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Extremal functions of the Nevanlinna–Pick problem and Douglas algebras

by

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Abstract. The Nevanlinna–Pick problem at the zeros of a Blaschke product B having a solution of norm smaller than one is studied. All its extremal solutions are invertible in the Douglas algebra D generated by B. If B is a finite product of sparse Blaschke products (Newman Blaschke products, Frostman Blaschke products) then so are all the extremal solutions. For a Blaschke product B a formula is given for the number $C(B)$ such that if the NP-problem has a solution of norm smaller than $C(B)$ then all its extremal solutions are Carleson Blaschke products, i.e. can be represented as finite products of interpolating Blaschke products.

1. Discussion of the results. Let H^∞ denote the algebra of all bounded analytic functions in the open unit disc $\mathbb{D}$ and let $\mathbb{U}$ be the unit ball of H^∞. Consider the Nevanlinna–Pick interpolation problem and the set E of its solutions:

$$E = \{f \in \mathbb{U} : f(z_n) = w_n\,,\ n = 1, 2, \ldots\}\,.$$

The main reference for H^∞ and NP-problems is Garnett's book [2].

For a point a in $\mathbb{D}$ let $b_a(z)$ be the Blaschke factor, and let $B = B_Z$ be the Blaschke product with zeros $Z = \{z_n\}_{n=1}^{\infty}$:

$$b_a(z) = \frac{|a|}{a}\frac{a-z}{1-\overline{a}z}\,, \qquad B_Z(z) = \prod_{n=1}^{\infty} b_{z_n}(z)\,.$$

Then the NP-problem is determined by the pair (Z, f) of its zero set and some solution. We have the following description of the set E:

$$E = (f + BH^\infty) \cap \mathbb{U} = B(f\overline{B} + H^\infty) \cap \mathbb{U}\,. \tag{1}$$

If the NP-problem has more than one solution, then for some functions P, Q, R, S analytic in $\mathbb{D}$, we have the representation

$$E = \left\{ f = \frac{P + Qw}{R + Sw} : w \in \mathbb{U} \right\}, \qquad PS - RQ = B\,. \tag{2}$$

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Consider the sets

$$\Delta(z) = \{f(z) : f \in E\}, \quad z \in \mathbb{D}.$$

From (2) it is clear that each $\Delta(z)$ is a closed disc, and the problem has a unique solution iff all those discs degenerate to points. A solution I of the NP-problem will be called *extremal* iff $I(z) \in \partial\Delta(z)$ for some z in $\mathbb{D}\setminus Z$; then the same is true for all z in $\mathbb{D}$. A function f is extremal for the NP-problem iff the corresponding w in the parametrization (2) is a constant of modulus one. If the NP-problem has more than one solution then all its extremal functions are inner functions.

For a subset T of the algebra L^∞ we denote by $[T]$ the closed subalgebra generated by T. If B is a Blaschke product, then $[H^\infty, \overline{B}]$ is the *Douglas algebra* generated by B. An NP-problem (Z, f) is called *scaled* iff there is a g in E such that $\|g\|_\infty < 1$, and *semiscaled* if for some N the NP-subproblem with zeros $\{z_n\}_{n=N}^\infty$ is scaled; this terminology is taken from [10].

In the following theorems we study the connection between the Blaschke product B of some NP-problem and its extremal solutions.

THEOREM 1. *If an NP-problem (Z, f) is semiscaled then all its extremal solutions are inner functions invertible in the Douglas algebra $D = [H^\infty, \overline{B}_Z]$ generated by B_Z. If it is semiscaled, but not scaled, then the NP-problem (Z, f) has a unique solution.*

We have two corollaries from this theorem. Let X denote the maximal ideal space of the algebra L^∞ and $QC = (H^\infty + C) \cap (\overline{H^\infty + C})$. If D is a Douglas algebra, then $QD = D \cap \overline{D}$ is the maximal symmetric subalgebra of D, CD is the closed symmetric algebra generated by all inner functions invertible in D, $QDA = H^\infty \cap QD$, $CDA = H^\infty \cap CD$. In [8] such algebras are called *Sarason algebras*. For a function f in L^∞, the set of "nonanalyticity"

$$N(f) = \operatorname{clos}\bigcup\{Q : f|_Q \notin H^\infty|_Q\} \subset X$$

was introduced in [6, 7], where the union is taken over all QC-level sets Q, i.e. maximal subsets of X where functions from QC are constant.

