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On polynomials in primes and J. Bourgain's circle method approach to ergodic theorems II

by

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Abstract. We show that if q is greater than one, T is a measure preserving transformation of the measure space (X,β,μ) and f is in $L^q(X,\beta,\mu)$ then if ϕ is a non-constant polynomial mapping the natural numbers to themselves, the averages

$$\pi_N^{-1}\sum_{1\le p\le N} f(T^{\phi(p)}x)\qquad (N=1,2,\ldots)$$

converge μ almost everywhere. Here p runs over the primes and π_N denotes their number in $[1,N]$.

1. Introduction. The purpose of this paper is to prove the following theorem:

Theorem 1. *Suppose that ϑ is a non-constant polynomial mapping the natural numbers to themselves, that (X,β,μ,T) is a measure preserving dynamical system and that p is a real number greater than one. Then for each function f in $L^p(X,\beta,\mu)$ the averages*

$$A_N f(x)=\frac{1}{\pi_N}\sum_{1\le q\le N} f(T^{\vartheta(q)}x)\qquad (N=1,2,\ldots)\tag{1}$$

converge almost everywhere in x with respect to the measure μ. Here q runs over the rational primes and π_N denotes their number in $[1,N]$.

The most important auxiliary result needed for the proof of Theorem 1 is the following maximal inequality, which may be of interest in its own right. Here and henceforth the letter C possibly with subscripts always refers to a positive constant not necessarily the same at each occurrence.

Theorem 2. *Suppose that the real number p is greater than 1, that the function f is in $L^p(X,\beta,\mu)$ and that the averages $A_N f$ $(N=1,2,\ldots)$ are*

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as in Theorem 1. *Then letting*

$$(2) \qquad Mf = \sup_{N\geq 1} |A_N f|$$

and letting $\| \ \|_p$ *denote the* $L^p(X,\beta,\mu)$ *norm we have*

$$\|Mf\|_p \leq C\|f\|_p.$$

The analogue of Theorem 1, where q runs along the natural numbers instead of the primes, at least in the case $\vartheta(x) = x^d$ with d in $\mathbb{N}$ was proved by Bourgain [3]. His proof uses a Fourier analysis approach, which has at its centre an application of the Hardy–Littlewood circle method from analytic number theory. The Hardy–Littlewood method depends in an essential way on estimates for exponential sums of the form

$$(3) \qquad \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\vartheta(a_n)\alpha}.$$

In the case of Bourgain's results $a_n = n$ $(n = 1,2,\ldots)$. In this paper a_n denotes the nth rational prime. Theorem 1 in the special case $p = 2$ has already been proved by the author [5] as has the case $\vartheta(x) = x$ with $p > 1$ by Wierdl [8]. In all these results L^2 estimates play a central role and extension to L^p where $p > 1$ is achieved by the methods of interpolation theory. Complementing these positive results is the fact that the corresponding ergodic theorems fail for all $p \geq 1$ including $p = \infty$, when a_n is lacunary (i.e. there exists $c > 1$ such that $\liminf_{n\to\infty} a_{n+1}/a_n > c$). This result in the case $p < \infty$ is due to Bellow and Losert [1]. The case $p = \infty$ is due to J. Rosenblatt and is not as yet published.

The plan of the paper is as follows. In Section 2 we show how to use Theorem 2 to deduce Theorem 1. This necessitates the proof of another maximal estimate. We postpone this proof till Section 12. In Section 3, following A. P. Calderón we show how the proof of Theorem 2 can be reduced to the very special case where T is translation on the integers and f is a finitely supported function also on the integers. This reduction is used again in this paper in Sections 10 and 12 where new maximal functions are estimated. In the simplified context of translation on the integers, using the Fourier inversion theorem, estimating these maximal functions is equivalent to estimating maximal functions of certain Fourier multipliers. These Fourier multipliers turn out to be exponential sums of the form (3).

To make further progress we need certain standard facts from analytic number theory relating to these exponential sums. These facts are collected in Section 4. The basic method from here on is to approximate these complicated Fourier multipliers sufficiently closely by other more easily managed if less elegant multipliers. The most important for these new multipliers, by analogy with the corresponding situation of Warings' problem, we call the singular series. This is introduced in Section 5 and its basic relation to the exponential sums (3) established.

In estimating the maximal function (2), it turns out to be effective to break it up into parts each of whose maximal functions are then approximated by multipliers best adapted to the individual characteristics of these parts. This process is carried out in Sections 6–10. Because of their intricacy we forgo a detailed description of these maximal functions and the appropriate multipliers here. Finally, in Section 11 this information is put together and the proof of Theorem 2 completed. The basic method just described is the same as that developed in [3].

Added in proof. As a consequence of Theorem 4.1 of R. L. Jones and J. Olsen, *Subsequence ergodic theorems for operators in* L^p, Israel J. Math. 77 (1992), 33–54, Theorem 1 immediately extends to the case where T is a Dunford–Schwartz operator.

2. Pointwise convergence. In the context of Birkhoff's ergodic theorem the T-invariance of the L^1 limit of the sequence $A_N f$, together with Hopf's maximal ergodic theorem, readily imply almost everywhere convergence. In the context of Theorem 1, however, this T-invariance is not available. This difficulty is overcome using the following result.

THEOREM 3. *Let* $(N_j)_{j=1}^{\infty}$ *be any sequence of positive integers such that* $2N_j < N_{j+1}$ *and for each* $\varepsilon > 0$, *if* $[x]$ *denotes the integer part of* x, *let* $Z_\varepsilon = \{[(1+\varepsilon)^n] : n = 1,2,\ldots\}$. *Then for every* f *with* L^2 *and* L^∞ *norms bounded by one*, *if we let* $Z_{j,\varepsilon} = [N_j, N_{j+1}) \cap Z_\varepsilon$ *and*

$$M_{j,\varepsilon} f = \max_{N\in Z_{j,\varepsilon}} |A_N f - A_{N_j} f|,$$

we have

$$\sum_{1\leq j\leq J} \|M_{j,\varepsilon} f\|_2 = o(J)\|f\|_2.$$

Henceforth, for convenience, we suppress reference to ε and just write M_j for $M_{j,\varepsilon}$ $(j = 1,2,\ldots)$. Theorem 2 reduces proving Theorem 1 to verifying it on a dense subset, and rescaling if necessary, we can assume f has L^2 and L^∞ norms bounded by one. Suppose the averages $(A_N f)_{N=1}^{\infty}$ fail to converge almost everywhere. Then there exists $\varepsilon_0 > 0$ and an increasing sequence $(N_j)_{j=1}^{\infty}$ of positive integers with $N_{j+1} > 2N_j$ such that

$$\mu[\max_{N_j<N<N_{j+1}} |A_N f - A_{N_j} f| > \varepsilon_0] > \varepsilon_0.$$

Suppose $M = [(1+\varepsilon)^n] \leq N \leq [(1+\varepsilon)^{n+1}]$. Then because $\|f\|_\infty \leq 1$ we have

$$|A_N f - A_M f| \leq 4\varepsilon,$$

except possibly on a set of measure zero. Thus, also off this exceptional set,

$$\max_{N_j<N<N_{j+1}} |A_N f - A_{N_j} f| \le \max_{N\in Z_{j,\varepsilon}} |A_N f - A_{N_j} f| + 4\varepsilon_0$$

and so if we set $\varepsilon = \varepsilon_0/20$ we have

$$\mu_0 = \inf_{j\ge 0} \mu[\max_{N\in Z_{j,\varepsilon}} |A_N f - A_{N_j} f| > \varepsilon] > \varepsilon.$$

This, however, means that $\mu_0^{1/2}\varepsilon \le \|M_j f\|_2$ uniformly in j. This contradicts Theorem 3 and hence proves Theorem 2.

3. Reduction to $l^p(\mathbb{Z})$. For $g:\mathbb{Z}\to\mathbb{R}$, let $g_n:\mathbb{Z}\to\mathbb{R}$ be defined by $g_n(j) = g(j+n)$. The proof of Theorem 2 can be reduced to proving the following lemma. Here and henceforth in this paper $\|g\|_p$ $(1\le p\le\infty)$ denotes the $l^p(\mathbb{Z})$ norm.

LEMMA 4. *For $g:\mathbb{Z}\to\mathbb{R}$, bounded and finitely supported, and each positive integer N_0,*

$$\Big\|\max_{1\le N\le N_0}\Big|\frac{1}{\pi_N}\sum_{1\le q\le N} g_{\vartheta(q)}\Big|\Big\|_p \le C\|g\|_p$$

where the constant C is independent of N_0.

To see how Theorem 2 follows from Lemma 4 we argue as follows. Fix a particular x in X and for J large compared to $\vartheta(N_0)$, let $g(j) = f(T^j x)$ if j is in $\mathbb{Z}\cap[1,J)$ and $g(j)=0$ elsewhere on $\mathbb{Z}$. Then by Lemma 4,

$$\sum_{1\le j\le J-\vartheta(N_0)}\Big|\max_{1\le N\le N_0}\Big|\frac{1}{\pi_N}\sum_{1\le q\le N} f(T^{\vartheta(q)+j}x)\Big|\Big|^p \le C\sum_{1\le j\le J}|f(T^j x)|^p.$$

Now integrating and using measure preservation we have

$$\frac{J-\vartheta(N_0)}{J}\Big\|\max_{1\le N\le N_0}\Big|\frac{1}{\pi_N}\sum_{1\le q\le N} f(T^{\vartheta(p)}x)\Big|\Big\|_p \le C\|f\|_p.$$

Letting $J\to\infty$ and then letting $N_0\to\infty$ proves Theorem 2. The argument we have just given is a special case of the transference principle of Calderón [4]. Theorem 3 can similarly be reduced to the following lemma.

