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Summable families in nuclear groups

by

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Abstract. Nuclear groups form a class of abelian topological groups which contains LCA groups and nuclear locally convex spaces, and is closed with respect to certain natural operations. In nuclear locally convex spaces, weakly summable families are strongly summable, and strongly summable are absolutely summable. It is shown that these theorems can be generalized in a natural way to nuclear groups.

Nuclear groups were introduced in [1]. They form a class of abelian topological groups which contains LCA groups and nuclear locally convex spaces, and is closed with respect to certain natural operations. The aim of this paper is to give proofs of some assertions formulated in [1], Section 10, concerning the convergence of sequences and series in nuclear groups.

By the weak topology on an abelian topological group we shall mean the topology induced by continuous characters. If the group is a locally convex space, this topology is essentially weaker than the topology induced by continuous linear functionals, but defines the same class of convergent sequences. Our first goal is to show that weakly convergent sequences in nuclear groups are convergent in the original topology; this was conjectured in [1], (10.17). Then we prove that weakly summable families in nuclear groups are summable, and that summable families are absolutely summable (the definitions are given below). The latter result was announced in [1], (10.16).

If every (weakly) summable family in a locally convex space is absolutely summable, then the space is nuclear ([4], 4.2.4). It seems conceivable that "locally convex space" may be replaced here by "locally quasi-convex group".

We have to introduce some notation and terminology. Let G be an abelian topological group. The family of all neighbourhoods of zero in G

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will be denoted by $\mathcal{N}_0(G)$. By a *character* of G we mean a homomorphism of G into the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We identify $\mathbb{T}$ with the interval $(-1/2, 1/2]$. Given a real number x, we denote by $\langle x \rangle$ the number $y \in (-1/2, 1/2]$ such that $x - y \in \mathbb{Z}$. By the *weak topology* on G we mean the topology induced by the family $G^\wedge$ of continuous characters. The original topology on G will be called the *strong topology*.

Let X, Y be two symmetric convex subsets of a vector space F (all vector spaces are assumed to be real). Suppose that $X \subset Y$. The *Kolmogorov diameters* of X with respect to Y are defined by

$$d_k(X,Y) = \inf_L \ \inf\{t > 0 : X \subset tY + L\} \quad (k = 1, 2, \ldots),$$

where the infimum is taken over all linear subspaces L of F with $\dim L < k$.

A Hausdorff abelian group G is called *nuclear* if it satisfies the following condition: given arbitrary $U \in \mathcal{N}_0(G)$, $c > 0$ and $m = 1, 2, \ldots$, there exist a vector space F, two symmetric and convex subsets X, Y of F with $d_k(X,Y) \le ck^{-m}$ for every k, a subgroup K of F and a homomorphism $\varphi : K \to G$ such that $\varphi(K \cap X) \in \mathcal{N}_0(G)$ and $\varphi(K \cap Y) \subset U$.

Now, we are ready to formulate our first assertion:

THEOREM 1. *Every weakly convergent series in a nuclear group is strongly convergent.*

The proof will be preceded by several lemmas.

Let F be a vector space and τ a topology on F such that F_τ is an additive topological group. We say that F_τ is a *locally convex vector group* if it is separated and has a base at zero consisting of symmetric convex sets. A locally convex vector group F is called a *nuclear vector group* if to each symmetric convex $U \in \mathcal{N}_0(F)$ there corresponds a symmetric convex $V \in \mathcal{N}_0(F)$ with $d_k(V, U) \le k^{-1}$ for every k. Evidently, every nuclear vector group is a nuclear group.

LEMMA 1. *Let G be a nuclear group. Then there exist a nuclear vector group F, a subgroup H of F and a closed subgroup K of H such that G is topologically isomorphic to H/K.*

This is the main assertion of Theorem (9.6) in [1].

Let p be a seminorm on a vector space E. We write $B_p = \{u \in E : p(u) \le 1\}$. The quotient space $E/p^{-1}(0)$ endowed with its canonical norm will be denoted by E_p, and the canonical projection of E onto E_p by ψ_p. We say that p is a *pre-Hilbert* seminorm if E_p is a pre-Hilbert space. If $q \le p$ is another seminorm on E, then the canonical operator from E_p to E_q will be denoted by T_{pq}. We have the following canonical commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{id}} & E \\ \downarrow{\scriptstyle \psi_p} & & \downarrow{\scriptstyle \psi_q} \\ E_p & \xrightarrow{T_{pq}} & E_q \end{array}$$

LEMMA 2. *Let F be a nuclear vector group. Given arbitrary $U \in \mathcal{N}_0(F)$, $c > 0$ and $m = 1, 2, \ldots$, we can find a linear subspace E of F and two pre-Hilbert seminorms p, q on E such that $B_q \subset U$, $B_p \in \mathcal{N}_0(F)$ and $d_k(B_p, B_q) \le ck^{-m}$ for every k.*

This is a direct consequence of Propositions (9.3) and (2.14) of [1].

