0, a \neq 1$ *and* $0 \leq \alpha \leq \frac{1}{2}$. *Suppose* (0.1), (1.1), (1.2)(b), (1.4), (1.5), (1.7)(c), (2.10) *hold and there exists a* $\beta > 1$ *so that*

$$(2.11) \qquad \sum_{l=1}^\infty l c_l + \sum_{j=1}^\infty \frac{1}{2^{j/2}} \sum_{l=j+4}^\infty c_l 2^{l/2} \leq c\,.$$

Then

$$\int_{3/16}^h \frac{dt}{t} \int_{|x-r|\leq 6h} |\varphi_t * \Omega(b)|\, dx \leq c\,.$$

P r o o f. Since (2.11) implies that Ω satisfies (1.3), by Proposition 1.2 we get

$$\int_{3/16}^h \frac{dt}{t} \int_{|x-r|\leq 6t} dx\, |\varphi_t * \Omega(b)| \leq c.$$

Also, we obtain

$$\int_{3/16}^1 \frac{dt}{t} \int_{|x-r|\leq 6h} dx\, |\varphi_t * \Omega(b)| \leq c$$

(note $h \geq \frac{3}{16}$). We need to estimate

$$I = \int\limits_1^h \frac{dt}{t} \int\limits_{6t \leq |x-r| \leq 6h} dx$$

$$\times \left| \int dy \, \varphi_t(x-y) \int du \, b(u) \Omega(y,u) \sum_{l=0}^\infty \psi_l(y-u) \sum_{s=l-2}^{l+2} \psi_s(y-u) \right|$$

where ψ, ψ_0 satisfy (2.9) and $\left(\sum_{l=0}^\infty \psi_l(x) \right)^2 = 1$ while $\psi_l \psi_s = 0$ if $l \leq s - 3$ or $l \geq s + 3$. By (0.1) since $\Omega(y,u) = 0$ if $|y-u| \leq \frac{3}{2}$, we get

$$I = \int\limits_1^h \frac{dt}{t} \int\limits_{6t \leq |x-r| \leq 6h} dx$$

$$\times \left| \int dy \, \varphi_t(x-y) \int du \, b(u) \Omega(y,u) \sum_{l=1}^\infty \psi_l(y-u) \sum_{s=l-2}^{l+2} \psi_s(y-u) \right|.$$

Furthermore, by (0.4) (and (0.1)) we get

$$\int \sum_{l=1}^\infty \psi_l(y-u) \sum_{s=l-2}^{l+2} \psi_s(y-u) \Omega(y,u) \, dy = 0 \quad \text{for each } u.$$

Thus,

$$I = \int\limits_1^h \frac{dt}{t} \int\limits_{6t \leq |x-r| \leq 6h} dx \left| \int du \, b(u) \right.$$

$$\times \left. \int dy \, (\varphi_t(x-y) - \varphi_t(x-u)) \Omega(y,u) \sum_{l=1}^\infty \psi_l(y-u) \sum_{s=l-2}^{l+2} \psi_s(y-u) \right|.$$

Now (with $2^{m-1} \leq h \leq 2^m$) we get

$$I \leq \sum_{j=1}^m \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \leq |x-r| \leq 6h} dx \left| \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} \int du \, b(u) \right.$$

$$\times \left. \int dy \, (\varphi_t(x-y) - \varphi_t(x-u)) \Omega(y,u) \psi_l(y-u) \psi_s(y-u) \right|$$

$$+ \sum_{j=1}^m \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \leq |x-r| \leq 6h} dx \left| \sum_{l=j+5}^\infty \sum_{s=l-2}^{l+2} \int du \, b(u) \right.$$

$$\times \left. \int dy \, (\varphi_t(x-y) - \varphi_t(x-u)) \Omega(y,u) \psi_l(y-u) \psi_s(y-u) \right|$$

$$= II + III.$$

Now let us consider II. In the integrand of II we can always take

$s \leq l \leq j+4$. Note that the y integral is supported in $u+2^{l-2} \leq y \leq u+2^{l+1}$ or $u - 2^{l+1} \leq y \leq u - 2^{l-2}$ for each fixed u. We shall argue the case of the first interval $(u \leq y)$; the second is similar and will be omitted here. Using integration by parts we get

$$\int\limits_{u+2^{l-2}}^{u+2^{l+1}} (\varphi_t(x-y) - \varphi_t(x-u)) \psi_s(y-u) \Omega(y,u) \psi_l(y-u) \, dy$$

$$= -\frac{1}{2^s} \int\limits_{u+2^{l-2}}^{u+2^{l+1}} dy \, \psi_s'(y-u)(\varphi_t(x-y) - \varphi_t(x-u))$$

$$\times \int \Omega(v,u) \psi_l(v-u) g(v) \, dv$$

$$+ \frac{1}{t} \int dy \, \varphi_t'(x-y) \int\limits_{y-2^{l+1}}^{y-2^{l-2}} du \, b(u) \psi_s(y-u)$$

$$\times \int \Omega(v,u) \psi_l(v-u) g(v) dv = A_{ls} + B_{ls}$$

where $g(v) = 1$ if $y - 2^{l+1} + 2^{l-2} \leq v \leq y$ and zero elsewhere. Notice that we are in the case where $l \leq j+4$, and $2^{l-2} \leq |y-u| \leq 2^{l+1}$, and so $|y-u| \leq 2^{j+5} \leq 2^6 t$. If $|x-y| \geq 128t$ then $|x-u| \geq |x-y| - |y-u| \geq 64t$ and so both $\varphi_t(x-u) = \varphi_t(x-y) = 0$. Thus, the integrand would be zero unless $|x-y| \leq 128t$. Therefore,

$$II \leq \sum_{j=1}^m \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \leq |x-r| \leq 6h} dx \, (|A_{ls}| + |B_{ls}|)$$

$$\leq c \sum_{j=1}^m \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} \frac{1}{2^l} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \leq |x-r| \leq 6h} dx \int\limits_{|x-y| \leq 128t} dy$$

$$\times \int du \, |b(u)| |\psi_s'(y-u)| |\Omega_l(g)| |\varphi_t(x-y) - \varphi_t(x-u)|$$

$$+ c \sum_{j=1}^m \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t^2} \int\limits_{6t \leq |x-r| \leq 6h} dx \int\limits_{|x-y| \leq 128t} dy$$

$$\times |\varphi_t'(x-y)| \int du \, |b(u)| |\psi_s(y-u)| |\Omega_l(g)|$$

$$\leq c \sum_{j=1}^m \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} \frac{1}{2^l} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \leq |x-r| \leq 6h} dx \int\limits_{|x-y| \leq 128t} dy$$

$$\times \int du \, |b(u)| \frac{|y-u|}{t^2} |\psi_s'(y-u)| |\Omega_l(g)|$$

$$+ c \sum_{j=1}^{m} \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t^2} \int_{6t \le |x-r| \le 6h} dx \int_{|x-y| \le 128t} dy$$

$$\times |\varphi_t'(x-y)| \left(\int_{y-2^{l+1}}^{y-2^{l-2}} |b(u)|^2 |\psi_s(y-u)|^2 du \right)^{1/2} \|\Omega_l(g)\|_2$$

$$\le c \sum_{j=1}^{m} \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t^3} \int_{6t \le |x-r| \le 6h} dx \int_{|x-y| \le 128t} dy$$

$$\times \frac{1}{h} \int du \, |\psi_s'(y-u)| \, |\Omega_l(g)|$$

$$+ c \sum_{j=1}^{m} \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t^2} 2^{l/2} c_l \|g\|_2 \quad \text{by (2.10)}$$

$$\le c \sum_{j=1}^{m} \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t^2} 2^{l/2} \|\Omega_l(g)\|_2 + c \sum_{j=1}^{m} \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} \frac{2^l}{2^j} c_l$$

$$\le c \sum_{j=1}^{m} \sum_{l=1}^{j+4} \sum_{s=l-2}^{l+2} 2^{l-j} c_l \le c \quad \text{by (2.11)}.$$

Next we estimate

$$III \le \sum_{j=1}^{m} \int_{2^{j-1}}^{2^j} \frac{dt}{t} \int_{6t \le |x-r| \le 6h} dx \left| \int_{|x-y| \le 2t} dy \, \varphi_t(x-y) \int du \right.$$

$$\times \left. \Omega(y,u) b(u) \sum_{l=j+4}^{\infty} \psi_l(y-u) \sum_{s=l-2}^{l+2} \psi_s(y-u) \right|$$

$$+ \sum_{j=1}^{m} \sum_{l=j+4}^{\infty} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t} \int_{6t \le |x-r| \le 6h} dx \left| \int_{|x-y| \ge 2t} dy \int du \right.$$

$$\times \left. \Omega(y,u) b(u) \psi_l(y-u) \psi_s(y-u) \varphi_t(x-u) \right| = III_1 + III_2.$$

Now,

$$III_1 \le \sum_{j=1}^{m} \sum_{l=j+4}^{\infty} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t} \int_{6t \le |x-r| \le 6h} dx \, |\varphi_t * \Omega_l(b)|$$

$$\le \sum_{j=1}^{m} \sum_{l=j+4}^{\infty} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t} h^{1/2} \|\varphi_t * \Omega_l(b)\|_2$$

$$\le \sum_{j=1}^{m} \sum_{l=j+4}^{\infty} \sum_{s=l-2}^{l+2} h^{1/2} c_l \|b\|_2 \quad \text{by (2.10)}$$

$$\le \sum_{j=1}^{m} \sum_{l=j+4}^{\infty} c_l \le c \sum_{l=1}^{\infty} l c_l \le c \quad \text{by (2.11)}.$$

For III_2, we first notice that with β as chosen in the hypothesis,

$$\sum_{j=1}^{m} \int_{2^{j-1}}^{2^j} \frac{dt}{t} \int_{6t \le |x-r| \le 6h} dx \left| \sum_{l=j+4}^{(j+4)\beta} \sum_{s=l-2}^{l+2} \int_{|y-x| \sim 2^l} dy \int du \, \Omega(y,u) \right.$$

$$\times \left. b(u) \varphi_t(x-u) \psi_l(y-u) \psi_s(y-u) \right|$$

$$\le \sum_{j=1}^{m} \int_{2^{j-1}}^{2^j} \frac{dt}{t} \int_{6t \le |x-r| \le 6h} dx \int_{|y-x| \sim 2^l} dy \sum_{l=j+4}^{(j+4)\beta} \sum_{s=l-2}^{l+2} |\Omega_l(b\varphi_t)|$$

$$\le \sum_{j=1}^{m} \sum_{l=j+4}^{(j+4)\beta} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t} \int_{6t \le |x-r| \le 6h} dx \, 2^{l/2} \|\Omega_l(b\varphi_t)\|_2$$

$$\le \sum_{j=1}^{m} \sum_{l=j+4}^{(j+4)\beta} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t} \int_{6t \le |x-r| \le 6h} dx \, 2^{l/2} c_l \|b\varphi_t\|_2$$

$$\le \sum_{j=1}^{m} \sum_{l=j+4}^{(j+4)\beta} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t^{3/2}} c_l 2^{l/2} \le c \sum_{j=1}^{m} \frac{1}{2^{j/2}} \sum_{l=j+4}^{(j+4)\beta} c_l 2^{l/2} \le c,$$

by (2.11). In order to complete the estimate for III_2, we need to estimate the piece

$$IV = \sum_{j=1}^{m} \sum_{l=(j+4)\beta}^{\infty} \sum_{s=l-2}^{l+2} \int_{2^{j-1}}^{2^j} \frac{dt}{t} \int_{6t \le |x-r| \le 6h} dx \left| \int_{|y-x| \sim 2^l} dy \right.$$

$$\times \left. \int du \, \Omega(y,u) b(u) \varphi_t(x-u) \psi_l(y-u) \psi_s(y-u) \right|.$$

We first suppose that $0 < a < 1$; then by (1.7)(c) we get

$$|\Omega(y,u)| \le \frac{c}{|y-u|^{2-a}}$$

since $|y-u| \ge |y-x| - |x-u| \ge \frac{1}{2}|y-x|$ with $|x-u| \le t \le 2^j$ and $|y-x| \ge 2^{l-2}$ with $l \ge (j+4)\beta$ and $\beta \ge 1$.