LEMMA 5. *For $g:\mathbb{Z}\to\mathbb{R}$, bounded and finitely supported,*

$$\sum_{1\le j\le J}\Big\|\max_{N\in Z_{j,\varepsilon}}\Big|\frac{1}{\pi_N}\sum_{1\le q\le N} g_{\vartheta(q)} - \frac{1}{\pi_{N_j}}\sum_{1\le q\le N_j} g_{\vartheta(q)}\Big|\Big\|_2 = o(J)\|g\|_2.$$

4. Analytic number theory. For g in $l^p(\mathbb{Z})$, if $*$ denotes convolution and δ_a denotes the delta function supported at a, note that

$$\frac{1}{\pi_N}\sum_{1\le q\le N} g_{\vartheta(q)} = g * K_N,$$

where

$$K_N = \frac{1}{\pi_N}\sum_{1\le q\le N}\delta_{\vartheta(q)}.$$

Hence if $\mathbb{F}(g)$ denotes the Fourier transform of g,

$$(f*K_N)(n) = \int_0^1 \mathbb{F}(f)(\alpha)\mathbb{F}(K_N)(\alpha)e^{2\pi i\alpha n}\,d\alpha,$$

where

$$\mathbb{F}(K_N)(\alpha) = \frac{1}{\pi_N}\sum_{1\le q\le N} e^{2\pi i\alpha\vartheta(q)}.$$

Use of the circle method requires that we collect information about the behaviour of $\mathbb{F}(K_N)(\alpha)$ for α near rationals $\xi = a/b$ in $[0,1)$ such that a and b are coprime and where b is "small". More precisely, let $|a_1-a_2|_1$ denote $\min(|a_1-a_2|, |1+a_1-a_2|)$ and for large N and an unspecified positive constant u, to be chosen when the proof of Lemma 11 is completed, suppose $1\le b\le(\log N)^u$. Set

$$M_N(\xi) = \{\alpha\in\mathbb{T} : |\alpha-\xi|_1 < (\log N)^u/N\}.$$

Classically these arcs on the unit circle will be referred to as the *major arcs* (of order N). Arcs in the complementary part of the circle are referred to as the *minor arcs.*

LEMMA 6. *Suppose ϕ denotes Euler totient function and if the polynomial*

$$\vartheta_0(x) = a_d x^d + \ldots + a_1 x$$

has integer coefficients such that

$$(a_d,\ldots,a_1,b) = 1,$$

set

$$S\Big(\frac{a}{b}\Big) = \frac{1}{\phi(b)}\sum_{\substack{r=0\\(r,b)=1}}^{b} e^{2\pi i a\vartheta_0(r)b^{-1}}. \tag{4}$$

Then there exists $\delta_1>0$ dependent only on d such that

$$|S(a/b)| \le Cb^{-\delta_1}.$$

LEMMA 7. *Suppose M_0 is an absolute positive constant and that $\vartheta(x)$ is a polynomial of degree d (say), with integer coefficients. Set*

$$(5)\qquad W_M(\beta)=\frac{1}{\pi_N}\int_{M_0}^{M}\frac{e^{2\pi i\beta\vartheta(x)}}{\log x}\,dx\,.$$

Now suppose that $\xi=a/b$ with $(a,b)=1$ and that α is in $M_N(\xi)$. Then if we let

$$\beta=\alpha-\xi\,,$$

we have

$$\mathbb{F}(K_N)(\alpha)=S\Big(\frac{a}{b}\Big)W_N(\beta)+O((\log N)^{-\delta_2})$$

for any $\delta_2>0$, where $S(a/b)$ is as in Lemma 6.

In the special case $\vartheta(x)=x^d$ Lemma 7 appears in [7, p. 292]. The proof of this more general case is virtually identical and omitted.

LEMMA 8. *For $W_M(\beta)$ as in Lemma 7,*

$$|1-W_M(\beta)|\le C|\beta|M^d\,.$$

Proof.

$$\Big|\int_{M_0}^{M}\frac{dx}{\log x}-\int_{M_0}^{M}\frac{e^{2\pi i\beta\vartheta_0(x)}}{\log x}\,dx\Big|\le C|\beta|M^d\int_{M_0}^{M}\frac{dx}{\log x}\,,$$

and by the prime number theorem,

$$\pi_M\sim\int_{M_0}^{M}\frac{dx}{\log x}\,,$$

as required. ∎

LEMMA 9. *Set $\beta=\delta_3N^{-d}$ and set*

$$D=\begin{cases}\min(1,|\delta_3|^{-1/d}) & \text{if } 0\le\|\delta_3\|\le N^{d-1}\,,\\ (\log N)|\delta_3|^{-1/d} & \text{if } N^{d-1}<\|\delta_3\|\,.\end{cases}$$

Then we have

$$\Big|\int_{M_0}^{M}\frac{e^{2\pi i\beta\vartheta(x)}}{\log x}\,dx\Big|\le C\frac{M}{\log M}D\,.$$

Again, in the special case $\vartheta(x)=x^d$, this lemma is proved in [7, p. 202]. The more general case needed here is proved in the same way.

We now state a lemma which will help us to treat α near rationals a/b with b large, that is, on the minor arcs [7, p. 285].

LEMMA 10. *For any α not in any major arc $M_N(\xi)$ and any $\delta_4>0$ (which may be chosen arbitrarily large) we have*

$$\Big|\frac{1}{\pi_N}\sum_{1\le q\le N}e^{2\pi i\vartheta(q)\alpha}\Big|\le C(\log N)^{-\delta_4}\,,$$

where C may depend on δ_4.

5. The singular series. In this section we approximate $\mathbb{F}(K_N)$ by $\mathbb{F}(L_N)$ where the definition of $\mathbb{F}(L_N)$ reflects the shape of $\mathbb{F}(K_N)$ on the major arcs. The definition of L_N is motivated by that of the singular series which appears in the analysis of Waring's problem or the Goldbach problems. To be more precise, we let $R_0=\{0\equiv1\}$ and for each natural number s let

$$R_s=\{\xi\in\mathbb{Q}\cap[0,1):\xi=a/b\ \text{ with } (a,b)=1 \text{ and } 2^s\le b<2^{s+1}\}\,.$$

For each natural number N now set

$$(6)\qquad \psi_{s,N}(\alpha)=\sum_{a/b\in R_s}S\Big(\frac{a}{b}\Big)W_N\Big(\alpha-\frac{a}{b}\Big)\zeta\Big(10^s\Big(\alpha-\frac{a}{b}\Big)\Big)$$

where ζ is any smooth bump function on $\mathbb{R}$, supported in $[-1/5,1/5]$ and equal to the identity on $[-1/10,1/10]$. As before, the functions $W_N(\alpha-a/b)$ and $S(a/b)$ are defined by (4) and (5) respectively. We now define

$$(7)\qquad \mathbb{F}(L_N)(\alpha)=\sum_{s=0}^{\infty}\psi_{s,N}(\alpha)\,.$$

Note that because the terms on the right of (7) are smooth the function L_N is defined pointwise via the inverse Fourier transform. The purpose of this section is to prove the following lemma.

LEMMA 11. *Given any $\delta_5>0$ there exists a constant $C>0$ dependent on δ_5 but independent of α such that*

$$|\mathbb{F}(K_N)(\alpha)-\mathbb{F}(L_N)(\alpha)|\le C(\log N)^{-\delta_5}\,.$$

Proof. The proof falls into two cases. Case 1 is where α belongs to a major arc of order N and case 2 where it does not.

Case 1: Assume α is in $M_N(\xi_0)$ for some ξ_0. Suppose ξ_0 is in R_{s_0}. This means that $2^{s_0}<(\log N)^u$. Let s_1 be a positive integer depending on N to be specified later. Using Lemmas 6 and 7 we see that

$$|\mathbb{F}(K_N)(\alpha)-\mathbb{F}(L_N)(\alpha)|\le|1-\zeta(10^{s_0}(\alpha-\xi_0))|$$
$$+\sum_{0\le s\le s_1}\sup_{a/b\in R_{s,0}}\Big|W_N\Big(\alpha-\frac{a}{b}\Big)\Big|+C_12^{-s_1\delta_1}+C_2(\log N)^{\delta_2}$$

where $\delta_2 > 0$ can be chosen arbitrarily large. Here $R_{s,0}$ denotes $R_s\setminus\{\xi_0\}$ if ξ_0 is in R_s and $R_{s,0}$ denotes R_s otherwise. Choose s_1 such that $2^{s_1}\approx(\log N)^{\delta_2}$.

Now $10^{s_0}\le(\log N)^{4u}$ and $|\alpha-\xi_0|<(\log N)^u/N^d<N^{\varepsilon-1}$ for any $\varepsilon>0$. Therefore $|1-\zeta(10^{s_0}(\alpha-\xi_0))|=0$. We need to show that

$$\sum_{0\le s\le s_1}\sup_{a/b\in R_{s,0}}\left|W_N\left(\alpha-\frac{a}{b}\right)\right|=O((\log N)^{-\delta_2}).$$

Note that

$$|\alpha-\xi|\ge|\xi-\xi_0|-|\alpha-\xi_0|.$$

If in reduced form, $\xi=a/b$ and $\xi_0=a_0/b_0$, then $|\xi-\xi_0|\ge1/(bb_0)$. Also $b_0^{-1}\ge(\log N)^{-u}$ and because $b\le2^{s_1+1}\approx(\log N)^{\delta_2}$ we have $b^{-1}\ge(\log N)^{-\delta_2}$, and thus

$$|\xi-\xi_0|\ge\frac{1}{bb_0}\ge(\log N)^{-u-\delta_2}.$$

Now, for any $\varepsilon>0$, $|\alpha-\xi_0|<N^{\varepsilon-1}$, therefore

$$|\alpha-\xi|\ge\tfrac12(\log N)^{-u-\delta_2}.$$

If we set $\delta_6=(u+\delta_2)/d$ then by Lemma 9,

$$|W_N(\alpha-\xi)|\le\frac{(\log N)^{\delta_6}}{N}.$$

This means that

$$\sum_{0\le s\le s_1}\sup_{\xi\in R_s}|W_N(\alpha-\xi)|\le C\frac{(\log N)^{\delta_7}}{N},$$

for appropriate $\delta_7>0$. We have therefore proved Lemma 11 in case 1.