Let A be a subset of a normed space E. The linear subspace spanned by A is denoted by $\operatorname{span} A$ and the distance of a point u to A by $d(u, A)$. The closed unit ball of E is denoted by B_E. Given an additive subgroup K of E, we denote by K_E^* the family of all continuous linear functionals f on E such that $f(K) \subset \mathbb{Z}$.

Let $T : E \to F$ be a bounded linear operator acting between normed spaces. We write $d_k(T) = d_k(T(B_E), B_F)$ for $k = 1, 2, \ldots$, and $\Sigma(T) = \sum_{k=1}^\infty k d_k(T)$.

LEMMA 3. *Let E, F be unitary spaces and $T : E \to F$ a bounded operator with $\Sigma(T) \le 1$. Let K be a subgroup of E.*

(a) *Suppose we are given some $a \in E$ and $r > 0$ such that $d(Ta, T(K)) \ge r$. Then we can find some $f \in K_E^*$ with $|\langle f(a) \rangle| \ge 1/4$ and $\|f\| \le 4r^{-1}$.*

(b) *Suppose we are given a sequence (a_n) in E such that $d(Ta_n, T(K)) \to \infty$. Then we can find some $f \in K_E^*$ such that $\langle f(a_n) \rangle \not\to 0$.*

Proof. Part (a) follows directly from Proposition (8.4) of [1]. We now prove (b).

We may assume E to be complete. Let $M = \operatorname{span} K$, let N be the orthogonal complement of $\overline{M}$ in E, and let φ and ψ be the orthogonal projections of E onto $\overline{M}$ and N, respectively.

Suppose first that $\limsup \|\psi(a_n)\| = \infty$. Then there is a bounded linear functional g on N such that $\limsup |g\psi(a_n)| = \infty$. Since weakly convergent sequences in $\mathbb{R}$ are strongly convergent, we can find some $\kappa \in \mathbb{R}^\wedge$ such that $\kappa g\psi(a_n) \not\to 0$. In other words, there is some $t \in \mathbb{R}$ such that $\langle tg\psi(a_n) \rangle \not\to 0$; then we may take $f = tg\psi$.

Next, suppose that $\limsup \|\psi(a_n)\| < \infty$. Choose a sequence (b_n) in M with $b_n - \varphi(a_n) \to 0$. For every n, we have

$$d(Ta_n, T(K)) \le \|Ta_n - T\varphi(a_n)\| + \|T\varphi(a_n) - Tb_n\| + d(Tb_n, T(K)),$$
$$\|Ta_n - T\varphi(a_n)\| \le \|T\| \cdot \|a_n - \varphi(a_n)\| = \|T\| \cdot \|\psi(a_n)\|,$$
$$\|T\varphi(a_n) - Tb_n\| \le \|T\| \cdot \|\varphi(a_n) - b_n\|.$$

As $d(Ta_n, T(K)) \to \infty$, it follows that $d(Tb_n, T(K)) \to \infty$.

Choose an index n_1 such that $d(Tb_{n_1}, T(K)) > 2$. Due to (a), there is some $g_1 \in K^*_{\overline{M}}$ with $|\langle g_1(b_{n_1})\rangle| \geq 1/4$ and $\|g_1\| \leq 4\cdot 2^{-1}$. As $b_{n_1} \in \operatorname{span} K$, we can find a finitely generated subgroup K_1 of K with $b_{n_1} \in M_1 := \operatorname{span} K_1$. Then we can find an index n_2 such that $d(Tb_{n_2}, T(K+M_1)) > 2^2$ and, by (a), some $g_2 \in (K+M_1)^*_{\overline{M}}$ with $|\langle g_2(b_{n_2})\rangle| \geq 1/4$ and $\|g_2\| \leq 4\cdot 2^{-2}$. By repeating this procedure, we construct by induction a sequence $M_1 \subset M_2 \subset \ldots$ of finite-dimensional subspaces of E, a sequence $b_{n_s} \in M_s$ and a sequence $g_s \in K^*_{\overline{M}}$ such that $|\langle g_s(b_{n_s})\rangle| \geq 1/4$, $g_s(M_{s-1}) = \{0\}$ and $\|g_s\| \leq 4\cdot 2^{-s}$ for every s.

If $x, y \in \mathbb{T}$ and $|\langle y\rangle| \geq 1/4$, then we can find a coefficient $t = 0, \pm 1$ such that $|\langle x + ty\rangle| \geq 1/4$. Therefore we can construct by induction a sequence $t_s = 0, \pm 1$ such that

$$|\langle t_1 g_1(b_{n_s}) + \ldots + t_s g_s(b_{n_s})\rangle| \geq \frac{1}{4}$$

for every s. Hence

$$|\langle (t_1 g_1 + \ldots + t_s g_s)(b_{n_r})\rangle| \geq \frac{1}{4}$$

if $r \leq s$. Set $g = \sum_{s=1}^{\infty} t_s g_s$. Then $g \in K^*_{\overline{M}}$, and $|\langle g(b_{n_r})\rangle| \geq 1/4$ for every r. Since $b_n - \varphi(a_n) \to 0$, it follows that $\langle g\varphi(a_n)\rangle \not\to 0$; then we may take $f = g\varphi$. ∎

Remark. In Lemma 3, the condition $\Sigma(T) \leq 1$ may be replaced by $\sum_{k=1}^{\infty} d_k(T) \leq c$ where c is some numerical constant. The proof will be given elsewhere.