For the integrand of IV we get

$$\left| \int\limits_{x-t}^{x+t} du\, \Omega(y,u)b(u)\varphi_t(x-u)\psi_l(y-u)\psi_s(y-u) \right| \le \frac{c}{|y-x|^{2-a}h}$$

and so

$$IV \le \frac{c}{h} \sum_{j=1}^{m} \sum_{l=(j+4)\beta}^{\infty} \sum_{s=l-2}^{l+2} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \le |x-r| \le 6h} dx \int\limits_{|y-x| \sim 2^l} \frac{dy}{|y-x|^{2-a}}$$

$$\le c \sum_{j=1}^{m} \sum_{l=(j+4)\beta}^{\infty} \sum_{s=l-2}^{l+2} \frac{1}{2^{l(1-a)}} \le c \quad \text{since } 0 < a < 1.$$

This completes the argument in case $0 < a < 1$.

In case $a > 1$, we are left with

$$IV \le \sum_{j=1}^{m} \sum_{l=(j+4)\beta}^{\infty} \sum_{s=l-2}^{l+2} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \le |x-r| \le 6h} dx$$

$$\times \left| \int\limits_{|x-y| \ge 2t} dy \int (k(y+\alpha,u) - k(y+\alpha,x)) \right.$$

$$\times g(y+\alpha-u)\varphi_t(x-u)\psi_l(y-u)\psi_s(y-u)b(u) \Big|$$

$$+ \sum_{j=1}^{m} \sum_{l=(j+4)\beta}^{\infty} \sum_{s=l-2}^{l+2} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \le |x-r| \le 6h} dx \left| \int\limits_{|x-y| \ge 2t} dy \int du\, b(u) \right.$$

$$\times k(y+\alpha,x)g(y+\alpha-u)\varphi_t(x-u)\psi_l(y-u)\psi_s(y-u) \Big| = IV_1 + IV_2.$$

By (1.2)(b) we conclude that

$$IV_1 \le \sum_{j=1}^{m} \sum_{l=(j+4)\beta}^{\infty} \sum_{s=l-2}^{l+2} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \le |x-r| \le 6h} dx \int\limits_{2t \le |x-y| \le 3 \cdot 2^l} dy$$

$$\times \int du\, \frac{|u-x|^{\varepsilon}}{|y-u|^{1+\varepsilon}} |b(u)||\varphi_t(x-u)||\psi_l(y-u)||\psi_s(y-u)|$$

$$\le \sum_{j=1}^{m} \sum_{l=(j+4)\beta}^{\infty} \frac{1}{2^{l\varepsilon}} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} t^{\varepsilon} \le c \quad \text{since } \beta > 1.$$

Finally, notice that

$$IV_2 = \sum_{j=1}^{m} \sum_{l=(j+4)\beta}^{\infty} \sum_{s=l-2}^{l+2} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \le |x-r| \le 6h} dx \left| \int\limits_{|x-y| \ge 2t} dy\, k(y+\alpha,x) \right.$$

$$\times \int du\, g(y+\alpha-u)b(u)\varphi_t(x-u)\psi_l(y-u)\psi_s(y-u) \Big|.$$

But $|u-r| \ge |x-r| - |x-u| \ge 5t$ and $x-t \le u \le x+t$, therefore u is always either above r or below r since if there are u_1, u_2 so that $u_1 > r$ and $u_2 < r$ and $x-t < u_2 < u_1 < x+t$ this would imply that $x-t \le r \le x+t$ and hence $|u_i - r| \le 2t$, which is a contradiction. Therefore, $b(u) = 1/h$ or $= -1/h$ but not both and so by (1.5) with $a > 1$ we get

$$IV_2 \le \frac{1}{h} \sum_{j=1}^{m} \sum_{l=(j+4)\beta}^{\infty} \sum_{s=l-2}^{l+2} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \le |x-r| \le 6h} dx$$

$$\times \left(\int\limits_{|x-y| \sim 2^l} \frac{1}{|x-y|^a} dy \right)^{1/2} \|g_l * \varphi_t\|_2$$

$$\le \frac{c}{h} \sum_{j=1}^{m} \sum_{l=(j+4)\beta}^{\infty} \sum_{s=l-2}^{l+2} \int\limits_{2^{j-1}}^{2^j} \frac{dt}{t} \int\limits_{6t \le |x-r| \le 6h} dx\, 2^{l(1-a)/2} \|\varphi_t\|_2 \quad \text{by (1.4)}$$

$$\le c \quad (\text{since } a > 1). \quad \blacksquare$$

PROPOSITION 2.5. *Let* $0 < h \le \frac{3}{16}$. *If* $\Omega_1 \in \Re$, *and* (0.1) *and* (0.2)(a) *hold, then*

$$\int\limits_{h}^{\infty} \frac{dt}{t} \int\limits_{|x-r| \le 6t} |\varphi_t * \Omega(b)|\, dx \le c.$$

P r o o f. Since Ω is supported in $|y-u| \ge \frac{3}{2}$ it follows that

$$\int\limits_{h}^{3/16} \frac{dt}{t} \int\limits_{|x-r| \le 6t} |\varphi_t * \Omega(b)|\, dx = 0$$

for $|y-u| \le |y-x| + |x-r| + |u-r| \le 7t + h \le \frac{3}{2}$. We are left with the term

$$I = \int\limits_{3/16}^{\infty} \frac{dt}{t} \int\limits_{|x-r| \le 6t} |\varphi_t * \Omega(b)|\, dx.$$

Because of (0.4) we get

$$I = \int\limits_{3/16}^{\infty} \frac{dt}{t} \int\limits_{|x-r| \le 6t} dx \left| \int dy\, (\varphi_t(x-y) - \varphi_t(x-r))\Omega(b) \right|$$

$$\le \int\limits_{3/16}^{\infty} \frac{dt}{t} \int\limits_{|x-r| \le 6t} dx \left| \int dy\, (\varphi_t(x-y) - \varphi_t(x-r))\chi(|y-r| \le 32t)\Omega(b) \right|$$

$$+ \int\limits_{3/16}^{\infty} \frac{dt}{t} \int\limits_{|x-r| \le 6t} dx \left| \int dy\, (\varphi_t(x-y) - \varphi_t(x-r))\chi(|y-r| \ge 32t)\Omega(b) \right|$$

$$= I_1 + I_2.$$

We begin with I_1; it suffices to estimate, with $0 \le \alpha \le \frac{1}{2}$,

$$\int_{3/16}^{\infty} \frac{dt}{t} \int_{|x-r|\le 6t} dx \left| \int du\, b(u) \int_{|y-r|\le 32t} dy\, (k(y+\alpha,u) - k(y+\alpha,r)) \right.$$

$$\left. \times g(y+\alpha-u)\psi(y+\alpha-u)(\varphi_t(x-y) - \varphi_t(x-r)) \right|$$

$$+ \int_{3/16}^{\infty} \frac{dt}{t} \int_{|x-r|\le 6t} dx \left| \int du\, b(u) \int_{|y-r|\le 32t} k(y+\alpha,r) \right.$$

$$\left. \times g(y+\alpha-u)\psi(y+\alpha-u)(\varphi_t(x-y) - \varphi_t(x-r)) \right|$$

$$= I_{11} + I_{12}.$$

By (1.2)(b) we get

$$I_{11} \le c \int_{3/16}^{\infty} \frac{dt}{t^3} \int_{|x-r|\le 6t} dx \int du\, |b(u)| \int_{1\le|y-r|\le 32t} \frac{|u-r|^\varepsilon}{|y-r|^{1+\varepsilon}} |y-r| \le c,$$

and since $|y+\alpha-u| \le |y-r|+\frac{1}{2}+h \le |y-r|+1$, we infer that $\psi(y+\alpha-u) = 0$ if $|y-r| \le 1$. Next notice that

$$I_{12} \le \int_{3/16}^{\infty} \frac{dt}{t} \int_{|x-r|\le 6t} dx \left| \int_{1\le|y-r|\le 32t} dy\, (\varphi_t(x-y) - \varphi_t(x-r))k(y+\alpha,r) \right.$$

$$\times \int du\, b(u)\psi(y+\alpha-u)(g(y+\alpha-u) - g(y+\alpha-r)) \Big|$$

$$+ \int_{3/16}^{\infty} \frac{dt}{t} \int_{|x-r|\le 6t} dx \left| \int_{1\le|y-r|\le 32t} dy\, (\varphi_t(x-y) - \varphi_t(x-r)) \right.$$

$$\times \Omega_1(y+\alpha,r) \int du\, b(u)\psi(y+\alpha-u) \Big| = I_{121} + I_{122}.$$

Since $h \le 1$, if $0 < a < 1$, by (1.5) and (1.6) we get

$$I_{121} \le c \int_{3/16}^{\infty} \frac{dt}{t^3} \int_{|x-r|\le 6t} dx \int_{|y-r|\le 32t} dy\, \frac{|y-r||u-r|}{|y-r|^{2-a}} \le c.$$

Next since $\int b(u)\psi(y+\alpha-u)\,du = \int du\, b(u)(\psi(y+\alpha-u) - 1)\,du$ we get

$$I_{122} = \int_{3/16}^{\infty} \frac{dt}{t} \int_{|x-r|\le 6t} dx \left| \int_{1\le|y-r|\le 32t} dy\, (\varphi_t(x-y) - \varphi_t(x-r)) \right.$$

$$\times \Omega_1(y+\alpha,r) \int du\, b(u)(\psi(y+\alpha-u) - 1) \Big|.$$

Now $|y+\alpha-u| \ge |y-u| - \frac{1}{2}$ and if $|y-u| \ge \frac{7}{2}$ then $I_{122} = 0$, thus $|y-u| \le \frac{7}{2}$ and so $|y-r| \le |y-u| + |u-r| \le 4$. Hence by (0.2)(a) we infer that

$$I_{122} \le c \int_{3/16}^{\infty} \frac{dt}{t^3} \int_{|x-r|\le 6t} dx \int_{1\le|y-r|\le 4} dy\, |y-r||\Omega_1(y+\alpha,r)| \int du\, |b(u)| \le c.$$

Notice that (for $0 < a < 1$) $|y+\alpha-u-h| \ge |y-r| - \frac{1}{2} - 2h \ge |y-r| - 1$ and so $|y+\alpha-u-h| \ge 3$ and $|y+\alpha-u| \ge 3$ if $|y-r| \ge 32t \ge 6$ since $t \ge \frac{3}{16}$. Hence

$$|\Omega_1(y+\alpha,u)\psi(y+\alpha-u) - \Omega_1(y+\alpha,u+h)\psi(y+\alpha-u-h)|$$

$$= |\Omega_1(y+\alpha,u) - \Omega_1(y+\alpha,u+h)| \le \frac{ch}{|y-r|^{2-a}}$$

by (1.7)(a) if $|y-r| \ge 32t$. Next by (0.1) we get

$$(2.12) \quad |\Omega(b)|\chi(|y-r| \ge 32t)$$

$$= \left| \frac{1}{h} \int_{r-h}^{r} \Omega(y,u) - \Omega(y,u+h)\,du \right| \chi(|y-r| \ge 32t)$$

$$\le \frac{ch}{|y-r|^{2-a}} \chi(|y-r| \ge 32t)$$

and hence

$$I_2 \le ch \left(\int_{3/16}^{\infty} \frac{dt}{t} \int_{|x-r|\le 6t} dx\, |\varphi_t(x-r)| \int_{|y-r|\ge 32t} \frac{dy}{|y-r|^{2-a}} \right.$$

$$\left. + \int_{3/16}^{\infty} \frac{dt}{t} \int_{|y-r|\ge 32t} \frac{dy}{|y-r|^{2-a}} \int_{|x-r|\le 6t} dx\, |\varphi_t(x-y)| \right) \le c,$$

since $h \le 1$ and $0 < a < 1$. This completes the proof for the case $0 < a < 1$.