Case 2: Suppose now that α is not in $M_N(\xi)$ for any ξ. By Lemma 10,

$$|\mathbb{F}(K_N)(\alpha)|\le C(\log N)^{-\delta_4}$$

and so we need only show that

$$|\mathbb{F}(L_N)(\alpha)|\le C(\log N)^{-\delta_4}.$$

Now using Lemma 6 we have for any positive integer s_2,

$$|\mathbb{F}(L_N)(\alpha)|\le\sum_{0\le s\le s_2}\sup_{a/b\in R_s}\left|W_N\left(\alpha-\frac{a}{b}\right)\right|+C2^{-s_2\delta_4}.$$

Now set $2^{s_2}\approx(\log N)^{\omega}$. Choosing ω large enough this reduces the proof of Lemma 11 in case 2 to showing that

$$\sum_{0\le s\le s_2}\sup_{a/b\in R_s}\left|W_N\left(\alpha-\frac{a}{b}\right)\right|\le C(\log N)^{-\delta_4}.$$

Because of the way we have defined $M_N(\xi)$, whenever ξ is in R_s and $0\le s\le s_2$ we have $|\alpha-\xi|\ge N^{-d}(\log N)^u$, so $|W_N(\alpha-\xi)|\le C(\log N)^{-u/d}$. Thus

$$\sum_{0\le s\le s_2}\sup_{\xi\in R_s}|W_N(\alpha-\xi)|\le C(\log N)^{2\omega}(\log N)^{-u/d},$$

which for u large enough compared to ω is $\le C(\log N)^{-\delta_4}$, completing the proof of case 2 and hence Lemma 11. ∎

6. Auxiliary L^2-maximal estimate I. In this section we prove the following lemma.

LEMMA 12. *Suppose D_2 denotes the natural numbers which are powers of two and that for each pair of natural numbers s and N the function $\psi_{s,N}$ is defined by* (6). *Then there exists $\delta_8>0$ such that*

$$\|\sup_{N\in D_2}|\mathbb{F}^{-1}[\psi_{s,N}\mathbb{F}(f)]|\,\|_2\le C2^{-s\delta_8}\|f\|_2.$$

Here $\mathbb{F}^{-1}$ denotes the inverse Fourier transform.

To prove this lemma we need the following maximal estimate due to Bourgain [3] that is central to this whole paper.

LEMMA 13. *Assume $0\le\lambda_1<\ldots<\lambda_K\le1$ and for each j in $\mathbb{N}$ define neighbourhoods*

$$T_j=\{\lambda\in\mathbb{T}:\min_{1\le k\le K}|\lambda-\lambda_k|_1\le2^{-j}\}.$$

Then for each function $f:\mathbb{Z}\to\mathbb{R}$ we have

$$\Big\|\sup_{j\ge1}\Big|\int_{T_j}\mathbb{F}(f)(\lambda)e^{2\pi i\lambda}\,d\lambda\Big|\Big\|_2\le C(\log K)^2\|f\|_2.$$

Before we return to the proof Lemma 12 we need the following subsidiary lemma.

LEMMA 14. *Suppose $\psi_{s,N}$ is defined by* (6), *let $\chi=\chi_{[-1,1]}$ denote the characteristic function of the interval $[-1,1]$ viewed as a function on $\mathbb{R}$, and if the functions $S(a/b)$ and ζ are as in the definition of $\psi_{s,N}$, let*

$$\psi_{s,N}^{(1)}(\alpha)=\sum_{a/b\in R_s}S\left(\frac{a}{b}\right)\chi\left(N^d\left(\alpha-\frac{a}{b}\right)\right)\zeta\left(10^s\left(\alpha-\frac{a}{b}\right)\right).$$

Then

$$\sum_{N\in D_2}|\psi_{s,N}-\psi_{s,N}^{(1)}|^2\le C2^{-2s\delta_1}.$$

Proof. We argue as follows:

$$\sum_{N\in D_2}|\psi_{s,N}-\psi_{s,N}^{(1)}|^2\le\sum_{k\ge1}\sum_{a/b\in R_s}\left|S\left(\frac{a}{b}\right)\right|^2\left|W_{2^k}\left(\alpha-\frac{a}{b}\right)-\chi\left(2^{kd}\left(\alpha-\frac{a}{b}\right)\right)\right|^2\left|\zeta\left(10^s\left(\alpha-\frac{a}{b}\right)\right)\right|^2$$

$$\le C2^{-2s\delta_1}\left(\sum_{2^k\le1/|\alpha-a/b|^{1/d}}\left|W_{2^k}\left(\alpha-\frac{a}{b}\right)-1\right|^2+\sum_{2^k>1/|\alpha-a/b|^{1/d}}\left|W_{2^k}\left(\alpha-\frac{a}{b}\right)\right|^2\right).$$

Here a/b is the unique $a/b\in R_s$ picked out by α and ζ.

Using Lemmas 8 and 9 this is

$$C2^{-2s\delta_1}\left(\sum_{2^k\le1/|\alpha-a/b|^{1/d}}\left|\alpha-\frac{a}{b}\right|^2 2^{2kd}+\sum_{2^k>1/|\alpha-a/b|^{1/d}}(\log2^k)^2 2^{-2k}\left|\alpha-\frac{a}{q}\right|^{-2/d}\right),$$

which is $\le C2^{-2s\delta_1}$, as required. ∎

We are now ready to complete the proof of Lemma 12. Note that

$$\|\sup_{N\in D_2}|\mathbb{F}^{-1}[\psi_{s,N}\mathbb{F}(f)]|\,\|_2\le\|\sup_{N\in D_2}|\mathbb{F}^{-1}[\psi_{s,N}^{(1)}\mathbb{F}(f)]|\,\|_2+\Big(\sum_{n\in D_2}\|\psi_{s,N}-\psi_{s,N}^{(1)}\|_\infty^2\Big)^{1/2}\|f\|_2,$$

which using Lemma 14 is

$$\le\|\sup_{N\in D_2}|\mathbb{F}^{-1}[\psi_{s,N}^{(1)}\mathbb{F}(f)]|\,\|_2+C2^{-s\delta_4}\|f\|_2.$$

This means that in proving Lemma 12 we may replace $\psi_{s,N}$ by $\psi_{s,N}^{(1)}$. For N in D_2 let $2^j=N^d$ and set T_j to be the 2^{-j} neighbourhood of R_s in $\mathbb{T}$. Thus if we set

$$\mathbb{F}(g_s)=\mathbb{F}(f)\sum_{a/b\in R_s}S\left(\frac{a}{b}\right)\zeta\left(10^s\left(\alpha-\frac{a}{b}\right)\right),$$

we have

$$\psi_{s,N}^{(1)}\mathbb{F}(f)=\mathbb{F}(g_s)\chi_{T_j}.$$

This means as a consequence of Lemma 13 that

$$\|\sup_{N\in D_2}|\mathbb{F}^{-1}[\psi_{s,N}^{(1)}\mathbb{F}(f)]|\,\|_2\le\|\sup_{j\ge1}|\mathbb{F}^{-1}[\mathbb{F}(g_s)\chi_{T_j}]|\,\|_2 \le C(\log|R_s|)^2\|g_s\|_2. \tag{8}$$

By definition $|R_s|\le4^s$. In addition, using Parseval's theorem,

$$\|g_s\|_2\le\Big\|\sum_{a/b\in R_s}S\left(\frac{a}{b}\right)\zeta\left(10^s\left(\alpha-\frac{a}{b}\right)\right)\Big\|_\infty\|f\|_2,$$

which using the fact that $|S(a/b)|<C2^{-s\delta_1}$ and that $\zeta(10^s(\alpha-a/b))$ has support near a/b tells us (8) is bounded by $C2^{-s\delta_1}s^2\|f\|_2$ as required, proving Lemma 12. ∎

7. Auxiliary L^2-maximal estimate II.

Our task in this section is to prove the following maximal inequality.

Lemma 15. *Let $Q_s=(2^s)!$ $(s=1,2,\ldots)$. Then there exists $\delta_9>0$ such that*

$$\Big\|\sup_{k\ge4^s}\Big|\mathbb{F}^{-1}\Big[\sum_{\substack{b\mid Q_s\\ b\ge2^{s+1}}}\sum_{\substack{1\le a<b\\(a,b)=1}}S\left(\frac{a}{b}\right)W_{2^k}\left(\alpha-\frac{a}{b}\right)\zeta\left(Q_s^2\left(\alpha-\frac{a}{b}\right)\right)\mathbb{F}(f)\Big]\Big|\Big\|_2\le C2^{-s\delta_9}\|f\|_2.$$

The proof of Lemma 15 will require the establishing of a number of lemmas which we prove first.