LEMMA 4. *Suppose we are given a sequence*

$$E \xrightarrow{R} F \xrightarrow{S} G \xrightarrow{T} H$$

of bounded operators acting between Hilbert spaces, with $\Sigma(R), \Sigma(T) \leq 1$ and $d_k(S) \to 0$. Let K be a subgroup of E and (a_n) a sequence in E such that $d(TSRa_n, TSR(K)) \geq 1$ for every n. Then there exists a functional $f \in K^*_E$ such that $\langle f(a_n)\rangle \not\to 0$.

Proof. In view of Lemma 3(b), we may assume that the sequence $d(Ra_n, R(K))$ is bounded. We may naturally assume G to be complete. Then S is compact, and we can choose a subsequence b_{n_k} of b_n such that SRb_{n_k} converges to some $u \in G$. We have

$$d(TSRb_{n_k}, TSR(K)) = d(TSRa_{n_k}, TSR(K)) \geq 1$$

for every k, which implies that $d(Tu, TSR(K)) \geq 1$. Hence, by Lemma 3(a), there is some $g \in SR(K)^*_G$ with $\langle g(u)\rangle \geq 1/4$. Then $\langle gSRa_{n_k}\rangle = \langle gSRb_{n_k}\rangle \to \langle g(u)\rangle \geq 1/4$, and we may take $f = gSR$. ∎

Proof of Theorem 1. Let $(g_n)_{n=1}^{\infty}$ be a weakly convergent series in a nuclear group G. We are to show that (g_n) is strongly convergent. Without loss of generality, we may assume that (g_n) is weakly convergent to zero.

Suppose that (g_n) is not strongly convergent. To obtain a contradiction, we have to find some $\chi \in G^\wedge$ such that $\chi(g_n) \not\to 0$.

In view of Lemma 1, we may assume that $G = F/K$ where K is a closed subgroup of some nuclear vector group F. Let $\beta : F \to F/K$ be the natural projection. For each $n = 1, 2, \ldots$, choose a vector $u_n \in F$ with $\beta(u_n) = g_n$. As (g_n) is not strongly convergent to zero in F/K, there exists some $U \in \mathcal{N}_0(F)$ such that

(1) $$u_n \notin K + U \quad \text{for infinitely many } n.$$

It follows easily from Lemma 2 that we can find a linear subspace E of F and pre-Hilbert seminorms $p \geq q \geq r \geq s$ on E such that $B_s \subset U$, $B_p \in \mathcal{N}_0(F)$,

$$\sum_{k=1}^{\infty} k d_k(B_p, B_q) \leq 1, \qquad \sum_{k=1}^{\infty} k d_k(B_r, B_s) \leq 1$$

and $d_k(B_q, B_r) \to 0$. Let us draw the canonical diagram

$$\begin{array}{ccccccc} E & \xrightarrow{\mathrm{id}} & E & \xrightarrow{\mathrm{id}} & E & \xrightarrow{\mathrm{id}} & E \\ \downarrow{\scriptstyle \psi_p} & & \downarrow{\scriptstyle \psi_q} & & \downarrow{\scriptstyle \psi_r} & & \downarrow{\scriptstyle \psi_s} \\ E_p & \xrightarrow{T_{pq}} & E_q & \xrightarrow{T_{qr}} & E_r & \xrightarrow{T_{rs}} & E_s \end{array}$$

We have $\Sigma(T_{pq}), \Sigma(T_{rs}) \leq 1$ and $d_k(T_{qr}) \to 0$.

Set $H = E \cap K$ and consider the following canonical commutative diagram:

$$\begin{array}{ccccc} E & \xrightarrow{\mathrm{id}} & F & \xrightarrow{\mathrm{id}} & F \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ E/H & \xrightarrow{\mu} & F/K & \xrightarrow{\nu} & F/(K+E) \end{array}$$

Since $B_p \in \mathcal{N}_0(F)$, the subspace E spanned by B_p is an open subgroup of F, and the group $F/(K+E)$ is discrete. Observe that μ is a topological embedding.

Suppose that $\gamma(u_n) \neq 0$ for infinitely many n. In discrete groups (and, in fact, in all LCA groups), weakly convergent series are strongly convergent. Therefore we can find a continuous character κ of $F/(K+E)$ such that $\kappa\gamma(u_n) \not\to 0$. Then $\chi = \kappa\nu \in (F/K)^\wedge$, and

$$\chi(g_n) = \kappa\nu(g_n) = \kappa\nu\beta(u_n) = \kappa\gamma(u_n) \not\to 0.$$

Thus we may assume that $\gamma(u_n) = 0$, i.e. that $u_n \in K + E$ for almost all n, say, for $n \geq n_0$.