Next suppose that $a > 1$ and choose the decomposition (1.1) where (1.4)(p), (1.5)(p), and (1.6)(p) are satisfied with $p' > a - 1$. We begin by estimating I_1 and note that the estimate for I_{11} works as above, whereas

$$I_{12} = \int_{3/16}^{h^{1/(1-a)}} + \int_{h^{1/(1-a)}}^{\infty} = I_{123} + I_{124}.$$

Now,

$$I_{123} \le \int_{3/16}^{h^{1/(1-a)}} \frac{dt}{t^3} \int_{|x-r|\le 6t} dx \int_{1\le|y-r|\le 32t} dy\, |k(y+\alpha,r)||y-r|$$

$$\times \left| \int (g(y+\alpha-u) - g(y+\alpha-r))b(u)\psi(y+\alpha-u) \right|$$

$$+ \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t^3} \int\limits_{|x-r|\leq 6t} dx \int\limits_{1\leq|y-r|\leq 32t} dy\, |\Omega_1(y+\alpha,r)||y-r|$$

$$\times \left| \int du\, b(u)\psi(y+\alpha-u) \right|$$

$$\leq ch \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t^3} \int\limits_{|x-r|\leq 6t} dx \int\limits_{1\leq|y-r|\leq 32t} dy\, |y-r|^{a-1}$$

$$\text{by } (1.5)(p) \text{ and } (1.6)(p)$$

$$+ ch \int\limits_{1}^{h^{1/(1-a)}} \frac{dt}{t^3} \int\limits_{|x-r|\leq 6t} dx \int\limits_{1\leq|y-r|\leq 4} dy\, |y-r||\Omega_1(y+\alpha,r)|$$

$$\times \left| \int du\, b(u)(\psi(y+\alpha-u)-1) \right| \leq c\,.$$

Furthermore, by $(1.5)(p)$ we get

$$I_{124} \leq c \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t^3} \int\limits_{|x-r|\leq 6t} dx \int\limits_{1\leq|y-r|\leq 32t} dy\, \frac{|y-r|}{|y-r|^{a/p'}} |g\psi * b(y+\alpha)|$$

$$\leq c \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t^3} \int\limits_{|x-r|\leq 6t} dx \left(\int\limits_{1\leq|y-r|\leq 32t} |y-r|^{p'-a}\, dy \right)^{1/p'} \|g\psi * b\|_p$$

$$\leq c \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t^2} t^{(p'-a+1)/p'} \|b\|_p \quad \text{by } (1.4)(p) \text{ and also since } p' > a-1$$

$$\leq \frac{c}{h^{1/p'}} \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t} t^{(1-a)/p'} \leq c \quad \text{since } a>1\,.$$

Now we estimate the term I_2 (for $a>1$); we return to the decomposition satisfying (1.4)–(1.6):

$$I_2 = \int\limits_{3/16}^{h^{1/(1-a)}} + \int\limits_{h^{1/(1-a)}}^{\infty}\,.$$

We notice with this decomposition that

$$\int\limits_{h^{1/(1-a)}}^{\infty} \leq \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t} \int\limits_{|x-r|\leq 6t} dx \left| \int dy\, (\varphi_t(x-y)-\varphi_t(x-r))\chi(|y-r|\geq 32t) \right.$$

$$\times \int (k(y+\alpha,u)-k(y+\alpha,r))g(y+\alpha-u)b(u)\, du \bigg|$$

$$+ \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t} \int\limits_{|x-r|\leq 6t} dx \left| \int dy\, (\varphi_t(x-y)-\varphi_t(x-r)) \right.$$

$$\left. \times \chi(|y-r|\geq 32t)k(y+\alpha,r)g*b(y+\alpha) \right|$$

$$= I_{21} + I_{22}\,.$$

By $(1.2)(b)$ we can conclude that

$$I_{21} \leq c \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t^2} \int\limits_{|x-r|\leq 6t} dx \int du\, |b(u)| h^\varepsilon \int\limits_{|y-r|\geq 32t} \frac{dy}{|y-r|^{1+\varepsilon}} \leq c,$$

since $h \leq 1$. Also,

$$I_{22} \leq c \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t^2} \int\limits_{|x-r|\leq 6t} dx \int\limits_{|y-r|\geq 32t} |k(y+\alpha,r)||g*b(y+\alpha)|\, dy$$

$$\leq c \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t} \left(\int\limits_{|y-r|\geq 32t} |k(y+\alpha,r)|^2\, dy \right)^{1/2} \|g*b\|_2\,,$$

which by (1.4) and (1.5) is (since $a>1$)

$$\leq c \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t} \frac{1}{t^{(a-1)/2}} \|b\|_2 \leq c\,.$$

Now we are left with estimating

$$\int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{|x-r|\leq 6t} dx \left| \int dy\, (\varphi_t(x-y)-\varphi_t(x-r))\chi(|y-r|\geq 32t)\Omega(b) \right|,$$

and since $|x-y| \geq |y-r| - |x-r| \geq 26t$ and so $\varphi_t(x-y)=0$, we are left with

$$I_{221} = \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{|x-r|\leq 6t} dx\, |\varphi_t(x-r)| \left| \int\limits_{|y-r|\geq 32t} \Omega(b)\, dy \right|.$$

By (0.4) we get

$$\int \Omega(y,u)\chi(|y-r|\geq 32t)\, dy = - \int \Omega(y,u)\chi(|y-r|\leq 32t)\, dy$$

$$= - \int \Omega(y,u)\chi(1\leq|y-r|\leq 32t)\, dy\,.$$

Therefore,

$$I_{221} = \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{|x-r|\leq 6t} dx\, |\varphi_t(x-r)|$$

$$\times \left| \int du\, b(u) \int \Omega(y,u)\chi(1\leq|y-r|\leq 32t)\, dy \right|$$

$$\leq \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{|x-r|\leq 6t} dx\, |\varphi_t(x-r)|$$

$$\times \left| \int du\, b(u) \int \Omega(y,u)\chi(1 \le |y-r| \le 4)\, dy \right|$$

$$+ \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{|x-r|\le 6t} dx\, |\varphi_t(x-r)|$$

$$\times \left| \int du\, b(u) \int \Omega(y,u)\chi(4 \le |y-r| \le 32t) \right|$$

$$= I_{2211} + I_{2212}.$$

Now,

$$\Omega(b) = \frac{1}{h} \int\limits_{r-h}^{r} (\Omega(y,u) - \Omega(y,u+h))\, dy$$

and by (1.7)(b) we get $|\Omega(b)| \le ch$ if $\frac{3}{2} \le |y-u| \le 5$. Thus

$$I_{2211} \le \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{|x-r|\le 6t} dx\, |\varphi_t(x-r)| \left| \int dy\, \chi(1 \le |y-r| \le 4)\Omega(b) \right|$$

$$\le ch \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \le c \quad (\text{since } h \le 1).$$

Note that $|y-u+\alpha| \ge |y-u| - \frac{1}{2} \ge |y-r| - |u-r| - \frac{1}{2} \ge |y-r| - 1$ and so $\psi(y+\alpha-u) = 1$ if $|y-r| \ge 4$ (note $0 \le \alpha \le \frac{1}{2}$). In order to estimate I_{2212} it suffices to estimate

$$\int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{|x-r|\le 6t} dx\, |\varphi_t(x-r)| \left| \int du\, b(u) \right.$$

$$\times \left. \int (k(y+\alpha,u) - k(y+\alpha,r))g(y+\alpha-u)\chi(4 \le |y-r| \le 32t) \right|$$

$$+ \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{|x-r|\le 6t} dx\, |\varphi_t(x-r)| \left| \int du\, b(u) \right.$$

$$\times \left. \int k(y+\alpha,r)g(y+\alpha-u)\chi(4 \le |y-r| \le 32t) \right| = I_{22121} + I_{22122}.$$

By (1.2)(b) we conclude that $I_{22121} \le c$ since $h \le 1$. At the same time,

$$I_{22122} = \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \left| \int dy\, k(y+\alpha,r)\chi(4 \le |y-r| \le 32t) \right.$$

$$\times \left. \int du\, (g(y+\alpha-u) - g(y+\alpha-r))b(u) \right|.$$

By (1.5) and (1.6) this is

$$\le ch \int\limits_{3/16}^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{4\le|y-r|\le 32t} dy\, |y-r|^{a-2} \le c$$

since $a > 1$. This completes the proof of the proposition. ∎

PROPOSITION 2.6. *Let* $h \ge \frac{3}{16}$. *If* $\Omega_1 \in \Re$ *and* (0.1) *holds, then*

$$\int\limits_{h}^{\infty} \frac{dt}{t} \int\limits_{|x-r|\le 6t} |\varphi_t * \Omega(b)|\, dx \le c.$$