Lemma 16. *As earlier, let χ denote the characteristic function of $[-1,1]$ on $\mathbb{R}$. Set*

$$P_{s,N}^{(1)}(\alpha)=\sum_{\substack{b\mid Q_s\\ b\ge2^{s+1}}}\sum_{\substack{1\le a<b\\(a,b)=1}}S\left(\frac{a}{b}\right)W_N\left(\alpha-\frac{a}{b}\right)\zeta\left(Q_s^2\left(\alpha-\frac{a}{b}\right)\right)$$

and set

$$P_{s,N}^{(2)}(\alpha)=\sum_{\substack{b\mid Q_s\\ b\ge2^{s+1}}}\sum_{\substack{1\le a<b\\(a,b)=1}}S\left(\frac{a}{b}\right)\chi\left(\alpha-\frac{a}{b}\right)\zeta\left(Q_s^2\left(\alpha-\frac{a}{b}\right)\right).$$

Then we have

$$T=\sum_{N\in D_2}|P_{s,N}^{(1)}(\alpha)-P_{s,N}^{(2)}(\alpha)|^2<C2^{-2s\delta_1}, \tag{9}$$

uniformly in α.

Proof. Writing out the left hand side of (9) more fully we have

$$(10)\qquad T=\sum_{N\in D_2}\Big|\sum_{\substack{b|Q_s\\ b\ge 2^{s+1}}}\sum_{\substack{1\le a<b\\ (a,b)=1}} S\Big(\frac{a}{b}\Big)\times\Big[W_N\Big(\alpha-\frac{a}{b}\Big)-\chi\Big(N^d\Big(\alpha-\frac{a}{b}\Big)\Big)\Big]\zeta\Big(Q_s^2\Big(\alpha-\frac{a}{b}\Big)\Big)\Big|^2.$$

For a fixed α suppose we are given distinct rationals a_1/b_1 and a_2/b_2 (in reduced form) such that $b_i\,|\,Q_s$, $b_i\ge 2^{s+1}$ and $1\le a_i<b_i$ for both $i=1$ and 2. To estimate (10), because the support of ζ is contained in $[-1/5,1/5]$, we need only consider a/b such that

$$\Big|\alpha-\frac{a}{b}\Big|\le\frac{1}{5Q_s^2}$$

in each summand. In addition, because b_1 and b_2 divide Q_s we know $(b_1b_2)^{-1}\ge Q_s^{-2}$. Now if

$$\Big|\alpha-\frac{a_1}{b_1}\Big|\le\frac{1}{5Q_s^2},$$

because

$$\Big|\alpha-\frac{a_2}{b_2}\Big|\ge\frac{1}{b_1b_2}-\Big|\alpha-\frac{a_1}{b_1}\Big|,$$

we have

$$\Big|\alpha-\frac{a_2}{b_2}\Big|\ge\frac{4}{5}\frac{1}{Q_s^2}.$$

In consequence, for each α the interval

$$I_\alpha=\Big[\alpha-\frac{1}{5Q_s^2},\alpha+\frac{1}{5Q_s^2}\Big]$$

contains at most one rational with the property assumed for the b_1 and b_2 above. We shall assume this rational does exist because if not we can ignore the corresponding term in (10). This means that

$$\sum_{\substack{b|Q_s\\ b\ge 2^{s+1}}}\sum_{\substack{1\le a<b\\ (a,b)=1}}\Big|S\Big(\frac{a}{b}\Big)\Big[W_N\Big(\alpha-\frac{a}{b}\Big)-\chi\Big(N^d\Big(\alpha-\frac{a}{b}\Big)\Big)\Big]\zeta\Big(Q_s^2\Big(\alpha-\frac{a}{b}\Big)\Big)\Big|^2 \le C2^{-2s\delta_1}\Big|W_N\Big(\alpha-\frac{a}{b}\Big)-\chi\Big(N^d\Big(\alpha-\frac{a}{b}\Big)\Big)\Big|^2,$$

where a/b is the lone possible rational in I_α. Thus we have

$$T\le C2^{-2s\delta_1}\sum_{N\in D_2}\Big|W_N\Big(\alpha-\frac{a}{b}\Big)-\chi\Big(N^d\Big(\alpha-\frac{a}{b}\Big)\Big)\Big|^2.$$

Using Lemmas 8 and 9 this is

$$\le C2^{-2s\delta_1}\Big(\sum_{2^k\le|\alpha-a/b|^{1/d}}\Big|\alpha-\frac{a}{b}\Big|^2 2^{2kd}+\sum_{2^k>|\alpha-a/b|^{1/d}}\log^2(2^k)\Big|\alpha-\frac{a}{b}\Big|^{-2/d}\Big),$$

which is $\le C2^{-2s\delta_1}$, as required. ∎

LEMMA 17. *Let $p_1,\ldots,p_r$ denote a set of coprime integers greater than one. Then if $\mathbb{Z}_+$ denotes the positive integers,*

$$(11)\qquad \#\{(m_1,\ldots,m_r)\in\mathbb{Z}_+^r : p_1^{m_1}\ldots p_r^{m_r}\le x\}\le\frac{1}{r!}\prod_{j=1}^{r}\Big(\frac{\log x}{\log p_j}+r^{1/2}\Big)^r.$$

Proof. Taking logarithms the left hand side of (11) is the number of $(m_1,\ldots,m_r)$ in $\mathbb{Z}_+^r$ such that

$$0\le\sum_{j=1}^{r}m_j\log p_j\le\log N.$$

With any $(m_1,\ldots,m_r)$ with this property associate the unit cube $(m_1,\ldots,m_r)+(\beta_1,\ldots,\beta_r)$ where $0\le\beta_j\le1$ $(j=1,\ldots,r)$. The union of these cubes is contained in the simplex which is the convex hull of the points $(0,\ldots,0)$ and $\big(\frac{\log N}{\log p_j}+r^{1/2}\big)e_j$. Here e_j $(j=1,\ldots,r)$ denotes the jth unit vector. The volume of this simplex is the right hand side of (11) and so Lemma 17 is proved. ∎

Note that the above lemma remains true irrespective of the base γ of the logarithm. In the proof of the next lemma it will be convenient to take the value of γ to be large.

LEMMA 18. *If ϕ denotes Euler's totient function there exists $\delta_9>0$ such that*

$$\sum_{\substack{b|Q_s\\ b\ge 2^{s+1}}}\frac{(\log\phi(b))^2}{b^{\delta_1}}\le\frac{C}{2^{s\delta_9}}.$$

Proof. Using the fact that $\phi(q)<q$ we have

$$\sum_{\substack{b|Q_s\\ b\ge 2^{s+1}}}\frac{(\log\phi(b))^2}{b^{\delta_1}}\le\sum_{\substack{b|Q_s\\ b\ge 2^{s+1}}}\frac{1}{b^{\delta_9}}=\sum_{2^{s+1}\le b\le Q_s}\frac{1}{b^{\delta_9}}\chi(b,Q_s)$$

where

$$\chi(b,Q_s)=\begin{cases}1 & \text{if } b \text{ divides } Q_s,\\ 0 & \text{otherwise}.\end{cases}$$

Using summation by parts we have

$$\sum_{2^{s+1}\le b\le Q_s}\frac{1}{b^{\delta_9}}\chi(b,Q_s)=\frac{1}{Q_s^{\delta_9}}\Big(\sum_{2^{s+1}\le b\le Q_s}\chi(b,Q_s)\Big)+\sum_{2^{s+1}\le n\le b}\left(\frac{1}{n^{\delta_9}}-\frac{1}{(n+1)^{\delta_9}}\right)\Big(\sum_{1\le k\le n}\chi(k,Q_s)\Big).$$

We now see that Lemma 18 is proved if we show that given $\varepsilon>0$, there exists $C_\varepsilon>0$ independent of s such that

$$\sum_{1\le k\le b}\chi(b,Q_s)\le C_\varepsilon b^\varepsilon. \tag{12}$$

Now all the prime factors p_i of Q_s satisfy $1\le p_i\le 2^s$. This means that if we set $r=\pi_{2^s}$, then

$$\sum_{1\le k\le b}\chi(b,Q_s)\le\#\{(m_1,\dots,m_r)\in\mathbb{Z}_+^r:p_1^{m_1}\dots p_r^{m_r}\le b\}.$$

By Lemma 17 this is

$$\le\frac{1}{r!}\prod_{j=1}^{r}\left(\frac{\log_\gamma x}{\log_\gamma p_j}+r^{1/2}\right)^r.$$

Because of the explicit dependence on $r=\pi_{2^s}$, it is perhaps not immediately clear that the bound (12) is attained. Nonetheless setting $\gamma>1/\varepsilon$ and using Stirling's formula together with a little elementary calculus establishes (12), as the reader will verify. ■

We now complete the proof of Lemma 15. First note that

$$\|\sup_{k\ge1}|\mathbb{F}^{-1}[P^{(1)}_{s,2^k}\mathbb{F}(f)]|\,\|_2\le\|\sup_{k\ge1}|\mathbb{F}^{-1}[P^{(2)}_{s,2^k}\mathbb{F}(f)]|\,\|_2+\Big(\sum_{k\ge1}\|P^{(1)}_{s,2^k}-P^{(2)}_{s,2^k}\|_\infty^2\Big)^{1/2}\|f\|_2,$$

which by Lemma 16 is

$$\le\|\sup_{k\ge1}|\mathbb{F}^{-1}[P^{(2)}_{s,2^k}\mathbb{F}(f)]|\,\|_2+C2^{-s\delta_1}\|f\|_2.$$

This means that in the proof of Lemma 15 we may assume $P^{(1)}_{s,N}$ is replaced by $P^{(2)}_{s,N}$. For N in D_2 let $N^d=2^j$ and let T_j be the 2^{-j} neighbourhood of

$$S_b=\{a/b:1\le a<b,\ (a,b)=1\}.$$

Note that $\#S_b=\phi(b)$. Suppose h_s is defined by

$$\mathbb{F}(h_s)=\mathbb{F}(f)\sum_{a/b\in S_b}S\Big(\frac{a}{b}\Big)\zeta\Big(Q_s^2\Big(\alpha-\frac{a}{b}\Big)\Big).$$