To each $n \geq n_0$ there corresponds some $v_n \in E$ with $v_n - u_n \in K$. Then $\beta(v_n) = g_n$ and, as $B_s \subset U$, condition (1) implies that $v_n \notin K + B_s$ for infinitely many n. This means that

$$d(T_{rs}T_{qr}T_{pq}\psi_p(v_n),\ T_{rs}T_{qr}T_{pq}\psi_p(H)) > 1$$

for infinitely many n. So, by Lemma 4, there is some $f \in \psi_p(H)^*_{E_p}$ such that $\langle f\psi_p(v_n)\rangle \not\to 0$. Then $f\psi_p \in H^*_E$, and the formula $\kappa\psi_p = \langle f\psi_p\rangle$ defines a continuous character κ of E/H with $\kappa\alpha(v_n) \not\to 0$. The character $\kappa\mu^{-1}$ of $\mu(E/H)$ can be extended to a continuous character χ of F/K ([1], (8.3)). Then

$$\chi(g_n) = \chi\beta(v_n) = \chi\mu\alpha(v_n) = \kappa\alpha(v_n) \not\to 0\,. \blacksquare$$

Theorem 1 implies that every nuclear group G satisfies the Orlicz–Pettis theorem: if a series of elements of G is subseries convergent in the weak topology, then it is subseries convergent in the original topology as well. This fact, however, follows directly from the result of Kalton [3]; cf. [1], (10.17).

Let G be an abelian topological group. Let I be a set of indices and $(g_i)_{i\in I}$ a family of elements of G. We say that $(g_i)_{i\in I}$ is *summable* (or *strongly summable*) if the following condition is satisfied: to each $U \in \mathcal{N}_0(G)$ there corresponds a finite subset J of I such that $\sum_{i\in K} g_i \in U$ for any finite subset K of $I \setminus J$ (cf. [2], Ch. III, Sect. 5, n° 1). We say that the family $(g_i)_{i\in I}$ is *weakly summable* if $\sum_{i\in I} |\chi(g_i)| < \infty$ for any $\chi \in G^\wedge$; clearly, this holds if and only if $(g_i)_{i\in I}$ is summable in the weak topology.

THEOREM 2. *Every weakly summable family of elements of a nuclear group is strongly summable.*

Proof. Let $(g_i)_{i\in I}$ be a family of elements of a nuclear group G. Suppose that $(g_i)_{i\in I}$ is not strongly summable. Then there is some $U \in \mathcal{N}_0(G)$ such that, to each finite subset J of I, there corresponds a finite subset K of $I\setminus J$ with $\sum_{i\in K} g_i \notin U$. So, we can construct by induction a sequence $K_1, K_2, \ldots$ of disjoint subsets of I such that $h_n := \sum_{i\in K_n} g_i \notin U$ for $n = 1, 2, \ldots$ According to Theorem 1, we can find some $\chi \in G^\wedge$ such that $\chi(h_n) \not\to 0$. Then

$$\sum_{i\in I} |\chi(g_i)| \geq \sum_{n=1}^{\infty} \sum_{i\in K_n} |\chi(g_i)| \geq \sum_{n=1}^{\infty} \Big|\chi\Big(\sum_{i\in K_n} g_i\Big)\Big| = \sum_{n=1}^{\infty} |\chi(h_n)| = \infty\,,$$

which means that $(g_i)_{i\in I}$ is not weakly summable. ■

Let A be a subset of an abelian topological group G. For each $g \in A$, we define

$$\mathrm{n}_A^g = \sup\{n : kg \in A \text{ for } k = 1, \ldots, n\} \quad \text{and} \quad g/A = (\mathrm{n}_A^g)^{-1}\,.$$

Thus, in particular, $g/A = 0$ if and only if $kg \in A$ for every k. Given a character χ of G, we write $|\chi(A)| = \sup\{|\chi(g)| : g \in A\}$. We say that A is a *quasi-convex* subset of G if, to each $g \in G/A$, there corresponds some $\chi \in G^\wedge$ with $|\chi(A)| \leq 1/4$ and $|\chi(g)| > 1/4$. Next, we say that G is a *locally quasi-convex* group if there is a basis of neighbourhoods of zero consisting of quasi-convex sets. Nuclear groups are locally quasi-convex ([1], (8.5)). A topological vector space is locally convex if and only if it is a Hausdorff locally quasi-convex group ([1], (2.4)).

Let $(g_i)_{i\in I}$ be a family of elements of G. We say that $(g_i)_{i\in I}$ is *absolutely summable* if, to each $U \in \mathcal{N}_0(G)$, there corresponds a finite subset J of I such that $g_i \in U$ for $i \notin J$, and $\sum_{i\in I\setminus J} (g_i/U) < \infty$. If G is a vector space and p_U the Minkowski functional of a radial subset U of G, then, clearly, $p_U(g) \leq g/U < 2p_U(g)$ for all $g \in G$. Therefore, if G is a locally convex space, the family $(g_i)_{i\in I}$ is absolutely summable if and only if it is absolutely summable in the usual sense.