P r o o f. Arguing as in Proposition 2.5, we notice that

$$I \le \int\limits_{h}^{\infty} \frac{dt}{t} \int\limits_{|x-r|\le 6t} dx \left| \int dy\, (\varphi_t(x-y) - \varphi_t(x-r)) \right.$$

$$\left. \times \chi(|y-r| \le 32h)\Omega(b) \right|$$

$$+ \int\limits_{h}^{\infty} \frac{dt}{t} \int\limits_{|x-r|\le 6t} dx \left| \int dy\, (\varphi_t(x-y) - \varphi_t(x-r)) \right.$$

$$\left. \times \chi(32h \le |y-r| \le 32t)\Omega(b) \right|$$

$$+ \int\limits_{h}^{\infty} \frac{dt}{t} \int\limits_{|x-r|\le 6t} dx \left| \int dy\, (\varphi_t(x-y) - \varphi_t(x-r)) \right.$$

$$\left. \times \chi(|y-r| \ge 32t)\Omega(b) \right|$$

$$= I_1 + I_2 + I_3.$$

First we get

$$I_1 \le ch \int\limits_{h}^{\infty} \frac{dt}{t^3} \int\limits_{|x-r|\le 6t} dx \int\limits_{|y-r|\le 32h} |\Omega(b)|\, dy$$

$$\le ch^{1/2}\|\Omega(b)\|_2 \le ch^{1/2}\|b\|_2 \le c \quad \text{by (1.3)}.$$

In order to estimate I_2, it suffices to estimate $(0 \le \alpha \le \frac{1}{2})$

$$\int\limits_{h}^{\infty} \frac{dt}{t} \int\limits_{|x-r|\le 6t} dx \left| \int dy\, (\varphi_t(x-y) - \varphi_t(x-r))\chi(32h \le |y-r| \le 32t) \right.$$

$$\times \left. \int (k(y+\alpha,u) - k(y+\alpha,r))g(y+\alpha-u)b(u)\, du \right|$$

$$+ \int\limits_{h}^{\infty} \frac{dt}{t} \int\limits_{|x-r|\le 6t} dx \left| \int dy\, (\varphi_t(x-y) - \varphi_t(x-r)) \right.$$

$$\times \left. \chi(32h \le |y-r| \le 32t)k(y+\alpha,r)g * b(\alpha+y) \right| = I_{21} + I_{22}.$$

Now by (1.2)(b) we get

$$I_{21} \le ch^\varepsilon \int\limits_h^\infty \frac{dt}{t^2} \int\limits_{32h \le |y-r| \le 32t} dy \, |y-r|^{-\varepsilon} \le c \quad \text{if } 0 < \varepsilon < 1.$$

In case $\varepsilon = 1$, since

$$|\varphi_t(x-y) - \varphi_t(x-r)| \le \frac{c}{t^{1-\beta}} |\varphi_t(x-y) - \varphi_t(x-r)|^\beta$$

for some $0 < \beta < 1$, we get

$$|\varphi_t(x-y) - \varphi_t(x-r)| \le \frac{c|y-r|^\beta}{t^{1+2\beta}}$$

and

$$I_{21} \le ch \int\limits_h^\infty \frac{dt}{t^{1+2\beta}} \int\limits_{32h \le |y-r| \le 32t} dy \, |y-r|^{\beta-2} \le chh^{-2\beta} h^{\beta-1} \le c.$$

Next we note that for $a > 1$,

$$I_{22} \le c \int\limits_h^\infty \frac{dt}{t^3} \int\limits_{|x-r| \le 6t} dx \int\limits_{32h \le |y-r| \le 32t} dy \, |y-r| |k(y+\alpha, r)| |g * b(\alpha + y)|$$

$$\le c \int\limits_h^\infty \frac{dt}{t^2} \Big(\int\limits_{32h \le |y-r| \le 32t} |y-r|^2 |k(y+\alpha, r)|^2 \, dy \Big)^{1/2} \|g * b\|_2$$

$$\le c\|b\|_2 \int\limits_h^\infty \frac{dt}{t^2} \Big(\int\limits_{32h \le |y-r| \le 32t} |y-r|^{2-a} \, dy \Big)^{1/2} \quad \text{by (1.4) and (1.5)}$$

$$\le c \quad \text{since } a > 1 \text{ and } h \ge \tfrac{3}{16}.$$

Now for $0 < a < 1$, we get

$$I_{22} \le c \Big(\int\limits_h^{h^{1/(1-a)}} + \int\limits_{h^{1/(1-a)}}^\infty \Big) = I_{221} + I_{222}.$$

By (1.4) and (1.5),

$$I_{221} \le c \int\limits_h^{h^{1/(1-a)}} \frac{dt}{t^2} \Big(\int\limits_{32h \le |y-r| \le 32t} |y-r|^2 |k(y+\alpha, r)|^2 \, dy \Big)^{1/2} \|g * b\|_2$$

$$\le c\|b\|_2 \int\limits_h^{h^{1/(1-a)}} \frac{dt}{t^2} t^{3/2 - a/2} \le c.$$

On the other hand,

$$I_{222} \le \int\limits_{h^{1/(1-a)}}^\infty \frac{dt}{t^2} \int\limits_{32h \le |y-r| \le 32t} dy \, |y-r| |k(y+\alpha, r)|$$

$$\times \Big| \int (g(y+\alpha - u) - g(y+\alpha - r)) b(u) \, du \Big|$$

$$\le c \int\limits_{h^{1/(1-a)}}^\infty \frac{dt}{t^2} \int du \, |b(u)| \int\limits_{32h \le |y-r| \le 32t} dy \, |y-r| |k(y+\alpha, r)|$$

$$\times |g(y+\alpha - u) - g(y+\alpha - r)|$$

$$\le ch \int\limits_{h^{1/(1-a)}}^\infty \frac{dt}{t^2} \int\limits_{32h \le |y-r| \le 32t} dy \, |y-r|^{a-1} \quad \text{by (1.5) and (1.6)}$$

$$\le c \quad \text{since } 0 < a < 1.$$

Finally, in order to estimate I_3 it suffices to estimate

$$\int\limits_h^\infty \frac{dt}{t} \int\limits_{|x-r| \le 6t} dx \, |\varphi_t(x-r)|$$

$$\times \Big| \int\limits_{|y-r| \ge 32t} dy \int du \, b(u)(k(y+\alpha, u) - k(y+\alpha, r)) g(y+\alpha - u) \Big|$$

$$+ \int\limits_h^\infty \frac{dt}{t} \int\limits_{|x-r| \le 6t} dx \, |\varphi_t(x-r)| \Big| \int\limits_{|y-r| \ge 32t} dy \, k(y+\alpha, r) g * b(\alpha + y) \Big|$$

$$= I_{31} + I_{32}.$$

Since $|x-y| \ge |y-r| - |x-r| \ge 26t$, therefore $\varphi_t(x-y) = 0$. Now by (1.2)(b) we get $I_{31} \le c$.

Next assume that $a > 1$. Then

$$I_{32} \le \int\limits_h^\infty \frac{dt}{t} \Big(\int\limits_{|y-r| \ge 32t} |k(y+\alpha, r)|^2 \, dy \Big)^{1/2} \|g * b\|_2$$

$$\le c \int\limits_h^\infty \frac{dt}{t} t^{(1-a)/2} \|b\|_2 \quad \text{by (1.4) and (1.5)}$$

$$\le c \quad \text{since } h \ge \tfrac{3}{16} \text{ and } a > 1.$$

This completes our argument in case $a > 1$.

Next suppose that $0 < a < 1$. Then

$$I_{32} \le \int\limits_h^{h^{1/(1-a)}} + \int\limits_{h^{1/(1-a)}}^\infty = I_{321} + I_{322}.$$

We notice that (as for I_{222})

$$\int\limits_{h^{1/(1-a)}}^{\infty}\frac{dt}{t}\int\limits_{|y-r|\geq 32t}|k(y+\alpha,r)||g*b(\alpha+y)|$$

$$\leq ch\int\limits_{h^{1/(1-a)}}^{\infty}\frac{dt}{t}\int\limits_{|y-r|\geq 32t}dy\,|y-r|^{a-2}\quad\text{by (1.5) and (1.6)}$$

$$\leq c\quad\text{since } 0<a<1.$$

Now putting this estimate together with I_{31}, we deduce that for $0<a<1$,

$$\int\limits_{h^{1/(1-a)}}^{\infty}\frac{dt}{t}\int\limits_{|x-r|\leq 6t}dx\,|\varphi_t(x-r)|\Big|\int\limits_{|y-r|\geq 32t}\Omega(b)\,dy\,\Big|\leq c.$$

Next by (1.7)(c) we get

$$\int\limits_{h}^{h^{1/(1-a)}}\frac{dt}{t}\int\limits_{|x-r|\leq 6t}dx\,|\varphi_t(x-r)|\Big|\int\limits_{|y-r|\geq 32t}\Omega(b)\,dy\,\Big|$$

$$\leq c\int\limits_{h}^{h^{1/(1-a)}}\frac{dt}{t}\int\limits_{|y-r|\geq 32t}dy\,\frac{1}{|y-r|^{2-a}}\leq c\quad\text{since } 0<a<1.$$

This completes the proof of the proposition. ∎

3. Main theorem.

In this section, we prove our main Theorem 3.2 and give two applications in Theorems 3.4 and 3.5. Next we show that our examples $\Omega_1(y,u)=K(y,u)e^{i|y-u|^a}$ satisfy these theorems; this is done in the Corollary at the end of the section.

LEMMA 3.1. *Let $\Omega_1\in\Re$, and assume that* (0.1), (0.2)(a), (2.3), (2.4), (2.10) *and* (2.11) *hold. Furthermore, if*

$$(3.1)\qquad\int\limits_{0}^{h}\frac{dt}{t}\int\limits_{|x-r|\geq 6h}|\varphi_t*\Omega(b)|\,dx+\int\limits_{h}^{\infty}\frac{dt}{t}\int\limits_{|x-r|\geq 6t}|\varphi_t*\Omega(b)|\,dx\leq c,$$

then Ω maps B into itself.

Proof. It suffices to show (1.10). We break up the argument into two cases, $h\leq\frac{3}{16}$ and $h\geq\frac{3}{16}$. For $0<h\leq\frac{3}{16}$, by (3.1) and since the support of $\Omega(y,u)$ is contained in $|y-u|\geq\frac{3}{2}$, we have

$$(3.2)\qquad\int\limits_{0}^{h}\frac{dt}{t}\int|\varphi_t*\Omega(b)|\,dx\leq c;$$

we argue as in the beginning of the proof of Proposition 2.5. Next by Proposition 2.5 and (3.1) we get

$$(3.3)\qquad\int\limits_{h}^{\infty}\frac{dt}{t}\int|\varphi_t*\Omega(b)|\,dx\leq c,$$

and so by (3.2) and (3.3) we get (1.10) for $h\leq\frac{3}{16}$.

Next for $h\geq\frac{3}{16}$, by (3.1) and Proposition 2.4 we deduce that

$$(3.4)\qquad\int\limits_{3/16}^{h}\frac{dt}{t}\int|\varphi_t*\Omega(b)|\,dx\leq c.$$

By (3.1) and Lemma 2.3 we get

$$(3.5)\qquad\int\limits_{0}^{3/16}\frac{dt}{t}\int|\varphi_t*\Omega(b)|\,dx\leq c.$$

Now by putting (3.4) and (3.5) together we conclude that

$$(3.6)\qquad\int\limits_{0}^{h}\frac{dt}{t}\int|\varphi_t*\Omega(b)|\,dx\leq c\quad\text{for } h\geq\frac{3}{16}.$$

By (3.1) and Proposition 2.6 we get

$$(3.7)\qquad\int\limits_{h}^{\infty}\frac{dt}{t}\int|\varphi_t*\Omega(b)|\,dx\leq c\quad\text{for } h\geq\frac{3}{16}.$$

Now by (3.6) and (3.7) we get (1.10) for $h\geq\frac{3}{16}$, and this completes our argument. ∎

Now we are in a position to state and prove the main result of this paper.