Also note that if

$$P_{b,s,N}(\alpha)=\sum_{\substack{1\le a<b\\(a,b)=1}}S\Big(\frac{a}{b}\Big)\zeta\Big(Q_s^2\Big(\alpha-\frac{a}{b}\Big)\Big)\chi\Big(N^d\Big(\alpha-\frac{a}{b}\Big)\Big)$$

then

$$P_{b,s,N}(\alpha)\mathbb{F}(f)=\chi_{T_j}\mathbb{F}(h_s).$$

Now

$$\|\sup_{k\ge1}|\mathbb{F}^{-1}[P^{(2)}_{s,2^k}\mathbb{F}(f)]|\,\|_2\le\sum_{\substack{b|Q_s^2\\b\ge2^{s+1}}}\|\sup_{k\ge1}|\mathbb{F}^{-1}[P_{b,s,2^k}\mathbb{F}(f)]|\,\|_2$$

and

$$\|\sup_{k\ge1}|\mathbb{F}^{-1}[P_{b,s,2^k}\mathbb{F}(f)]|\,\|_2\le\|\sup_{j\ge1}|\mathbb{F}^{-1}[\chi_{T_j}\mathbb{F}(h_s)]|\,\|_2.$$

Also by Lemma 13,

$$\|\sup_{j\ge0}|\mathbb{F}^{-1}[\chi_{T_j}\mathbb{F}(h_s)]|\,\|_2\le C(\log(\#S_b))^2\|h_s\|_2.$$

Finally, note that by Parseval's Theorem and Lemma 6 we have $\|h_s\|_2\le Cb^{-\delta_1}\|f\|_2$. This means that

$$\|\sup_{k\ge4^s}|\mathbb{F}^{-1}[P^{(1)}_{s,2^k}\mathbb{F}(f)]|\,\|_2\le C\Big(\sum_{\substack{b|Q_s\\b\ge2^{s+1}}}\frac{(\log\phi(b))^2}{b^{\delta_1}}\Big)\|f\|_2,$$

which by Lemma 18 is $\le C2^{-s\delta_9}\|f\|_2$, as required. ■

8. Auxiliary L^2-maximal estimate III. Let $B_s=\{k\in\mathbb{Z}:4^s\le k<4^{s+1}\}$ and set

$$\Gamma_{k,s}=\sum_{0\le a\le Q_s}S\Big(\frac{a}{Q_s}\Big)W_{2^k}\Big(\alpha-\frac{a}{Q_s}\Big)\zeta\Big(Q_s^2\Big(\alpha-\frac{a}{Q_s}\Big)\Big) \tag{13}$$

for each $s\le s'$ and each k in B_s. The purpose of this section is to prove the following lemma.

Lemma 19. *There exists $\delta_{10}>0$ such that*

$$\|\sup_{k\ge4^s}|f*K_{2^k}-\mathbb{F}^{-1}[\Gamma_{k,s}\mathbb{F}(f)]|\,\|_2\le C2^{-s\delta_{10}}\|f\|_2.$$

Our use of Lemma 19, will be in the form of two immediate corollaries.

Lemma 20. *For δ_{10} as in Lemma* 19,

$$\|\sup_{k\in B_s}|f*K_{2^k}-\mathbb{F}^{-1}[\Gamma_{k,s}\mathbb{F}(f)]|\,\|_2\le C2^{-s\delta_{10}}\|f\|_2.$$

LEMMA 21. *For δ_{10} as in Lemma* 19,

$$\Big\| \sup_{k\ge 4^s} |\mathbb{F}^{-1}[(\Gamma_{k,s}-\Gamma_{k,s-1})\mathbb{F}(f)]| \Big\|_2 \le C2^{-s\delta_{10}}\|f\|_2 .$$

To complete the proof of Lemma 19 we need another preparatory lemma.

LEMMA 22. *Let*

$$P^{(3)}_{s,N}(\alpha) = \sum_{0\le r\le s}\sum_{a/b\in R_r} S\Big(\frac{a}{b}\Big)W_N\Big(\alpha-\frac{a}{b}\Big)\Big[\zeta\Big(Q_s^2\Big(\alpha-\frac{a}{b}\Big)\Big)-\zeta\Big(10^r\Big(\alpha-\frac{a}{b}\Big)\Big)\Big].$$

Then

$$\Big\| \sup_{k\ge 4^s} |\mathbb{F}^{-1}[P^{(3)}_{s,2^k}(\alpha)\mathbb{F}(f)]| \Big\|_2 \le C2^{-s}\|f\|_2 . \tag{14}$$

Proof. Estimating the supremum by the sum, the left hand side of (14) is

$$\le \Big\| \sum_{k\ge 4^s} |\mathbb{F}^{-1}[P^{(3)}_{s,2^k}(\alpha)\mathbb{F}(f)]| \Big\|_2 ,$$

which from the definition of $P^{(3)}_{s,2^k}$ is

$$\le \sum_{k\ge 4^s}\sum_{0\le r\le s}\sum_{a/b\in R_r} \Big\|W_{2^k}\Big(\alpha-\frac{a}{b}\Big) \times\Big[\zeta\Big(Q_s^2\Big(\alpha-\frac{a}{b}\Big)\Big)-\zeta\Big(10^r\Big(\alpha-\frac{a}{b}\Big)\Big)\Big]\Big\|_\infty \|f\|_2 ,$$

and thus

$$\le \Big(\sum_{k\ge 4^s} 4^s \sup_{|\beta|>Q_s^{-2}} |W_{2^k}(\beta)|\Big)\|f\|_2 .$$

By Stirling's formula $Q_s^2 < C2^{s2^{s+1}}$ and so using Lemma 9, this finally is

$$\le \Big(\sum_{k\ge 4^s} 2^{-k/2}\Big)\|f\|_2 \le C2^{-s}\|f\|_2 ,$$

as required. ∎

We can now complete the proof of Lemma 19. First note that for L_N defined by (7),

$$\begin{aligned}\Big\| \sup_{k\ge 4^s} |f*K_{2^k}-\mathbb{F}^{-1}[\Gamma_{k,s}\mathbb{F}(f)]| \Big\|_2 &\le \Big\| \sup_{k\ge 4^s} |f*(K_{2^k}-L_{2^k})| \Big\|_2 \\ &\quad + \Big\| \sup_{k\ge 4^s} |f*L_{2^k}-\mathbb{F}^{-1}[\Gamma_{k,s}\mathbb{F}(f)]| \Big\|_2\end{aligned}$$

$$\begin{aligned}&\le \Big(\sum_{k\ge 4^s} \|K_{2^k}-L_{2^k}\|_\infty^2\Big)^{1/2}\|f\|_2 \\ &\quad + \Big\| \sup_{k\ge 4^s} \Big|\mathbb{F}^{-1}\Big[\Big(\sum_{0\le r\le s}\psi_{r,2^k}-\Gamma_{k,s}\Big)\mathbb{F}(f)\Big]\Big| \Big\|_2 \\ &\quad + \sum_{r>s} \Big\| \sup_{k\ge 4^s} |\mathbb{F}^{-1}[\psi_{r,2^k}\mathbb{F}(f)]| \Big\|_2 .\end{aligned} \tag{15}$$

Because by Lemma 11,

$$\sum_{k\ge 4^s} \|K_{2^k}-L_{2^k}\|_\infty^2 \le C4^{-s\delta_5} ,$$

the first term of (15) is bounded by $C2^{-s\delta_5}\|f\|_2$. In addition, note that

$$\Gamma_{k,s}-\sum_{0\le r\le s}\psi_{r,2^k} = P^{(1)}_{s,2^k}+P^{(3)}_{s,2^k} ,$$

so by Lemmas 15 and 22 the second term is bounded by $C2^{-s\delta_{11}}\|f\|_2$. Finally, by Lemma 12 the third term is $\le C\sum_{r>s}2^{-r\delta_8}\|f\|_2 \le C2^{-s\delta_8}\|f\|_2$, as required, completing the proof of Lemma 19. ∎

9. Auxiliary L^2-maximal estimate IV.

9. Auxiliary L^2-maximal estimate IV. The purpose of this section is to prove the following lemma.

LEMMA 23. *Suppose $p_0>1$ and that $1<b<\varepsilon D$, where $\varepsilon=o(1)$. Then*

$$\Big\| \sup_{N\ge1}\Big| \sum_{0\le a<b} S\Big(\frac{a}{b}\Big)\int W_N(\beta)\mathbb{F}(f)\Big(\beta+\frac{a}{b}\Big) \times\zeta(D\beta)e^{2\pi i(\beta+a/b)}\,d\beta\Big| \Big\|_{p_0} \le C\log\log b\|f\|_{p_0} . \tag{16}$$

As before the proof Lemma 23 will be achieved with the aid of a series of lemmas. The first two we quote from [3].

LEMMA 24. *Suppose $p_0>1$ and that $1<b<\varepsilon D$ with $\varepsilon=o(1)$. Then*

$$\|F(\beta)e^{2\pi i\beta qy}\zeta(D\beta)d\beta\|_{L^{p_0}(\mathbb{R})} \sim \|F(\beta)e^{2\pi i\beta yq}\zeta(D\beta)d\beta\|_{l^{p_0}(\mathbb{Z})} ,$$

that is, the $L^{p_0}(\mathbb{R})$ and $l^{p_0}(\mathbb{Z})$ norms are equivalent.