PROPOSITION. *Every absolutely summable family in a locally quasi-convex group is strongly summable.*

Proof. Let $(g_i)_{i\in I}$ be an absolutely summable family in a locally quasi-convex group G. Choose an arbitrary quasi-convex $U \in \mathcal{N}_0(G)$. There is a finite subset J of I such that $g_i \in U$ if $i \notin J$, and $\sum_{i\in I\setminus J} (g_i/U) \leq 1$. Now, take an arbitrary $\chi \in G^\wedge$ with $|\chi(U)| \leq 1/4$, and an arbitrary finite subset K of $I \setminus J$; we are to show that $|\chi(\sum_{i\in K} g_i)| \leq 1/4$. It is not hard to see that $|\chi(g)| \leq \frac{1}{4}(g/U)$ for each $g \in U$. Hence

$$\Big|\chi\Big(\sum_{i\in K} g_i\Big)\Big| \leq \sum_{i\in K} |\chi(g_i)| \leq \frac{1}{4}\sum_{i\in K} (g_i/U) \leq \frac{1}{4}\sum_{i\in I\setminus J} (g_i/U) \leq \frac{1}{4}\,. \blacksquare$$

THEOREM 3. *Every strongly summable family in a nuclear group is absolutely summable.*

In the proof we shall need several lemmas.

LEMMA 5. *Let $v_1, \ldots, v_m$ be a linearly independent system in $\mathbb{R}^m$. Let $w_1, \ldots, w_m$ be the orthogonalization of $v_1, \ldots, v_m$, and L the subgroup of $\mathbb{R}^m$ generated by $v_1, \ldots, v_m$. Then*

$$\max_{u\in\mathbb{R}^m} d(u, L) \leq \tfrac{1}{2}(\|w_1\|^2 + \ldots + \|w_m\|^2)^{1/2}\,. \tag{2}$$

Proof. For each $k = 1, \ldots, m$, let L_k be the subgroup generated by $v_1, \ldots, v_k$, and let $M_k = \operatorname{span} L_k$. Define

$$\mu_k = \max_{u\in M_k} d(u, L_k) \quad (k = 1, \ldots, m)$$

and $\mu_0 = 0$. It is enough to observe that

$$\mu_k^2 \leq \mu_{k-1}^2 + \tfrac{1}{4}\|w_k\|^2 \quad (k = 1, \ldots, m). \blacksquare$$

LEMMA 6. *Let D be an n-dimensional o-symmetric ellipsoid in $\mathbb{R}^n$ with principal semiaxes $\lambda_1, \ldots, \lambda_n$. Let M be an m-dimensional linear subspace of $\mathbb{R}^n$ and $\pi : \mathbb{R}^n \to M$ the orthogonal projection. If $\mu_1, \ldots, \mu_m$ are the principal semiaxes of the m-dimensional ellipsoid $\pi(D)$, then $\mu_1^2 + \ldots + \mu_m^2 \le \lambda_1^2 + \ldots + \lambda_n^2$.*

Proof. Let U be the euclidean unit ball in $\mathbb{R}^n$. We may assume that $\lambda_1 \ge \ldots \ge \lambda_n$ and $\mu_1 \ge \ldots \ge \mu_m$. Then we have

$$\lambda_k = d_k(D, U) \qquad (k = 1, \ldots, n),$$

$$\mu_k = d_k(\pi(D), U) = d_k(\pi(D), \pi(U)) \qquad (k = 1, \ldots, m).$$

As $d_k(\pi(D), \pi(U)) \le d_k(D, U)$ for every k (see e.g. [1], (2.8)(a)), it follows that $\mu_k \le \lambda_k$ for $k = 1, \ldots, m$, which yields the result. ■

LEMMA 7. *Let D be an o-symmetric n-dimensional ellipsoid in $\mathbb{R}^n$ with principal semiaxes $\lambda_1, \ldots, \lambda_n$. Given an arbitrary closed subgroup K of $\mathbb{R}^n$, we can find a closed subgroup H of $\mathbb{R}^n$, with $K \subset H$, such that*

(i) *all non-zero components of H are disjoint from D;*
(ii) $\sup_{u \in H} d(u, K) \le \frac{1}{2}(\lambda_1^2 + \ldots + \lambda_n^2)^{1/2}$.

Proof. Suppose first that K is discrete. If $K \cap D = \{0\}$, we may take $H = K$. In the other case, we can construct inductively a sequence $u_1, \ldots, u_m \in D$ such that

(iii) u_k belongs to some non-zero component of $K + \operatorname{span}\{u_j\}_{j<k}$ for each $k = 1, \ldots, m$ (the symbol $\operatorname{span}\{u_j\}_{j<1}$ denotes the zero subspace);
(iv) all non-zero components of $K + \operatorname{span}\{u_k\}_{k=1}^m$ are disjoint from D.