THEOREM 3.2. *Let Ω satisfy the hypothesis of Lemma 3.1. Then Ω maps B into itself if and only if both* (1.11) *and* (1.12) *are satisfied.*

Proof. We first assume that (1.11) and (1.12) are satisfied and then show that Ω maps B into itself. Since Ω_1 is regular, by (0.1) and (1.1) we get

$$\Omega(y,u)=k(y,u)g(y-u)\psi(y-u)-k(y+\tfrac{1}{2},u)g(y+\tfrac{1}{2}-u)\psi(y+\tfrac{1}{2}-u).$$

Thus

$$(3.8)\quad\Omega(b)=\sum_{i=0}^{1}(-1)^i\int(k(y+\alpha_i,u)-k(x+\alpha_i,u))$$

$$\times g(y+\alpha_i-u)\psi(y+\alpha_i-u)b(u)\,du$$

$$+\sum_{i=0}^{1}(-1)^i\int k(x+\alpha_i,u)g(y+\alpha_i-u)\psi(y+\alpha_i-u)b(u)\,du$$

with $\alpha_0 = 0, \alpha_1 = \frac{1}{2}$. It follows that

$$A = \int_0^h \frac{dt}{t} \int_{|x-r| \geq 6h} |\varphi_t * \Omega(b)| \, dx$$

$$\leq \int_0^h \frac{dt}{t} \int_{|x-r| \geq 6h} \left| \sum_{i=0}^1 (-1)^i \int dy \, \varphi_t(x-y) \right.$$

$$\times \int du \, (k(y + \alpha_i, u) - k(x + \alpha_i, u))$$

$$\times g(y + \alpha_i - u)\psi(y + \alpha_i - u)b(u) \, du \Big|$$

$$+ \int_0^h \frac{dt}{t} \int_{|x-r| \geq 6h} \left| \sum_{i=0}^1 (-1)^i \int dy \, \varphi_t(x-y) \right.$$

$$\times \int du \, k(x + \alpha_i, u)g(y + \alpha_i - u)\psi(y + \alpha_i - u)b(u) \Big| = I + II.$$

But $I \leq c$ by Proposition 2.1 and $II \leq c$ by (1.11), therefore we get

$$(3.9) \qquad\qquad\qquad A \leq c.$$

Also,

$$(3.10) \qquad \Omega(b) = \sum_{i=0}^1 (-1)^i \int (k(y + \alpha_i, u) - k(y + \alpha_i, r))$$

$$\times g(y + \alpha_i - u)\psi(y + \alpha_i - u)b(u) \, du$$

$$+ \sum_{i=0}^1 (-1)^i k(y + \alpha_i, r)g\psi * b(y + \alpha_i)$$

with $\alpha_0 = 0, \ \alpha_1 = \frac{1}{2}$. It follows that

$$B = \int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} |\varphi_t * \Omega(b)| \, dx$$

$$\leq \int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} dx \left| \sum_{i=0}^1 (-1)^i \int dy \, \varphi_t(x-y) \right.$$

$$\times \int (k(y + \alpha_i, u) - k(y + \alpha_i, r))$$

$$\times g(y + \alpha_i - u)\psi(y + \alpha_i - u)b(u) \, du \Big|$$

$$+ \int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} dx \left| \sum_{i=0}^1 (-1)^i \right.$$

$$\times \int dy \, \varphi_t(x-y)k(y + \alpha_i, r)g\psi * b(y + \alpha_i) \Big| = III + IV.$$

But $III \leq c$ by Proposition 2.1 and $IV \leq c$ by (1.12), therefore

$$(3.11) \qquad\qquad\qquad B \leq c.$$

Now by (3.9) and (3.11) we see that Lemma 3.1 applies and hence we obtain the sufficiency of (1.11) and (1.12).

To show the necessity of (1.11) and (1.12), we assume that Ω maps B into itself. We notice from (3.8) that

$$\int_0^h \frac{dt}{t} \int_{|x-r| \geq 6h} dx \left| \sum_{i=0}^1 (-1)^i \int dy \, \varphi_t(x-y) \right.$$

$$\times \int k(x + \alpha_i, u)g(y + \alpha_i - u)\psi(y + \alpha_i - u)b(u) \, du \Big|$$

$$\leq \int_0^h \frac{dt}{t} \int_{|x-r| \geq 6h} dx \, |\varphi_t * \Omega(b)|$$

$$+ \sum_{i=0}^1 \int_0^h \frac{dt}{t} \int_{|x-r| \geq 6h} dx \left| \int dy \, \varphi_t(x-y) \right.$$

$$\times \int (k(y + \alpha_i, u) - k(x + \alpha_i, u))$$

$$\times \psi(y + \alpha_i - u)g(y + \alpha_i - u)b(u) \, du \Big| = V + VI.$$

Since Ω maps B into itself, we infer that $V \leq c$, while by Proposition 2.1 we get $VI \leq c$. Hence (1.11) now holds.

In order to see (1.12), we notice from (3.10) that

$$\int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} dx \left| \sum_{i=0}^1 (-1)^i \int dy \, \varphi_t(x-y)k(y + \alpha_i, r)g\psi * b(y + \alpha_i) \right|$$

$$\leq \int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} |\varphi_t * \Omega(b)| \, dx$$

$$+ \int_h^\infty \frac{dt}{t} \int_{|x-r| \geq 6t} dx \left| \sum_{i=0}^1 (-1)^i \int (k(y + \alpha_i, u) - k(y + \alpha_i, r)) \right.$$

$$\times g(y + \alpha_i - u)\psi(y + \alpha_i - u)b(u) \, du \Big| = VII + VIII.$$

Once again, $VII \leq c$ since Ω maps B into itself, and $VIII \leq c$ by Proposition 2.1. This implies (1.12). Putting all these estimates together, we get our result. ∎

We need the following technical lemma.

LEMMA 3.3. *Let* $0 < h \leq 1$, $0 \leq \alpha \leq \frac{1}{2}$, $a > 0$, $a \neq 1$ *and* $N = g(y + \alpha - u)\psi(y + \alpha - u)$. *Suppose that* (1.1), (1.4)-(1.6) *and* (2.4)(a) *hold. Then if*

$$(3.12) \quad |k(y,r) - k(x,r)| \leq c \frac{|x-y|}{|x-r|^{1+a/2}} \quad \text{for } |x-r| \geq \max(1, 2|y-x|)$$

and

$$(3.13) \qquad\qquad |g(x)| \leq c \quad \text{for } 2 \leq |x| \leq 10,$$

then

$$\text{(i)} \quad I = \int_0^h \frac{dt}{t} \int_{6h \leq |x-r| \leq 6} dx \left| \int dy\, \varphi_t(x-y) \right.$$
$$\left. \times \int du\, k(x+\alpha, u) N b(u) \right| \leq c,$$

and

$$\text{(ii)} \quad II = \int_h^1 \frac{dt}{t} \int_{6t \leq |x-r| \leq 6} dx \left| \int dy\, \varphi_t(x-y) \right.$$
$$\left. \times k(y+\alpha, r) \int du\, N b(u) \right| \leq c.$$

Proof. We begin with (i). Notice that

$$I \leq \int_0^h \frac{dt}{t} \int_{6h \leq |x-r| \leq 6} dx \left| \int dy\, \varphi_t(x-y) \int du\, k(x+\alpha, u) g(y+\alpha-u) \right.$$
$$\times (\psi(y+\alpha-u) - \psi(x+\alpha-u)) b(u) \Big|$$
$$+ \int_0^h \frac{dt}{t} \int_{6h \leq |x-r| \leq 6} dx \left| \int dy\, \varphi_t(x-y) \psi(x+\alpha-u) \right.$$
$$\times \int du\, k(x+\alpha, u) g(y+\alpha-u) b(u) \Big| = I_1 + I_2.$$

We first estimate I_1. Because of (2.4)(a) we get (2.5) if we can show (2.6). Notice for I_1 that $|x+\alpha-u| \geq 1$. Now if $0 < h \leq \frac{1}{2}$, then $|x+\alpha-u| \geq 1 \geq 2|y-x|$ and so (2.6) holds. Next suppose $\frac{1}{2} \leq h \leq 1$. Then

$$|x+\alpha-u| \geq |x-u| - \tfrac{1}{2} \geq |x-r| - |u-r| - \tfrac{1}{2} \geq 5h - \tfrac{1}{2} \geq 2 \geq 2|y-x|$$

and so again (2.6) holds. Thus,

$$I_1 \leq \int_0^h \frac{dt}{t} \int_{6h \leq |x-r| \leq 6} dx \int dy\, |\varphi_t(x-y)||x-y| \int du\, |b(u)| \quad \text{by (2.5)}$$
$$\leq c.$$

Since $2|y-x| \leq 2 \leq |x+\alpha-u| \leq 7\frac{1}{2}$ just as in (2.7) we get

$$I_2 \leq c \int_0^h \frac{dt}{t} \int_{6h \leq |x-r| \leq 6} dx \int du\, |\psi(x+\alpha-u)|$$
$$\times |x+\alpha-u|^{a-2} |b(u)| \int dy\, |\varphi_t(x-y)||x-y| \leq c.$$

Now to see (ii), first assume that $0 < h \leq \frac{1}{8}$. Then

$$II \leq \int_h^{1/8} \frac{dt}{t} \int_{6t \leq |x-r| \leq 6} dx \left| \int dy\, \varphi_t(x-y) k(y+\alpha, r) \right.$$
$$\times \int du\, (g(y+\alpha-u) - g(x+\alpha-u)) \psi(y+\alpha-u) b(u) \Big|$$
$$+ \int_h^{1/8} \frac{dt}{t} \int_{6t \leq |x-r| \leq 6} dx \left| \int dy\, \varphi_t(x-y) k(y+\alpha, r) \right.$$
$$\times \int du\, g(x+\alpha-u) \psi(y+\alpha-u) b(u) \Big|$$
$$+ \int_{1/8}^1 \frac{dt}{t} \int_{6t \leq |x-r| \leq 6} dx \left| \int dy\, \varphi_t(x-y) k(y+\alpha, r) \right.$$
$$\times \int du\, g(y+\alpha-u) \psi(y+\alpha-u) b(u) \Big| = II_1 + II_2 + II_3.$$

For the term II_3 we note that $|k(y+\alpha, r)\, g(y+\alpha-r+r-u)| \leq c$ by (2.4)(a) since

$$|y+\alpha-r| \geq |y+\alpha-u| - |u-r| \geq 2 - \tfrac{1}{8} \geq 1 \geq 2|u-r|,$$

and so

$$II_3 \leq c \int_{1/8}^1 \frac{dt}{t} \int_{|x-r| \leq 6} dx \int dy\, |\varphi_t(x-y)| \int du\, |b(u)| \leq c.$$

For II_1, since $2|y-x| \leq 2 - \frac{1}{8} \leq |y+\alpha-u| - |x-y| \leq |x+\alpha-u|$, we deduce as in (2.7) that

$$II_1 \leq c \int_h^{1/8} \frac{dt}{t} \int_{6t \leq |x-r| \leq 6} dx \int dy\, |\varphi_t(x-y)||y+\alpha-r|^{a-2} t$$
$$\times \int du\, |b(u)||\psi(y+\alpha-u)| \leq c.$$

Next notice that

$$II_2 \leq \int_h^{1/8} \frac{dt}{t} \int_{6t \leq |x-r| \leq 6} dx \left| \int dy\, \varphi_t(x-y)(k(y+\alpha, r) - k(x+\alpha, r)) \right.$$

$$\times \int du\, g(y + \alpha - u)\psi(y + \alpha - u)b(u)\Big|$$

$$+ \int\limits_{h}^{1/8} \frac{dt}{t} \int\limits_{6t \le |x-r| \le 6} dx \left| \int dy\, \varphi_t(x - y)k(x + \alpha, r)\right.$$

$$\times \int du\, g(y + \alpha - u)\psi(y + \alpha - u)b(u)\Big| = II_{21} + II_{22}.$$

Now by (3.12) since $|y + \alpha - r| \ge |y + \alpha - u| - |u - r| \ge 2 - \frac{1}{8}$ and $|x + \alpha - r| \ge |y + \alpha - r| - |x - y| \ge 2 - \frac{1}{8} - \frac{1}{8} \ge 1\frac{3}{4} \ge 2|y - x|$, we get

$$II_{21} \le c \int\limits_{h}^{1/8} \frac{dt}{t} \int\limits_{6t \le |x-r| \le 6} dx \int dy\, \frac{|\varphi_t(x - y)||x - y|}{|y + \alpha - r|^{1 + a/2}}$$

$$\times \int du\, |g(y + \alpha - u)\psi(y + \alpha - u)b(u)|.$$

As also $2 \le |y + \alpha - u| \le 8\frac{1}{2}$, thus by (3.13) we get $II_{21} \le c$.