LEMMA 25. *Suppose $p_0>1$ and that $1<b<\varepsilon D$ with $\varepsilon=o(1)$. Then if $U_N(\beta)$ is uniformly bounded in N, we have*

$$\Big\| \sup_{N\ge1}\Big| \sum_{0\le a<b} \int U_N(\beta)\mathbb{F}(f)\Big(\beta+\frac{a}{b}\Big) \times\zeta(D\beta)e^{2\pi i(\beta+a/b)}d\beta\Big| \Big\|_{l^{p_0}(\mathbb{Z})} \le C\|f\|_{l^{p_0}(\mathbb{Z})} .$$

LEMMA 26. *The $l^1(\mathbb{Z})$ norm of the function*

$$\sum_{0\le a<b} S\left(\frac{a}{b}\right)G\left(\alpha-\frac{a}{b}\right)$$

is bounded by

$$C(\log\log b)b\sum_{j\in\mathbb{Z}}\sup_{0\le x<b}|\mathbb{F}(G)(bj+x)|.$$

Proof. We first note that

$$\begin{aligned}\mathbb{F}\Big(\sum_{0\le a<b} S\left(\frac{a}{b}\right)G\left(\alpha-\frac{a}{b}\right)\Big)(x) &= \Big(\sum_{0\le a<b} S\left(\frac{a}{b}\right)e^{2\pi i a x/b}\Big)\mathbb{F}(G)(x)\\ &= \mathbb{F}(G)(x)\frac{1}{\phi(b)}\sum_{\substack{r=0\\(r,b)=1}}^{b-1}\sum_{0\le a<b} e^{-2\pi i(\vartheta(r)-x)ab^{-1}}\\ &= \mathbb{F}(G)(x)\#\{r\in\mathbb{Z}/b\mathbb{Z} : x\in\mathbb{Z}/b\mathbb{Z}+\vartheta(r)\}.\end{aligned}$$

Thus if

$$\chi(r,x)=\begin{cases}1 & \text{if } x \text{ is in } \mathbb{Z}/b\mathbb{Z}+\vartheta(r),\\ 0 & \text{otherwise},\end{cases}$$

we have

$$\Big\|\mathbb{F}\Big(\sum_{0\le a<b} S\left(\frac{a}{b}\right)G\left(\alpha-\frac{a}{b}\right)\Big)\Big\|_{l^1(\mathbb{Z})}=\sum_{x\in\mathbb{Z}}|\mathbb{F}(G)(x)|\sum_{r\in\mathbb{Z}/b\mathbb{Z}}\chi(r,x)\frac{b}{\phi(b)},$$

which, using the fact that $\phi(b)>C(\log\log b)b^{-1}$, is

$$\le C(\log\log b)b\sum_{j\in\mathbb{Z}}\sup_{0\le x<b}|\mathbb{F}(G)(bj+x)|,$$

as required. ∎

LEMMA 27. *Suppose $1<b<D$. Then*

$$\Big\|\sum_{0\le a<b} S\left(\frac{a}{b}\right)\int W_N\left(\alpha-\frac{a}{b}\right)\zeta\left(D\left(\alpha-\frac{a}{b}\right)\right)e^{2\pi i\alpha x}\,dx\Big\|_{l^1(\mathbb{Z})}\le C\log\log b.$$

Proof. Note that for $W_M(\beta)$ defined by (5) if we take M_0 large enough to ensure that $\vartheta(x)$ is increasing for $x\ge M_0$ and that $\vartheta^{-1}(M_0)\ge 1$, then if

$$A_M(x)=\frac{\vartheta^{-1}(M)}{\pi_M}\left(\frac{\chi_{[\vartheta^{-1}(M_0)/\vartheta^{-1}(M),1]}}{\log(\vartheta^{-1}(\vartheta^{-1}(M)x))\vartheta'(\vartheta^{-1}(M)x))}\right)(x) \tag{17}$$

where, for a set B, we have used χ_B to denote its characteristic function, we have

$$W_N(\beta)=\mathbb{F}(A_N)\left(\frac{\beta}{\vartheta^{-1}(N)}\right),$$

and in addition

$$\mathbb{F}(\zeta)(D\cdot)=D^{-1}(\mathbb{F}(\zeta))\left(\frac{\cdot}{D}\right),$$

so applying Lemma 26 with $G(\beta)=W_N(\beta)\zeta(D\beta)$ we have

$$\mathbb{F}(G)=D^{-1}\mathbb{F}(A_N)\left(\frac{\beta}{\vartheta^{-1}(N)}\right)*\mathbb{F}(\zeta)\left(\frac{\beta}{D}\right).$$

We therefore have

$$\begin{aligned}&\Big\|\sum_{0\le a<b} S\left(\frac{a}{b}\right)\int W_N\left(\alpha-\frac{a}{b}\right)\zeta\left(D\left(\alpha-\frac{a}{b}\right)\right)e^{2\pi i\alpha x}\,d\alpha\Big\|_{l^1(\mathbb{Z})}\\ &\qquad\le C\frac{\log\log b}{D}\sum_{j\in\mathbb{Z}}\sup_{0\le x<b}\int_0^1\Big|\mathbb{F}^{-1}[\zeta]\left(\frac{x+t-y}{D}\right)\Big|\,dy\end{aligned}$$

for certain t. Therefore, using the fact that ζ is smooth and using the implication of that for the decay of Fourier coefficients, we see this is

$$\le Cb\frac{\log\log b}{D}\sum_{j\in\mathbb{Z}}\sup_{0\le x<b}\frac{1}{1+\left(\frac{x+jb}{D}\right)^2}, \tag{18}$$

which is $\le C\log\log b$, as required. ∎

LEMMA 28. *If $p_0>1$ and f is in $L^{p_0}(\mathbb{R})$ then we have*

$$\|\sup_{N\ge1}|\mathbb{F}^{-1}[W_N\mathbb{F}(f)]|\,\|_{L^{p_0}(\mathbb{R})}\le C\|f\|_{L^{p_0}(\mathbb{R})}.$$

Proof. Note that the left hand side is $\|\sup_{N\ge1}|f*V_N|\,\|_{L^{p_0}(\mathbb{R})}$ where $V_N(\beta)=A_N(-\beta/\vartheta^{-1}(N))$ and A_N is defined by (17). As the reader will readily verify letting $s_1=s_1(N)=\vartheta^{-1}(N)$, $s_2=s_1$,

$$\begin{aligned}K_1(x)&=K_1^N(x)\\ &=(\log N)^{-1}\left(\frac{\vartheta^{-1}(N)}{\vartheta^{-1}(N_0)}\chi_{[0,\vartheta^{-1}(N_0)/\vartheta^{-1}(N)]}+\chi_{[\vartheta^{-1}(N_0)/\vartheta^{-1}(N),1]}(x)x^{-1}\right)\end{aligned}$$

and

$$K_2(x)=\chi_{[0,1]}(x)x^{1/d-1}$$

and setting $(K_i)_{s_i}(t)=(1/s_i)K_i(|t|/s_i)$ $(i=1,2)$ we have

$$\sup_{N\ge1}|f*V_N|\le C(\sup_{s_1\ge0}|f*(K_1)_{s_1}(t)|+\sup_{s_2\ge0}|f*(K_2)_{s_2}(t)|)$$

for $f \geq 0$. Here we have used the fact that K_1 and K_2 are non-negative. Note that both K_1 and K_2 have integrals bounded above absolutely, that is, independent of N or s. Splitting f into its positive and negative parts we have

$$\|\sup_{N\geq 1}|f * V_N|\|_{L^{p_0}(\mathbb{R})} \leq C(\|\sup_{s_1\geq 0}|f*(K_1)_{s_1}(t)|\|_{L^{p_0}(\mathbb{R})} + \|\sup_{s_2\geq 0}|f*(K_2)_{s_2}(t)|\|_{L^{p_0}(\mathbb{R})}).$$

We need to show that this is bounded by $C\|f\|_{L^{p_0}(\mathbb{R})}$. Suppose K denotes either K_1 or K_2. Again because K is non-negative, in proving Lemma 28 we may assume f is non-negative. Define the function K_L to be K on the set $\{x : 2^{-L-1} < K(x) \leq 2^L\}$, zero on $\{x : K(x) \leq 2^{-L-1}\}$ and 2^L on $\{x : K(x) > 2^L\}$. Also define B_j to be the smallest interval centred at the origin containing all x such that $K(x) > 2^{j-1}$. Then as we readily see, if as before χ_{B_j} denotes the characteristic function of B_j, we have

$$2^{-L-1}\chi_{B_{-L}} + \sum_{j=-L}^{L}(2^j - 2^{j-1})\chi_{B_j} \leq K_L \leq 2^{-L}\chi_{B_{-L}} + \sum_{j=-L}^{L}(2^{j+1}-2^j)\chi_{B_j},$$

Let $K_{L,s}(x)$ denote $s^{-1}K_L(x/s)$; then if sB denotes the set $\{sx : x \in B\}$ and $|sB|$ denotes its Lebesgue measure, we have

$$(K_{L,s} * f)(x) \leq 2^{-L}|B_{-L}|(1/|sB_{-L}|)\int_{sB_{-L}} f(x-y)\,dy + \sum_{j=-L}^{L}(2^{j+1}-2^j)|B_{-j}|(1/|sB_{-j}|)\int_{sB_{-j}} f(x-y)\,dy$$

which is

$$\leq 2M(f)(x)\Big(\int K\Big).$$

Here $M(f)$ denotes the Hardy–Littlewood maximal function. Lemma 28 now follows from the L^{p_0} boundedness of this maximal function. ∎