Then we can choose vectors $v_1, \ldots, v_m \in K$ such that

$$v_k - u_k \in \operatorname{span}\{u_j\}_{j<k} \qquad (k = 1, \ldots, m). \tag{3}$$

It follows from (iii) that both systems $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$ are linearly independent, and $M := \operatorname{span}\{u_k\}_{k=1}^m = \operatorname{span}\{v_k\}_{k=1}^m$. Take $H = K + M$ and set

$$L = \{t_1 v_1 + \ldots + t_m v_m : t_1, \ldots, t_m \in \mathbb{Z}\},$$

$$Q = \{t_1 v_1 + \ldots + t_m v_m : 0 \le t_1, \ldots, t_m \le 1\}.$$

It is obvious that $M = L + Q$. Hence $H = K + L + Q = K + Q$. As K is closed and Q compact, it follows that H is closed. Condition (iv) yields (i). We shall prove (ii).

Let $w_1, \ldots, w_m$ be the orthogonalization of $u_1, \ldots, u_m$. Since $u_1, \ldots, u_m \in D$, we have

$$r := (\|w_1\|^2 + \ldots + \|w_m\|^2)^{1/2} \le (\lambda_1^2 + \ldots + \lambda_n^2)^{1/2} \tag{4}$$

(see [1], (3.12)). Let U be the unit ball in $\mathbb{R}^n$. It follows from (3) that $w_1, \ldots, w_m$ is, in fact, the orthogonalization of $v_1, \ldots, v_m$. So, due to Lemma 5, we have $M \subset L + \frac{1}{2}rU$. Hence

$$H = K + M \subset K + L + \tfrac{1}{2}rU = K + \tfrac{1}{2}rU.$$

In view of (4), this proves (ii).

Now, suppose that K is not discrete, and let N be the zero component of K. Let then M be the orthogonal complement of N and $\pi : \mathbb{R}^n \to M$ the orthogonal projection. We may identify M with $\mathbb{R}^m$ (if $m = 0$, then $K = \mathbb{R}^n$ and there is nothing to prove). Then $\pi(D)$ is an m-dimensional ellipsoid in $\mathbb{R}^m$; let $\mu_1, \ldots, \mu_m$ be its principal semiaxes. By Lemma 6, we have

$$\mu_1^2 + \ldots + \mu_m^2 \le \lambda_1^2 + \ldots + \lambda_n^2. \tag{5}$$

Since our lemma is true for discrete groups, there exists a closed subgroup H' of $\mathbb{R}^m$ such that

(v) all non-zero components of H' are disjoint from $\pi(D)$;
(vi) $\sup_{u \in H'} d(u, \pi(K)) \le \frac{1}{2}(\mu_1^2 + \ldots + \mu_m^2)^{1/2}$.

Take $H = \pi^{-1}(H')$. Then (i) follows from (v), while (ii) is a direct consequence of (vi) and (5). ■

LEMMA 8. *Let D be an o-symmetric n-dimensional ellipsoid in $\mathbb{R}^n$ with principal semiaxes $\lambda_1, \ldots, \lambda_n$ such that*

$$\varrho := (\lambda_1^2 + \ldots + \lambda_n^2)^{1/2} < 1. \tag{6}$$

Let K be an arbitrary additive subgroup of $\mathbb{R}^n$, and $u_1, \ldots, u_s$ a finite system of vectors such that

$$\sum_{i \in I} u_i \in K + D \quad \textit{for each } I \subset \{1, \ldots, s\}. \tag{7}$$

If U is the euclidean unit ball in $\mathbb{R}^n$, then

$$\sum_{i=1}^{s} u_i/(K + 3U) \le 11\varrho. \tag{8}$$

Proof. Suppose first that K is a closed subgroup of $\mathbb{R}^n$. It follows from Lemma 7 that there exists a closed subgroup H of $\mathbb{R}^n$, with $K \subset H$, such that

(i) all non-zero components of H are disjoint from $3D$;
(ii) $H \subset K + \frac{3}{2}\varrho U$.

Let N be the zero component of H and M the orthogonal complement of N in $\mathbb{R}^n$. If $m := \dim M = 0$, then $H = \mathbb{R}^n$ and from (ii) and (6) we get $\mathbb{R}^n \subset K + U$, whence

$$\sum_{i=1}^{s} u_i/(K + 3U) \le \sum_{i=1}^{s} u_i/\mathbb{R}^n = 0.$$

So, assume that $m > 0$ and let $\pi : \mathbb{R}^n \to M$ be the canonical projection. Let then $\mu_1, \ldots, \mu_m$ be the principal semiaxes of $C := \pi(D)$. By (6) and Lemma 6, we have

$$(\mu_1^2 + \ldots + \mu_m^2)^{1/2} \le \varrho . \tag{9}$$

It follows from (7) that there exist vectors $v_1, \ldots, v_s \in D$ such that

$$v_i - u_i \in K \quad (i = 1, \ldots, s). \tag{10}$$

From (7) and (10) we obtain

$$\sum_{i \in I} v_i \in K + D \quad \text{for each } I \subset \{1, \ldots, s\}. \tag{11}$$