In order to estimate II_{22}, we argue as we did in Lemma 2.3, but in place of (2.5) we need to show that $|k(x + \alpha, r)g(x + \alpha - r + y - x + r - u)| \le c$, and in place of (2.6) we get $|x + \alpha - r| \ge |x + \alpha - u| - |u - r| \ge \frac{7}{8} \ge 2(|y - x| + |u - r|)$. Once again we get $II_{22} \le c$.

Finally, suppose that $\frac{1}{8} \le h \le 1$. Then

$$II \le \int\limits_{1/8}^{1} \frac{dt}{t} \int\limits_{|x-r| \le 6} dx \int dy\, \frac{|\varphi_t(x - y)|}{|y + \alpha - r|^{a/2}} |g\psi * b|$$

by (1.5); but $|y + \alpha - r| \ge |y + \alpha - u| - |u - r| \ge 1$, therefore

$$II \le c \int\limits_{1/8}^{1} \frac{dt}{t} \int\limits_{|x-r| \le 6} dx\, \|\varphi_t\|_2 \|g\psi * b\|_2$$

$$\le c \int\limits_{1/8}^{1} \frac{dt}{t^{3/2}} \int\limits_{|x-r| \le 6} dx\, \|b\|_2 \qquad \text{by (1.4)}$$

$$\le \frac{c}{h^{1/2}} \int\limits_{1/8}^{1} \frac{dt}{t^{3/2}} \le c \qquad \text{since } h \ge \frac{1}{8}. \quad \blacksquare$$

THEOREM 3.4. *Let* $0 < a < 1$. *If* Ω *satisfies the hypothesis of Theorem 3.2 along with (3.12) and (3.13), then* Ω *maps* B *into itself.*

Proof. Arguing as in (2.7) we get

$$\int\limits_{0}^{h} \frac{dt}{t} \int\limits_{|x-r| \ge 6} dx \left| \int du\, b(u)k(x + \alpha, u) \int dy\, \varphi_t(x - y)g(y + \alpha - u)\right|$$

$$\le c \int\limits_{0}^{h} dt \int\limits_{|x-r| \ge 6} dx \int dy\, |\varphi_t(x - y)| \int du\, |b(u)||x - u|^{a-2}$$

$$\le c \int\limits_{0}^{h} dt \int\limits_{|x-r| \ge 6} dx\, |x - r|^{a-2}$$

$$\times \int dy\, |\varphi_t(x - y)| \int du\, |b(u)| \le c \qquad \text{for } 0 < h \le 1.$$

Putting this together with Lemma 3.3(i), we conclude that

$$(3.14) \qquad \int\limits_{0}^{h} \frac{dt}{t} \int\limits_{|x-r| \ge 6h} dx \left| \int du\, b(u)k(x + \alpha, u)\right.$$

$$\times \int dy\, \varphi_t(x - y)g(y + \alpha - u)\psi(y + \alpha - u)\Big| \le c \qquad \text{for } 0 < h \le 1.$$

For $h \ge 1$, arguing as above, we get

$$(3.15) \qquad \int\limits_{0}^{1} \frac{dt}{t} \int\limits_{|x-r| \ge 6h} dx \left| \int du\, b(u)k(x + \alpha, u)\right.$$

$$\times \int dy\, \varphi_t(x - y)g(y + \alpha - u)\Big| \le c.$$

By Proposition 2.1 and (3.14) we get

$$(3.16) \qquad \int\limits_{0}^{h} \frac{dt}{t} \int\limits_{|x-r| \ge 6h} |\varphi_t * \Omega(b)|\, dx \le c \qquad \text{for } 0 < h \le 1,$$

and again by Proposition 2.1 and (3.15)

$$(3.17) \qquad \int\limits_{0}^{1} \frac{dt}{t} \int\limits_{|x-r| \ge 6h} |\varphi_t * \Omega(b)|\, dx \le c \qquad \text{for } h \ge 1.$$

Furthermore, note that since $|y - u| \ge |x - r| - |u - r| - |x - y| \ge \frac{2}{3}|x - r|$ and $|x - r| \ge 6$, we have $|y - u| \ge 4$ and so by (1.7)(c) we get

$$|\Omega(y, u)| \le \frac{c}{|y - u|^{2-a}} \le \frac{c}{|x - r|^{2-a}}.$$

Thus,

$$(3.18) \qquad \int\limits_{1}^{h} \frac{dt}{t} \int\limits_{|x-r| \ge 6h} |\varphi_t * \Omega(b)|\, dx$$

$$\leq \int\limits_1^h \frac{dt}{t} \int\limits_{|x-r|\geq 6h} \frac{dx}{|x-r|^{2-a}} \int dy\, |\varphi_t(x-y)| \int du\, |b(u)|$$

$$\leq c \qquad \text{for } h \geq 1\,.$$

Now by (3.17) and (3.18) we get

$$(3.19) \qquad \int\limits_0^h \frac{dt}{t} \int\limits_{|x-r|\geq 6h} |\varphi_t * \Omega(b)|\, dx \leq c \qquad \text{for } h \geq 1\,.$$

By (3.16) and (3.19),

$$(3.20) \qquad \int\limits_0^h \frac{dt}{t} \int\limits_{|x-r|\geq 6h} |\varphi_t * \Omega(b)|\, dx \leq c \qquad \text{for } h > 0\,.$$

Now for $0 < h \leq 1$ we obtain

$$I = \int\limits_h^\infty \frac{dt}{t} \int\limits_{|x-r|\geq 6t} dx \left| \int dy\, \varphi_t(x-y)k(y+\alpha,r) \right.$$

$$\times \left. \int du\, g(y+\alpha-u)\psi(y+\alpha-u)b(u) \right|$$

$$\leq \int\limits_h^1 \frac{dt}{t} \int\limits_{6t\leq|x-r|\leq 6} dx \ldots + \int\limits_h^1 \frac{dt}{t} \int\limits_{|x-r|\geq 6} dx \ldots + \int\limits_1^\infty \frac{dt}{t} \int\limits_{|x-r|\geq 6t} dx \ldots$$

$$= I_1 + I_2 + I_3\,.$$

By Lemma 3.3(ii), $I_1 \leq c$ for $0 < h \leq 1$.

Since $|x-u| \geq \frac{5}{6}|x-r|$ and for the terms $I_2, I_3, |x-u| \geq 5$, we get $|y+\alpha-u| \geq |x-u| - |x-y| - \frac{1}{2} \geq \frac{4}{5}|x-u| - \frac{1}{2} \geq 3$ and thus $\psi(y+\alpha-u) = 1$. Our problem reduces to showing that

$$(3.21) \qquad II = \int\limits_h^\infty \frac{dt}{t} \int\limits_{|x-r|\geq 6t_1} dx \left| \int dy\, \varphi_t(x-y)k(y+\alpha,r)g*b(y+\alpha) \right| \leq c$$

for all $h > 0$ where $t_1 = \max(t,1)$. We first consider

$$(3.22) \qquad \int\limits_h^\infty \frac{dt}{t} \int\limits_{|x-r|\geq 6t_1^{1/(1-a)}} \left| \int dy\, \varphi_t(x-y)k(y+\alpha,r)g*b(y+\alpha) \right|$$

$$\leq \int\limits_h^\infty \frac{dt}{t} \int\limits_{|x-r|\geq 6t_1^{1/(1-a)}} \int dy\, |\varphi_t(x-y)||k(y+\alpha,r)|$$

$$\times \left| \int (g(y+\alpha-u) - g(y+\alpha-r))b(u)du \right|$$

$$\leq ch \int\limits_h^\infty \frac{dt}{t} \int\limits_{|x-r|\geq 6t^{1/(1-a)}} dx \int dy\, \frac{|\varphi_t(x-y)|}{|x-r|^{2-a}} \qquad \text{by (1.5) and (1.6)}$$

$$\leq ch \int\limits_h^\infty \frac{dt}{t}\frac{1}{t} \leq c\,.$$

Notice that if $h \leq 1$, arguing as above we get

$$(3.23) \qquad II \leq c \qquad \text{for } 0 < h \leq 1\,.$$

Next suppose that $h \geq 1$. Then

$$\int\limits_h^\infty \frac{dt}{t} \int\limits_{6t\leq|x-r|\leq 6t^{1/(1-a)}} dx \left| \int dy\, \varphi_t(x-y)k(y+\alpha,r)g*b(y+\alpha) \right|$$

$$\leq \int\limits_h^\infty \frac{dt}{t} \int\limits_{6t\leq|x-r|\leq 6t^{1/(1-a)}} dx$$

$$\times \left| \int dy\, \varphi_t(x-y)(k(y+\alpha,r) - k(x+\alpha,r))g*b(y+\alpha) \right|$$

$$+ \int\limits_h^\infty \frac{dt}{t} \int\limits_{6t\leq|x-r|\leq 6t^{1/(1-a)}} dx\, |k(x+\alpha,r)||\varphi_t * g*b(x)| = II_1 + II_2\,.$$

By (1.14) of Proposition 1.1 we get $II_2 \leq c$, while (disregard the first term for $h \leq 1$)

$$II_1 \leq \int\limits_h^{h^{1/(1-a)}} + \int\limits_{h^{1/(1-a)}}^\infty = II_{11} + II_{12}\,.$$