We can now complete the proof of Lemma 23. Suppose g is defined indirectly by

$$\mathbb{F}(g)(\alpha) = \sum_{0\leq a<b} S\Big(\frac{a}{b}\Big)\zeta\Big(\frac{D}{4}\Big(\alpha - \frac{a}{b}\Big)\Big)\mathbb{F}(f)(\alpha).$$

Then

$$\|g\|_{L^{p_0}(\mathbb{R})} = \Big\|f * \mathbb{F}\Big[\sum_{0\leq a<b}\zeta\Big(\frac{D}{4}\Big(\alpha-\frac{a}{b}\Big)\Big)\Big]\Big\|_{L^{p_0}(\mathbb{R})}, \tag{19}$$

which by Young's inequality is

$$\leq \Big\|\mathbb{F}\Big[\sum_{0\leq a<b} S\Big(\frac{a}{b}\Big)\zeta\Big(\frac{D}{4}\Big(\alpha-\frac{a}{b}\Big)\Big)\Big]\Big\|_{l^1(\mathbb{Z})}\|f\|_{L^{p_0}(\mathbb{R})},$$

and by Lemma 27 this is

$$\leq C(\log\log b)\|f\|_{L^{p_0}(\mathbb{R})}. \tag{20}$$

Now note that because

$$\zeta\Big(\frac{D}{4}\Big(\alpha - \frac{a^1}{b^1} - \frac{a}{b}\Big)\Big)\zeta\Big(D\Big(\alpha-\frac{a}{b}\Big)\Big) = \begin{cases}\zeta(D(\alpha - a/b)) & \text{if } a^1/b^1 = 0,\\ 0 & \text{otherwise},\end{cases}$$

we have the left hand side of (16) equal to

$$\Big\|\sup_{N\geq 1}\Big|\sum_{0\leq a<b}\int W_N(\beta)\mathbb{F}(g)\Big(\beta+\frac{a}{b}\Big)\zeta(D\beta)e^{2\pi i(\beta+a/b)}\,d\beta\Big|\Big\|,$$

which together with (20) completes the proof of Lemma 23. ∎

10. The restricted maximal function. In this section we show that the following lemma is true.

LEMMA 29. *Suppose $p > 1$. Then for each $N_0 > 1$,*

$$\|\sup_{N_0\leq N\leq N_0^2}|f * K_N|\|_p \leq C(\log\log N_0)\|f\|_p.$$

Let G denote the cyclic group $\mathbb{Z}/J\mathbb{Z}$ for large J. Endow G with its normalised counting measure. Using the fact that for each $N \geq 1, K_N$ is a positive kernel we see that Lemma 29 follows from the following inequality:

$$\|\sup_{k_0\leq k\leq 2k_0}|f * K_{2^k}|\|_{L^p(G)} \leq C(\log k)\|f\|_{L^p(G)}. \tag{21}$$

The next lemma is a well known tool in harmonic analysis due to E. Stein (see [6]). Suppose G is a compact group with Haar measure indicated by $\nu = dg$. Suppose $1 \leq p \leq 2$ and that $(T_m)_{m=1}^{\infty}$ is a sequence of bounded linear transformations of $L^p(G)$ to itself such that each T_m commutes with translation by any group element. We set

$$T^*(f)(x) = \sup_{m\geq 0}|T_m f(x)| \quad\text{and}\quad E_\alpha = \{x : T^*f(x) > \alpha\}.$$

LEMMA 30. *Suppose that for any function f in $L^p(G)$ we have*

$$T^*(f)(x) < \infty \tag{22}$$

almost everywhere. Then there is a constant $A_p > 0$ such that

$$\nu(E_\alpha) \le \frac{A_p}{\alpha^p}\|f\|^p_{L^p(G)}.$$

Note that this means that showing

$$(23)\qquad \|\sup_{N_0\le N\le N_0^2}|f*K_N|\,\|_p \le C\log\log N_0\|f\|_p$$

is reduced to showing

$$(24)\qquad \|\sup_{k_0\le k\le 2k_0}|f*K_{2^k}|\,\|_{L^1(G)} \le C\log k\|f\|_{L^p(G)}.$$

Since (24) weakens as p increases we assume $q = 1/(p-1)$ is an integer. We now estimate (23) in its dual form. We therefore need to show that whenever we have functions $(g_k)_{k\ge0}$ satisfying $g_k \ge 0$ and $\sum_{k\ge0} g_k \le 1$ then

$$(25)\qquad \Big\|\sum_{k=k_0}^{2k_0} g_k * K_{2^k}\Big\|_q \le C_q\log k_0.$$

To prove this statement we need the following lemma proved using major arcs information.

LEMMA 31. *Suppose $k_0 < k_1 < \ldots < k_q < 2k_0$. Then for an appropriate choice of $M \approx \log k_0$ there exists B greater than one such that if we set $N_l = K_{2^{lM}}$ then*

$$(26)\qquad \|(g_{k_2}*N_{k_2})\ldots(g_{k_q}*N_{k_q})*(N_{k_1}-N_{k_0})\|_{L^2(G)} \le k_0^{-B}.$$

Proof. Denote g_{k_r} by h_r and 2^{Mk_r} by P_r. For D and s_t to be defined later let

$$\Gamma_t(\alpha) = \sum_{0\le r\le s_t} S\Big(\frac{a}{b}\Big)W_{P_r}\Big(\alpha-\frac{a}{b}\Big)\zeta\Big(10^r\Big(\alpha-\frac{a}{b}\Big)\Big)\zeta\Big(D^{-1}P_r^d\Big(\alpha-\frac{a}{b}\Big)\Big).$$

It follows from Lemmas 6, 9 and 11 that

$$(27)\qquad |\mathbb{F}(N_{k_r})(\alpha)-\Gamma_t(\alpha)| \le C((\log P_r)^{-\delta_5}+2^{-s_r\delta_1}+(\log P_r)D^{-1/d}).$$

From the definition of R_s we have $\|\mathbb{F}^{-1}(\Gamma_t)\|_{l^1(\mathbb{Z})} \le C4^{s_t}$, giving the uniform estimate

$$(28)\qquad |\mathbb{F}^{-1}[\Gamma_t\mathbb{F}(h_t)]| \le C4^{s_t}.$$

This means as a consequence that

$$(29)\qquad \|h_t*N_{k_t}-\mathbb{F}^{-1}[\Gamma_t\mathbb{F}(h_t)]\|_{L^2(G)} \le C((\log P_t)^{-\delta_5}+2^{-s_t\delta_1}+(\log P_t)D^{-1/d}).$$

We estimate the left hand side of (26) as

$$\begin{aligned}
&\le \|(h_2*N_{k_2})-\mathbb{F}^{-1}[\Gamma_2\mathbb{F}(h_2)]\|_{L^2(G)}\\
&\quad+\|\mathbb{F}^{-1}[\Phi_2\mathbb{F}(h_2)]\|_\infty\|(h_3*N_{k_3})-\mathbb{F}^{-1}[\Gamma_3\mathbb{F}^{-1}(h_3)]\|_{L^2(G)}\\
(30)\qquad&\quad\begin{cases}+\ldots+\|\mathbb{F}^{-1}[\Gamma_2\mathbb{F}(h_2)]\|\ldots\|\mathbb{F}^{-1}[\Gamma_{q-1}\mathbb{F}(h_{q-1})]\|\\ \times\|(h_q*N_{k_q})-\mathbb{F}^{-1}[\Gamma_q\mathbb{F}(h_q)]\|_{L^2(G)}\end{cases}\\
(31)\qquad&\quad+\|\mathbb{F}^{-1}[\Gamma_2\mathbb{F}(h_2)]\ldots\mathbb{F}^{-1}[\Gamma_q\mathbb{F}(h_q)]*(N_{k_1}-N_{k_0})\|_{L^2(G)}.
\end{aligned}$$

By (28) and (29) we know (30) is bounded by

$$C_t\sum_{t=2}^q 4^{s_1+\ldots+s_q}((\log P_t)^{-\delta_5}+2^{-s_t\delta_1}+(\log P_q)D^{-1/d}).$$

Making appropriate choice of the numbers $s_t \approx \log k_0$, this is

$$(32)\qquad \le \frac{1}{10}k_0^{-q}+k_0^c((\log P_2)^{-\delta_5}+(\log P_q)D^{-1/d}).$$

Note that the Fourier transform of

$$\mathbb{F}^{-1}[\Gamma_2\mathbb{F}(h_2)]\ldots\mathbb{F}^{-1}[\Gamma_q\mathbb{F}(h_q)]$$

vanishes outside a DP_2^{-d} neighbourhood of the set

$$U = \{a/b \in \mathbb{T}\cap\mathbb{Q} : 0 \le b < 2^{qs_q}\}.$$

Therefore (31) is bounded by

$$(33)\qquad \|\mathbb{F}^{-1}[\Gamma_2\mathbb{F}(h_2)]\ldots\mathbb{F}^{-1}[\Gamma_q\mathbb{F}(h_q)]\|_{L^2(G)}\sup_{\alpha\in U}|\mathbb{F}(N_{k_1}-N_{k_0})(\alpha)|.$$

As with (30), the first factor is bounded by $C4^{s_2+\ldots+s_q} < k_0^c$.