Condition (i) implies that $\pi(H) \cap 3C = \{0\}$, whence

$$\pi(K) \cap 3C = \{0\}. \tag{12}$$

Define $w_i = \pi(v_i)$ for $i = 1, \ldots, s$. Then, by (11),

$$\sum_{i \in I} w_i \in \pi(K) + C \quad \text{for each } I \subset \{1, \ldots, s\}. \tag{13}$$

As $v_1, \ldots, v_s \in D$, we have

$$w_i \in C \quad \text{for each } i = 1, \ldots, s. \tag{14}$$

It is not hard to conclude from (12)–(14) that

$$\sum_{i \in I} w_i \in C \quad \text{for each } I \subset \{1, \ldots, s\}. \tag{15}$$

Given an arbitrary system $\varepsilon_1, \ldots, \varepsilon_s = \pm 1$, we may write

$$\sum_{i=1}^{s} \varepsilon_i w_i = \sum_{i=1}^{s} w_i - 2 \sum_{i \in I} w_i$$

where $I = \{ i : \varepsilon_i = -1 \}$. Thus, by (15), we have

$$\sum_{i=1}^{s} \varepsilon_i w_i \in 3C \quad \text{for any } \varepsilon_1, \ldots, \varepsilon_s = \pm 1. \tag{16}$$

Let $\| \ \|$ be the euclidean norm on M, and $\| \ \|_C$ the Minkowski functional of C. Let T be the identity operator acting from the normed space $(M, \| \ \|)$ to $(M, \| \ \|_C)$. Let then $\|T\|_{\mathrm{AS}}$ and $\|T\|_{\mathrm{HS}}$ denote the absolutely summing and the Hilbert–Schmidt norms of T, respectively. It is clear that $\|T\|_{\mathrm{HS}} = (\sum_{k=1}^{m} \mu_k^2)^{1/2}$, and (16) implies that $\sum_{i=1}^{s} \|w_i\| \le 3 \|T\|_{\mathrm{AS}}$. Since $\|T\|_{\mathrm{AS}} \le \sqrt{3} \|T\|_{\mathrm{HS}}$ (see e.g. [4], 2.5.5), it follows that

$$\|w_1\| + \ldots + \|w_s\| \le 3\sqrt{3} (\mu_1^2 + \ldots + \mu_m^2)^{1/2}. \tag{17}$$

Now, fix an arbitrary $i = 1, \ldots, s$. From (6) we have $D \subset U$. As $v_i \in D$, it follows that $w_i = \pi(v_i) \in \pi(D) \subset \pi(U)$, whence $\|w_i\|^{-1} \ge 1$. Let r be the integer part of $\|w_i\|^{-1}$. Then $\|tw_i\| \le 1$, i.e. $tw_i \in U$ for $t \le r$. Hence $\pi(tv_i) \in \pi(U)$, i.e. $tv_i \in U + N$ for $t \le r$. This means that $v_i/(U + N) \le r^{-1} < 2\|w_i\|$. Since this is true for every i, from (17) and (9) we obtain

$$\sum_{i=1}^{s} v_i/(U + N) < 6\sqrt{3}\varrho < 11\varrho . \tag{18}$$

In view of (ii) and (6), we have

$$M + U \subset H + U \subset K + \tfrac{3}{2}\varrho U + U \subset K + 3U .$$

Condition (10) implies that $u_i/(K + 3U) = v_i/(K + 3U)$ for every i. Thus, by (18),

$$\sum_{i=1}^{s} u_i/(K + 3U) = \sum_{i=1}^{s} v_i/(K + 3U) \le \sum_{i=1}^{s} v_i/(M + U) < 11\varrho .$$

It remains to consider the case when K is not closed. Take an arbitrary $\vartheta < 1$. Replacing in our lemma K by $\overline{K}$ and U by ϑU, we obtain

$$\sum_{i=1}^{s} u_i/(\overline{K} + 3\vartheta U) \le 11\vartheta^{-1}\varrho . \tag{19}$$

Now, $\overline{K} \subset K + 3(1 - \vartheta)U$, whence $\overline{K} + 3\vartheta U \subset K + 3U$ and

$$\sum_{i=1}^{s} u_i/(K + 3U) \le \sum_{i=1}^{s} u_i/(\overline{K} + 3\vartheta U). \tag{20}$$

As $\vartheta < 1$ was arbitrary, from (20) and (19) we obtain (8). ∎

LEMMA 9. *Let p, q be pre-Hilbert seminorms on a vector space E such that $\sum_{k=1}^{\infty} d_k(B_p, B_q) < 1$. Let K be a subgroup of E, and $u_1, \ldots, u_s$ a finite system of vectors such that*

$$\sum_{i \in I} u_i \in K + B_p \quad \textit{for each } I \subset \{1, \ldots, s\}. \tag{21}$$

Then $\sum_{i=1}^{s} u_i/(K + 3B_q) \le 11$.