By (3.12), since

$$(3.24) \qquad |x+\alpha-r| \geq |x-r| - \tfrac{1}{2} \geq 6t_1 - \tfrac{1}{2}t_1 \geq \tfrac{11}{2}t_1 \geq 2|y-x|$$

we get

$$II_{12} \leq c \int\limits_{h^{1/(1-a)}}^\infty \frac{dt}{t} \int\limits_{6t_1\leq|x-r|\leq 6t_1^{1/(1-a)}} dx \int dy\, \frac{|\varphi_t(x-y)|t}{|x-r|^{1+a/2}}$$

$$\times \left| \int (g(y+\alpha-u) - g(y+\alpha-r))b(u)\, du \right|$$

$$\leq ch \int\limits_{h^{1/(1-a)}}^\infty dt \int\limits_{6t\leq|x-r|\leq 6t^{1/(1-a)}} dx\, \frac{1}{|x-r|^{3-a}} \int du\, |b(u)| \qquad \text{by (1.6)}$$

$$\leq c\,.$$

By (3.12) and (3.24) we get

$$II_{11} \leq c \int\limits_{h}^{h^{1/(1-a)}} dt \int\limits_{6t_1 \leq |x-r| \leq 6t_1^{1/(1-a)}} \frac{dx}{|x-r|^{1+a/2}}$$

$$\times \int dy \, |\varphi_t(x-y)| |g * b(y+\alpha)|$$

$$\leq c \int\limits_{h}^{h^{1/(1-a)}} dt \int\limits_{6t \leq |x-r| \leq 6t^{1/(1-a)}} \frac{dx}{|x-r|^{1+a/2}} \|\varphi_t\|_2 \|g * b\|_2$$

$$\leq c\|b\|_2 \int\limits_{h}^{h^{1/(1-a)}} \frac{dt}{t^{(1+a)/2}} \qquad \text{by (1.4)}$$

$$\leq c.$$

Therefore,

$$(3.25) \qquad \int\limits_{h}^{\infty} \frac{dt}{t} \int\limits_{6t \leq |x-r| \leq 6t^{1/(1-a)}} dx \left| \int dy \, \varphi_t(x-y) \right.$$

$$\left. \times k(y+\alpha, r) g * b(y+\alpha) \right| \leq c \qquad \text{for } h \geq 1.$$

Putting (3.22) and (3.25) together and using (3.23) shows that (3.21) holds. Now this implies that $I \leq c$. By Proposition 2.1 we get

$$(3.26) \qquad \int\limits_{h}^{\infty} \frac{dt}{t} \int\limits_{|x-r| \geq 6t} |\varphi_t * \Omega(b)| \, dx \leq c \qquad \text{for all } h > 0.$$

By Lemma 3.1 we get our result from (3.20) and (3.26). ∎

THEOREM 3.5. *Let $a > 1$. If Ω satisfies the hypothesis of Theorem 3.4 and*

(i) $g \in L^\infty$ *in case $a = 2$, and*

(ii) $|k(x,u) - k(x,r)| \leq c \dfrac{|u-r|}{|x-r|^{1+a/2}}$ *if $|x-r| \geq \max(1, 2|u-r|)$,*

then Ω maps B into itself.

Proof. In fact, it suffices by Theorem 3.2 for $\alpha = 0, \frac{1}{2}$ to show that with

$$N = g(y+\alpha-u)\psi(y+\alpha-u),$$

we have

$$(3.27) \quad \begin{cases} \text{(a)} \quad A = \int\limits_{0}^{h} \frac{dt}{t} \int\limits_{|x-r| \geq 6h} dx \left| \int dy \, \varphi_t(x-y) \right. \\ \qquad\qquad \left. \times \int du \, k(x+\alpha, u) N b(u) \right| \leq c, \\[2ex] \text{(b)} \quad B = \int\limits_{h}^{\infty} \frac{dt}{t} \int\limits_{|x-r| \geq 6t} dx \left| \int dy \, \varphi_t(x-y) \right. \\ \qquad\qquad \left. \times \int du \, k(y+\alpha, r) N b(u) \right| \leq c. \end{cases}$$

We first show (3.27)(a). We begin by assuming that $0 < h \leq 1$; then

$$A = \int\limits_{0}^{h} \frac{dt}{t} \int\limits_{6h \leq |x-r| \leq 6} dx \ldots + \int\limits_{0}^{h} \frac{dt}{t} \int\limits_{|x-r| \geq 6} dx \ldots = I_1 + I_2.$$

By Lemma 3.3(i), $I_1 \leq c$. For the term I_2 we get $|y+\alpha-u| \geq 3$ and so $\psi(y+\alpha-u) = 1$. In order to see (3.27)(a), it suffices to prove that

$$(3.28) \quad II = \int\limits_{0}^{h} \frac{dt}{t} \int\limits_{|x-r| \geq 6h_1} \left| \int dy \, \varphi_t(x-y) \right.$$

$$\left. \times \int du \, k(x+\alpha, u) g(y+\alpha-u) b(u) \right| \leq c,$$

where $h_1 = \max(h, 1)$ and $h > 0$.

We note that

$$II \leq \int\limits_{0}^{h} \frac{dt}{t} \int\limits_{|x-r| \geq 6h_1} dx \left| \int dy \, \varphi_t(x-y) \right.$$

$$\times \int du \, (k(x+\alpha, u) - k(x+\alpha, r)) g(y+\alpha-u) b(u) \Big|$$

$$+ \int\limits_{0}^{h} \frac{dt}{t} \int\limits_{|x-r| \geq 6h_1} dx \, |k(x+\alpha, r)| |g * \varphi_t * b(x+\alpha)| = II_1 + II_2.$$

By (1.13) of Proposition 1.1 we get $II_2 \leq c$. Next (disregard the second term for $h \leq 1$),

$$II_1 = \int\limits_{0}^{h^{1-a}} + \int\limits_{h^{1-a}}^{h} = II_{11} + II_{12}.$$

But by (ii) and since $|x+\alpha-r| \geq |x-r| - \frac{1}{2} \geq 6h_1 - \frac{1}{2}h_1 \geq 2|u-r|$ we get

$$II_{12} \leq ch \int\limits_{h^{1-a}}^{h} \frac{dt}{t} \int\limits_{|x-r| \geq 6h} \frac{dx}{|x-r|^{1+a/2}} \int du \, |b(u)| |\varphi_t * g(x+\alpha-u)|$$

$$\leq ch \int_{h^{1-a}}^{h} \frac{dt}{t} \int_{|x-r|\geq 6h} \frac{dx}{|x-r|^{1+a/2}} \|b\|_2 \|\varphi_t * g\|_2$$

$$\leq ch^{1/2} \int_{h^{1-a}}^{h} \frac{dt}{t} \frac{1}{h^{a/2}} \|\varphi_t\|_2 \qquad \text{by (1.4)}$$

$$\leq \frac{ch^{1/2}}{h^{a/2}} \int_{h^{1-a}}^{h} \frac{dt}{t^{3/2}} \leq c.$$

Now by (1.6) since $|x+\alpha-u| \geq |x+\alpha-r| - |u-r| \geq 6h_1 - \frac{1}{2}h_1 - h_1 = 5h_1 - \frac{1}{2}h_1$ we get

(3.29)(a) $\quad |\varphi_t * g(x+\alpha-u)|$

$$= \left| \int \varphi_t(s)(g(x+\alpha-u-s) - g(x+\alpha-u))\,ds \right|$$

$$\leq \frac{ct}{|x-r|^{2-3a/2}} \qquad \text{if } a \neq 2, \qquad \text{and}$$

(3.29)(b) $\quad |\varphi_t * g(x+\alpha-u)|$

$$\leq c \left(\int |\varphi_t(s)| |g(x+\alpha-u-s) - g(x+\alpha-u)|\,ds \right)^{1/2} \qquad \text{by (i)}$$

$$\leq ct^{1/2}|x-r|^{1/2} \qquad \text{by (1.6) if } a = 2.$$

Now by (3.29)(a) and (ii) as above we get

$$II_{11} \leq ch \int_0^{h^{1-a}} \frac{dt}{t} t \int_{|x-r|\geq 6h} \frac{dx}{|x-r|^{3-a}} \int du\,|b(u)|$$

$$\leq c \qquad \text{if } a < 2.$$

By (3.29)(b) and (ii) we infer that

$$II_{11} \leq ch \int_0^{1/h} \frac{dt}{t} \int_{6h\leq|x-r|} \frac{dx}{|x-r|^2} \int du\,|b(u)||x-r|^{1/2}t^{1/2}$$

$$\leq ch \int_0^{1/h} \frac{dt}{t^{1/2}} \int_{6h\leq|x-r|} \frac{dx}{|x-r|^{3/2}} \leq c \qquad \text{for } a = 2,$$

while for $a > 2$,

$$II_{11} = \int_0^{h^{1-a}} \frac{dt}{t} \int_{6h_1\leq|x-r|\leq 6t^{1/(1-a)}} dx \ldots + \int_0^{h^{1-a}} \frac{dt}{t} \int_{|x-r|\geq 6t^{1/(1-a)}} dx \ldots$$

$$= II_{111} + II_{112}.$$

By (3.29)(a) and (ii) we get

$$II_{111} \leq ch \int_0^{h^{1-a}} \frac{dt}{t} t \int_{6h\leq|x-r|\leq 6t^{1/(1-a)}} dx\,|x-r|^{a-3}$$

$$\leq ch \int_0^{h^{1-a}} dt\, t^{(a-2)/(1-a)} \leq c \qquad \text{since } a > 2 > 1.$$

Also by (ii) we get

$$II_{112} \leq ch \int_0^{h^{1-a}} \frac{dt}{t} \int_{|x-r|\geq 6t^{1/(1-a)}} dx \int du \frac{|b(u)|}{|x-r|^{1+a/2}} |\varphi_t * g(x+\alpha-u)|$$

$$\leq ch \int_0^{h^{1-a}} \frac{dt}{t} \int_{|x-r|\geq 6t^{1/(1-a)}} \frac{dx}{|x-r|^{1+a/2}} \|b\|_2 \|g * \varphi_t\|_2$$

$$\leq ch^{1/2} \int_0^{h^{1-a}} \frac{dt}{t^{3/2}t^{a/(2(1-a))}} \qquad \text{by (1.4)}$$

$$\leq c \qquad \text{since } a > 1.$$

This completes the proof of (3.27)(a).