From the definition of U and Lemmas 7 and 8 one easily verifies that if

$$D < (\log P_0)^{\delta_5}\quad\text{and}\quad 2^{qs_q} < (\log P_0)^{\delta_5},$$

as can easily be arranged, we have

$$\sup_{\alpha\in U}|\mathbb{F}(N_{k_1}-N_{k_0})(\alpha)| \le CD\Big(\frac{P_1}{P_2}\Big)^d + C(\log P_0)^{-\delta_2}.$$

This means that (33) is

$$(34)\qquad \le Ck_0^c[D2^{-Md}+\log(2^Mk_0)^{-\delta_2}].$$

Combining (32) and (34), Lemma 31 is proved. ■

We now return to the proof of (25) with $M \approx \log k_0$. Clearly (25) follows if we show

$$(35)\qquad \Big\|\sum_{k_0<k<2k_0} g_k*L_k\Big\|_q \le C_q,$$

whenever $g_k \ge 0$ and $\sum_{k\ge0} g_k \le 1$. Let η be the least possible C_q. Expanding the qth power of the left hand side of (35) and integrating,

$$\Big\|\sum_{k_0<k<2k_0} g_k * N_k\Big\|_q^q$$

$$\leq C \sum_{k_0<k_1<\ldots<k_q<2k_0} \int_G (g_{k_1} * N_{k_1}) \ldots (g_{k_q} * N_{k_q})\, dg \tag{36}$$

$$+ C \int_G \Big[\sum_{k_0<k<2k_0} g_k * N_k\Big]^{q-1} dg\,. \tag{37}$$

Clearly (36) is bounded by η^{q-1} and as a consequence of Lemma 31,

$$\Big|\int_G (g_{k_1} * (N_{k_1} - N_{k_0}))(g_{k_2} * N_{k_2}) \ldots (g_{k_q} * N_{k_q})\, dg\Big| < Ck_0^{-q}\,,$$

so we estimate (36) by

(38)

$$C + \sum_{k_0<k_2<\ldots<k_q<2k_0} \int_G \Big[\Big(\sum_{k_0<k<2k_0} g_k\Big) * N_{k_0}\Big](g_{k_2} * N_{k_2}) \ldots (g_{k_q} * N_{k_q})\, dg\,.$$

Since the first factor in the integral is bounded by 1, (38) turns out to be bounded by (37) and hence by $C\eta^{q-1}$. Consequently, one gets $\eta^q < C + C\eta^{q-1}$, giving $\eta < C$ and completing the proof of Lemma 29. ∎

11. Proof of Theorem 2. We now complete the proof of the central maximal estimate of this paper, which we stated as Theorem 2. First note that for $\Gamma_{k,s}$ defined by (13) we have

$$\begin{aligned} f * K_{2^k} &= \mathbb{F}^{-1}[\Gamma_{k,1}\mathbb{F}(f)] + \mathbb{F}^{-1}[(\Gamma_{k,2} - \Gamma_{k,1})\mathbb{F}(f)] \\ &\quad + \ldots + \mathbb{F}^{-1}[(\Gamma_{k,s} - \Gamma_{k,s-1})\mathbb{F}(f)] + [f * K_{2^k} - \mathbb{F}^{-1}[\Gamma_{k,s}\mathbb{F}(f)]]\,, \end{aligned}$$

so for $p > 1$,

$$\sup_{k\geq 1} |f * K_{2^k}|_p$$

$$\leq \sum_{s=1}^{\infty} \sup_{k\geq 4^s} |\mathbb{F}^{-1}[(\Gamma_{k,s} - \Gamma_{k,s-1})\mathbb{F}(f)]| \tag{39}$$

$$+ \sum_{s=0}^{\infty} \sup_{k\in B_s} |f * K_{2^k} - \mathbb{F}^{-1}[\Gamma_{k,s}\mathbb{F}(f)]|\,. \tag{40}$$

Suppose $1 < p_0 < p < 2$. It follows from Lemma 23 and subtraction that

$$\|\sup_{k\geq 4^s} |\mathbb{F}^{-1}[(\Gamma_{k,s} - \Gamma_{k,s-1})\mathbb{F}(f)]|\,\|_{p_0} \leq Cs\|f\|_{p_0}\,.$$

We also showed in Lemma 21 that

$$\|\sup_{k\geq 4^s} |\mathbb{F}^{-1}(\Gamma_{k,s} - \Gamma_{k,s-1})\mathbb{F}(f)|\,\|_2 \leq C2^{-\delta_{10}s}\|f\|_2\,.$$

Interpolation gives

$$\|\sup_{k\geq 4^s} |\mathbb{F}^{-1}(\Gamma_{k,s} - \Gamma_{k,s-1})\mathbb{F}(f)|\,\|_p \leq C2^{-\delta_p^{(1)}s}\|f\|_p\,.$$

Now,

$$\|\sup_{k\in B_s} |f * K_{2^k} - \mathbb{F}^{-1}[\Gamma_{k,s}\mathbb{F}(f)]|\,\|_{p_0}$$

$$\leq \|\sup_{k\in B_s} |f * K_{2^k}|\,\|_{p_0} \tag{41}$$

$$+ \|\sup_{k\in B_s} |\mathbb{F}^{-1}[\Gamma_{k,s}\mathbb{F}(f)]|\,\|_{p_0}\,. \tag{42}$$

Using the restricted maximal function estimate, (41) is bounded by $s\|f\|_p$ and by Lemma 23 a similar bound can be found for (42). Also by Lemma 20,

$$\|\sup_{k\in B_s} |f * K_{2^k} - \mathbb{F}^{-1}[\Gamma_{k,s}\mathbb{F}(f)]|\,\|_2 \leq C2^{-s\delta_{10}}\|f\|_2\,.$$

This means that by interpolation we have

$$\|\sup_{k\in B_s} |f * K_{2^k} - \mathbb{F}^{-1}[\Gamma_{k,s}\mathbb{F}(f)]|\,\|_p \leq C2^{-s\delta_p^{(2)}}\|f\|_p\,.$$

Returning to (39) and (40) we have

$$\|\sup_{k\geq 0} |f * K_{2^k}|\,\|_p \leq C\Big(\sum_{s\geq 0} 2^{-s\delta_p^{(1)}} + \sum_{s\geq 0} 2^{-s\delta_p^{(2)}}\Big)\|f\|_p \leq C\|f\|_p\,,$$

as required, for all p such that $1 < p < 2$, thereby completing the proof of Theorem 2. ∎

12. Proof of Theorem 3. We need the following lemma [2].

Lemma 32. *Suppose $\omega : \mathbb{R} \to [0,1]$ is smooth and supported in the interval $[-\tau, \tau]$. For a non-negative function $K : \mathbb{R} \to \mathbb{R}$ in $L^1(\mathbb{R})$, decreasing as $|x| \to \infty$, set $K_t(x) = t^{-1}K(x/t)$. For $f \in l^2(\mathbb{Z})$ and $t > 0$, let*

$$D_t f(x) = \int_{-\infty}^{\infty} \mathbb{F}(f)(\alpha)\omega(\alpha)\mathbb{F}(K_t)(\alpha)e^{2\pi i\alpha x}\, d\alpha\,.$$

Then

$$\Big(\sum_{k=1}^{\infty} \|\sup_{N\in Z_{k,\varepsilon}} |D_N f - D_{N_k} f|\,\|_2^2\Big)^{1/2} \leq C\|f\|_2\,.$$

Note that

$$M_j f \le \sup_{N\in Z_{j,\varepsilon}} |f*(L_N - L_{N_j})| + 2\Big\{\sum_{N\in Z_{j,\varepsilon}} |f*(K_N - L_N)|^2\Big\}^{1/2}.$$

Therefore

$$\sum_{1\le k\le J} \|M_k f\|_2^2 \le C\Big(\sum_{1\le k\le J} \|\sup_{N\in Z_{k,\varepsilon}} |f*(L_N - L_{N_k})|\,\|_2^2 + \sum_{N\in Z_\varepsilon} \|f*(K_N - L_N)\|_2^2\Big).$$

By Parseval's inequality

$$\sum_{N\in Z_\varepsilon} \|f*(K_N - L_N)\|_2^2 \le \Big(\sum_{N\in Z_\varepsilon} \|\mathbb{F}(K_N) - \mathbb{F}(L_N)\|_\infty^2\Big)\|f\|_2^2.$$

Using Lemma 11, this is less than $\frac{\pi^2}{6\varepsilon^2}\|f\|_2^2$. Now let

$$\mathbb{F}(P_{n,N})(\alpha) = \sum_{y=1}^{n} \psi_{y,N}(\alpha).$$

Then by Lemma 6,

$$\|\sup_{N\ge 1} |f*(L_N - P_{n,N})|\,\|_2 \le C\Big(\sum_{y>n} 2^{-y\delta_1}\Big)\|f\|_2 \le C2^{-n\delta_1}\|f\|_2.$$

Hence

$$\sum_{1\le k\le J} \|M_k f\|^2 \le \sum_{1\le k\le J} \|\sup_{Z_{k,\varepsilon}} |f*(P_{n,N} - P_{n,N_k})|\,\|_2^2 + CJ2^{-n\delta_1}\|f\|_2^2,$$

and

$$\sum_{1\le k\le J} \|\sup_{Z_{k,\varepsilon}} |f*(P_{n,N} - P_{n,N_k})|\,\|_2^2$$

$$\le 4^n \sup_{a,b,y} \sum_{1\le k\le J} \Big\| \sup_{N\in Z_{j,\varepsilon}} \Big| \int_{-\infty}^{\infty} \mathbb{F}(f)(\alpha)\zeta\Big(10^y\Big(\alpha - \frac{a}{q}\Big)\Big) \times (W_N - W_{N_k})\Big(\alpha - \frac{a}{q}\Big) e^{2\pi i\alpha x}\,d\alpha \Big|\Big\|_2^2.$$

Here the supremum is taken over all pairs a, b such that $1 \le a < b$ and $(a,b) = 1$ and all R_y they belong to. Applying Lemma 32, we have

$$\sum_{1\le k\le J} \|M_k f\|_2^2 \le C(4^n + J2^{-n\delta_1})\|f\|_2^2.$$

Thus for an appropriate choice of n, we have Theorem 3 and hence Theorem 1.

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