Proof. Condition (21) says that to each $I \subset \{1, \ldots, s\}$ there corresponds some $v_I \in K$ such that $\sum_{i \in I} u_i \in v_I + B_p$. Let K' be the subgroup generated by the vectors v_I, $I \subset \{1, \ldots, s\}$. Let then $E' = \operatorname{span}(K \cup \{u_i\}_{i=1}^{s})$, and let p' and q' be the restrictions to E' of p and q, respectively. We have

$$\sum_{k=1}^{\infty} d_k^2(B_{p'}, B_{q'}) \le \sum_{k=1}^{\infty} d_k^2(B_p, B_q) < 1 ,$$

$$\sum_{i \in I} u_i \in K' + B_{p'} \quad \text{for each } I \subset \{1, \ldots, s\},$$

$$\sum_{i=1}^{s} u_i/(K+3B_q) \le \sum_{i=1}^{s} u_i/(K'+3B_{q'}).$$

This means that, without loss of generality, we may assume $\dim E < \infty$. In this case, however, our lemma follows easily from the preceding one. ∎

Proof of Theorem 3. Let $(g_i)_{i\in I}$ be a strongly summable family in a nuclear group G. We are to show that $(g_i)_{i\in I}$ is absolutely summable. As in the proof of Theorem 1, we may assume that $G = F/K$ where K is a closed subgroup of some nuclear vector group F. Let $\varphi : F \to F/K$ be the natural projection.

Take an arbitrary $U \in \mathcal{N}_0(F)$. Due to Lemma 2, we can find a linear subspace E of F and pre-Hilbert seminorms p, q on E such that $3B_q \subset U$, $B_p \in \mathcal{N}_0(F)$ and $\sum_{k=1}^{\infty} d_k^2(B_p, B_q) < 1$. As $(g_i)_{i\in I}$ is strongly summable, there is a finite subset J of I such that $\sum_{i\in L} g_i \in \varphi(B_p)$ for each finite subset L of $I \setminus J$. It is enough to prove that

$$\sum_{i\in I\setminus J} g_i/\varphi(3B_q) < \infty. \tag{22}$$

For each $i \in I$, choose some $u_i \in \varphi^{-1}(g_i)$. Then $\sum_{i\in L} u_i \in K + B_p$ for each finite $L \subset I \setminus J$. Lemma 9 yields

$$\sum_{i\in I\setminus J} u_i/(K+3B_q) \le 11,$$

whence (22) follows because $g_i/\varphi(3B_q) = u_i/(K+3B_q)$ for every i. ∎

References

[1] W. Banaszczyk, *Additive Subgroups of Topological Vector Spaces*, Lecture Notes in Math. 1466, Springer, Berlin 1991.

[2] N. Bourbaki, *Topologie générale*, Chap. III, *Groupes topologiques* (*Théorie élémentaire*), Hermann, Paris 1960.

[3] N. Kalton, *Subseries convergence in topological groups and vector spaces*, Israel J. Math. 10 (1971), 402–412.

[4] A. Pietsch, *Nuclear Locally Convex Spaces*, Springer, Berlin 1972.

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An inverse Sidon type inequality

by

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Abstract. Sidon proved the inequality named after him in 1939. It is an upper estimate for the integral norm of a linear combination of trigonometric Dirichlet kernels expressed in terms of the coefficients. Since the estimate has many applications for instance in L^1 convergence problems and summation methods with respect to trigonometric series, newer and newer improvements of the original inequality has been proved by several authors. Most of them are invariant with respect to the rearrangement of the coefficients. Although the newest results are close to best possible, no nontrivial lower estimate has been given so far. The aim of this paper is to give the best rearrangement invariant function of coefficients that can be used in a Sidon type inequality. We also show that it is equivalent to an Orlicz type and a Hardy type norm. Examples of applications are also given.

1. Introduction. Let $L^1[-\pi,\pi]$ denote the set of 2π-periodic Lebesgue-integrable functions with norm denoted by $\|\ \|_{L^1[-\pi,\pi]}$. Furthermore, let the real Hardy space $H[-\pi,\pi]$ be defined as the Banach space of functions $f \in L^1[-\pi,\pi]$ the trigonometric conjugate $\widetilde{f}$ of which is integrable, and

$$\|f\|_{H[-\pi,\pi]} = \|f\|_{L^1[-\pi,\pi]} + \|\widetilde{f}\|_{L^1[-\pi,\pi]}.$$

We will also need the Banach spaces $L^p[0,1]$ $(1 \le p \le \infty)$ with the usual norm denoted by $\|\ \|_p$, and the dyadic Hardy space $H[0,1]$. For any $f \in L^1[0,1]$ let the dyadic maximal function f^* be defined as follows:

$$f^*(x) = \sup\left\{\frac{1}{\mu(I)}\left|\int_I f(t)\,dt\right| : I \in \mathcal{I}, I \ni x\right\} \quad (x \in [0,1)),$$

where $\mathcal{I}$ is the set of dyadic intervals, i.e.

$$\mathcal{I} = \{[k2^{-n}, (k+1)2^{-n}) : k, n \in \mathbb{N},\ 0 \le k < 2^n\},$$

and $\mu(I)$ denotes the length of I. ($\mathbb{N}$ stands for the set of natural numbers.) Then, $H[0,1]$ is the set of integrable functions f for which f^* is integrable,

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