Now to show (3.27)(b), we suppose that $0 < h \leq 1$, and as we estimated I in Theorem 3.4 we again get

$$B \leq \int_h^1 \frac{dt}{t} \int_{6t\leq|x-r|\leq 6} dx \ldots + \int_h^1 \frac{dt}{t} \int_{|x-r|\geq 6} dx \ldots + \int_1^\infty \frac{dt}{t} \int_{|x-r|\geq 6t} dx \ldots$$

$$= III_1 + III_2 + III_3.$$

By Lemma 3.3(ii), $III_1 \leq c$. For the terms III_2 and III_3 again we get $|y+\alpha-u| \geq 3$ and so $\psi(y+\alpha-u) = 1$. Thus in order to see (3.27)(b), it suffices to prove (3.21). We begin with

$$IV = \int_h^\infty \frac{dt}{t} \int_{|x-r|\geq 6t_1} dx \left| \int dy\, \varphi_t(x-y)k(y+\alpha,r)g * b(y+\alpha) \right|$$

$$\leq \int_h^\infty \frac{dt}{t} \int_{|x-r|\geq 6t_1} dx \left| \int dy\, \varphi_t(x-y) \right.$$

$$\times (k(y+\alpha,r) - k(x+\alpha,r))g * b(y+\alpha) \bigg|$$

$$+ \int_h^\infty \frac{dt}{t} \int_{|x-r|\geq 6t_1} dx\, |k(x+\alpha,r)||\varphi_t * g * b(x+\alpha)| = IV_1 + IV_2.$$

By (1.14), $IV_2 \leq c$. Next (disregard the first term if $h \geq 1$),

$$IV_1 = \int\limits_h^{h^{1/(1-a)}} + \int\limits_{h^{1/(1-a)}}^{\infty} = IV_{11} + IV_{12}.$$

Notice by (3.12) and (3.24) that

$$IV_{12} \leq c \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t} \int\limits_{|x-r| \geq 6t} dx \, \frac{t}{|x-r|^{1+a/2}} \int dy \, |\varphi_t(x-y)| \, |g*b(y+\alpha)|$$

$$\leq c \int\limits_{h^{1/(1-a)}}^{\infty} dt \int\limits_{|x-r| \geq 6t} \frac{dx}{|x-r|^{1+a/2}} \|\varphi_t\|_2 \|g*b\|_2$$

$$\leq c\|b\|_2 \int\limits_{h^{1/(1-a)}}^{\infty} \frac{dt}{t^{(1+a)/2}} \qquad \text{by (1.4)}$$

$$\leq c.$$

We are left with estimating IV_{11} only when $0 < h \leq 1$. By (3.12) (see (3.24)) we get

$$IV_{11} \leq ch \int\limits_h^{h^{1/(1-a)}} \frac{dt}{t} t \int\limits_{|x-r| \geq 6t} dx \int dy \, \frac{|\varphi_t(x-y)|}{|x-r|^{3-a}} \int du \, |b(u)|$$

$$\leq ch \int\limits_h^{h^{1/(1-a)}} dt \int\limits_{|x-r| \geq 6t} \frac{dx}{|x-r|^{3-a}} \leq c \qquad \text{if } 1 < a < 2.$$

Next,

$$IV_{11} = \int\limits_h^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{6t_1 \leq |x-r| \leq 6h^{1/(1-a)}} dx \, \ldots + \int\limits_h^{h^{1/(1-a)}} \frac{dt}{t} \int\limits_{|x-r| \geq 6h^{1/(1-a)}} dx \, \ldots$$

$$= IV_{111} + IV_{112}.$$

By (1.6), (3.12) and (3.24),

$$IV_{111} \leq ch \int\limits_h^{h^{1/(1-a)}} \frac{dt}{t} t \int\limits_{6t \leq |x-r| \leq 6h^{1/(1-a)}} dx \int dy \, \frac{|\varphi_t(x-y)|}{|x-r|^{3-a}} \int du \, |b(u)|$$

$$\leq ch \int\limits_h^{h^{1/(1-a)}} dt \int\limits_{6t \leq |x-r| \leq 6h^{1/(1-a)}} dx \, |x-r|^{a-3} \leq c \qquad \text{for } a > 2.$$

Next by (3.12) and (3.24) we get

$$IV_{112} \leq c \int\limits_h^{h^{1/(1-a)}} \frac{dt}{t} t \int\limits_{|x-r| \geq 6h^{1/(1-a)}} dx \int dy \, \frac{|\varphi_t(x-y)|}{|x-r|^{1+a/2}} |g*b(y+\alpha)|$$

$$\leq c \int\limits_h^{h^{1/(1-a)}} dt \int\limits_{|x-r| \geq 6h^{1/(1-a)}} \frac{dx}{|x-r|^{1+a/2}} \|\varphi_t\|_2 \|g*b\|_2$$

$$\leq c\|b\|_2 \int\limits_h^{h^{1/(1-a)}} \frac{dt}{t^{1/2}} \int\limits_{|x-r| \geq 6h^{1/(1-a)}} \frac{dx}{|x-r|^{1+a/2}} \qquad \text{by (1.4)}$$

$$\leq c.$$

Finally, we do the case where $a = 2$:

$$\int\limits_h^{1/h} \frac{dt}{t} \int\limits_{6t_1 \leq |x-r|} dx \left| \int dy \, \varphi_t(x-y)(k(y+\alpha,r) - k(x+\alpha,r)) \right.$$

$$\left. \times \int du \, b(u)(g(y+\alpha-u) - g(y+\alpha-r)) \right|$$

$$\leq \int\limits_h^{1/h} dt \int\limits_{6t_1 \leq |x-r|} dx \int dy \, \frac{|\varphi_t(x-y)|}{|x-r|^2}$$

$$\times \int du \, |b(u)| |g(y+\alpha-u) - g(y+\alpha-r)|$$

by (3.12) and (3.24); but $|g(y+\alpha-u) - g(y+\alpha-r)| \leq c|y-r|^{1/2} h^{1/2}$ by (i) and (1.6), and therefore, the above is

$$\leq ch^{1/2} \int\limits_h^{1/h} dt \int\limits_{|x-r| \geq 6t} \frac{dx}{|x-r|^{3/2}} \leq c.$$

This completes the proof of (3.27)(b). Putting all these results together, we get the proof of the theorem. ∎

As a consequence of the Corollary below we conclude that $\Omega_1 \in \Re$, where $\Omega_1(y,u) = K(y,u)e^{i|y-u|^a}$ and $K(y,u)$ satisfies (0.2).

COROLLARY. *Let $a > 0$, $a \neq 1$. Then if $\Omega_1(y,u) = K(y,u)e^{i|y-u|^a}$ where $K(y,u)$ satisfies (0.2), then*

$$\|\Omega(f)\|_B \leq c\|f\|_B,$$

where Ω is defined by (0.1).

Proof. Most of what we show here is routine; however, there are lots of things to show. We need to show (1.1), (1.2), (1.3), (1.4)(p), (1.5)(p),

(1.6)(p), (1.7), (2.3), (2.4), (2.10), (2.11), (3.12), (3.13) and (i) and (ii) of Theorem 3.5.

By Theorem 2.2 of [20] we find that (1.3) holds and by (2.2) of [20] we get $c_l = 1/2^{la/4}$ in (2.10). Thus (2.11) holds when $a \geq 2$ and any $\beta > 1$ works, while when $0 < a < 2$ we need to choose β so that $1 < \beta < 1/(1 - a/2)$.

Next take

$$k(y, u) = K(y, u)(1 + |y - u|)^{1 - a/p'} \quad \text{and} \quad g(y) = \frac{e^{i|y|^a}}{(1 + |y|)^{1 - a/p'}} \,.$$

By Theorem 5 of [12] we deduce that (1.4)(p) holds (in Examples 7', 8', 9' of [17] we also showed (1.4)). Next consider

$$(3.30) \quad \frac{\partial k}{\partial y} = \left(1 - \frac{a}{p'}\right) K(y, u)(1 + |y - u|)^{-a/p'} + \frac{\partial K}{\partial y}(1 + |y - u|)^{1 - a/p'} \,,$$

$$(3.31) \quad \frac{\partial k}{\partial u} = \left(1 - \frac{a}{p'}\right) K(y, u)(1 + |y - u|)^{-a/p'} + \frac{\partial K}{\partial u}(1 + |y - u|)^{1 - a/p'} \,,$$

$$(3.32) \quad \frac{\partial g}{\partial y} = e^{i|y|^a}\left(1 - \frac{a}{p'}\right)(1 + |y|)^{a/p' - 2} + (1 + |y|)^{a/p' - 1} ia|y|^{a-1} e^{i|y|^a} \,,$$

$$(3.33) \quad |\nabla \Omega_1| \leq \frac{c}{|y - u|^{2 - a}} \quad \text{if } |y - u| \geq 1 \,.$$

By (3.30)–(3.33) we conclude that (1.1), (1.2), (1.5)(p), (1.6)(p), (1.7)(a), (1.7)(b), (3.12), (3.13), and (i) and (ii) of Theorem 3.5 hold.

To see (1.7)(c), since K satisfies (0.2)(a) and by (0.1), we infer that (1.7)(c) holds if $2 \leq |y - u| \leq 5$. If $|y - u| \geq 5$ then $|y + \frac{1}{2} - u| \geq 3$ and so

$$\Omega(y, u) = \Omega_1(y, u) - \Omega_1(y + \tfrac{1}{2}, u) \,,$$

and now (1.7)(c) follows from (3.33). Hence we get (1.7)(c) for all $|y - u| \geq 2$.

In order to see (2.4) note that

$$K_\lambda(y, u) = \frac{K(y, u)(1 + |y - u|)^{1 - a/p'}}{|y - u - \lambda|^{1 - a/p'}}$$

and so again $K_\lambda(y, u)$ satisfies (0.2) as long as $|y - u| \geq \max(1, 2|\lambda|)$.

Finally, to see (2.3) we need to check

$$\int_c^d e^{i|x + \alpha - u - \lambda|^a} \chi(|x - u| \geq 5t^\beta)\, du$$

where $|\lambda| \leq t \leq 1$, $a > 1$ and $\beta = 1/(1 - 2a) < 0$. We rewrite it as

$$\int_c^d e^{i|x - u + \alpha - \lambda|^a} \frac{|x - u + \alpha - \lambda|^{a-1}}{|x - u + \alpha - \lambda|^{a-1}} \chi(|x - u| \geq 5t^\beta)\, du \,.$$

Since

$$|x - u + \alpha - \lambda| \geq |x - u| - \tfrac{1}{2} - |\lambda|$$
$$\geq |x - u| - \tfrac{1}{10}|x - u| - \tfrac{1}{5}|x - u| = \tfrac{7}{10}|x - u| \,,$$

and since $|x - u| \geq 5t^\beta$ and $|x - u + \alpha - \lambda|^{a-1}$ is monotonic in u we conclude that (2.3) holds. This completes our proof. ∎

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MATHEMATICS DEPARTMENT
AUBURN UNIVERSITY
AUBURN, ALABAMA 36849-5201
U.S.A.

Ergodic properties of skew products
with Lasota–Yorke type maps in the base

by

ZBIGNIEW S. KOWALSKI (Wrocław)

Abstract. We consider skew products $T(x,y) = (f(x), T_{e(x)}y)$ preserving a measure which is absolutely continuous with respect to the product measure. Here f is a 1-sided Markov shift with a finite set of states or a Lasota–Yorke type transformation and T_i, $i = 1, \ldots, \max e$, are nonsingular transformations of some probability space. We obtain the description of the set of eigenfunctions of the Frobenius–Perron operator for T and consequently we get the conditions ensuring the ergodicity, weak mixing and exactness of T. We apply these results to random perturbations.

0. Introduction. Let $\{T_i\}_{i=1}^s$ be a finite family of nonsingular transformations of a probability space $(Y, \mathcal{B}, p)$. Given a nonsingular transformation f of a probability space $(X, \mathcal{A}, \mu)$ and a mapping e from X to $\{1, \ldots, s\}$, we define the *skew product transformation*

$$T(x,y) = (f(x), T_{e(x)}y).$$

The purpose of this paper is the description of the ergodic properties of T. To this end we use our results on eigenfunctions of the Frobenius–Perron operator for T. The above problem was considered in [10] and [11] where the transformation f preserves the Bernoulli measure μ and the family of transformations may be infinite.

The paper consists of two parts. In the first part we assume that f is a 1-sided Markov shift preserving the measure μ with a finite set of states. In the second part we assume f to be a general Lasota–Yorke type transformation, i.e. f is piecewise C^1 and uniformly expanding.

PART I

1. Introduction. Let σ be the shift endomorphism in a space $X \subset \{1, \ldots, s\}^{\mathbb{N}}$ preserving μ. The measure μ is Markov and it is determined by