COROLLARY 2. *If I is an extremal solution of a scaled NP-problem at the zeros of a Blaschke product B then $N(\overline{I}) = N(\overline{B})$. If B can be continuously extended to some point of the unit circle, then the same is true for I.*

To formulate the next corollary and theorem, we introduce some classes of inner functions. We call a finite product of interpolating Blaschke products a *Carleson Blaschke product* and denote the set of all such products by Carl. Let $\mathcal{M}(H^\infty)$ be the maximal ideal space of the algebra H^∞. Denote by $\mathcal{MP}$ the set of all points in $\mathcal{M}(H^\infty)$ with trivial Gleason part. For an inner function I let

$$C(I) = \min\{|I(m)| : m \in \mathcal{MP}\}. \tag{3}$$

Then $I \in$ Carl iff $C(I) > 0$ [3]. Denote by $\mathcal{P}$ the subclass of Carl of Blaschke products such that $C(B) = 1$. It is easy to prove from (3) that $B \in \mathcal{P}$ iff

$$\forall a \in \mathbb{D} \quad \frac{B-a}{1-\overline{a}B} \in \text{Carl}.$$

In [11] the author has given many examples of Blaschke products in $\mathcal{P}$ and their zero sets $Z = \{z_n\}_{n=1}^\infty$. These include:

(a) *sparse Blaschke products*:

$$\lim_{n\to\infty} \prod_{n\neq m} |z_n - z_m|/|1 - z_n\overline{z}_m| = 1,$$

(b) *Newman Blaschke products*:

$$\sup\left\{\frac{1-|a|}{1-|b|} : a, b \in Z,\ |a| \geq |b|,\ a \neq b\right\} < 1,$$

(c) *Frostman Blaschke products*:

$$\sup_{z\in\mathbb{T}} \sum_{a\in Z} \frac{1-|a|}{|1-az|} < \infty,$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$,

(d) Carleson Blaschke products with zeros in a *Stolz angle*, i.e. in the convex hull of some disc $\{z : |z| \leq c\}$, $0 < c < 1$, and a point $t \in \mathbb{T}$.

More information on those types of Blaschke products can be found in [5, 9].

COROLLARY 3. *Let I be an extremal solution of a semiscaled NP-problem at the zeros of a Blaschke product B. Then:*

(i) *If $B \in \mathcal{P}$ then $I \in \mathcal{P}$, in particular $I \in$* Carl.
(ii) *If B is a finite product of sparse Blaschke products then so is I.*
(iii) *If B is a finite product of Newman Blaschke products then so is I.*
(iv) *If B is a Frostman Blaschke product then so is I.*
(v) *If B is Carleson Blaschke product with zeros in a Stolz angle then so is I.*

In the following theorem it is proved that for every zero set Z the constant $C(B_Z)$ is the largest number C for which all extremal functions of NP-problems (Z, f) with $\|f\|_\infty < C$ are Carleson Blaschke products.

THEOREM 4. (i) *Let $B \in$* Carl *with zeros Z and $C = C(B)$. For any f in H^∞ with norm smaller than C, all extremal functions I of the NP-problem (Z, f) belong to* Carl.

(ii) *Conversely, if the above constant C is smaller than 1, then there exists an extremal function I for the NP-problem (Z, C) which is not a Carleson Blaschke product.*

The existence of some constant in the first part of this theorem was proved in [10]. The following problem was formulated there: Is this theorem valid with some constant independent of $\{z_n\}_{n=1}^{\infty}$? Theorem 3 gives a negative answer to this question. Indeed, if

$$S(z) = \exp\left(\frac{z+1}{z-1}\right), \quad a \in \mathbb{D}, \quad S_a = \frac{S+a}{1+\overline{a}S},$$

then it is easy to prove by direct computation that S_a is an interpolating Blaschke product for all nonzero values a in $\mathbb{D}$ and then $C(S_a) = |a|$. So the constant $C(B)$ for $B \in$ Carl may be as small as we want. Stray's conjecture was independently disproved by Nicolau [8].

We can make some conclusions about the coefficients of the Nevanlinna's parametrization (2) from the properties of extremal functions. Let us introduce new functions:

$$R_1 = 1/R, \quad P_1 = P/R, \quad Q_1 = Q/R,$$
$$S_1 = S/R, \quad T = P_1 S_1 - Q_1 = R_1^2 B.$$

Nevanlinna proved that all these functions are in H^∞ and have norms not greater than one.

THEOREM 5. *Let the NP-problem (Z, f) be scaled and let $D = [H^\infty, \overline{B}]$. Then the functions*

$$P_1, \quad TS_1^k \quad (k \geq 0) \tag{4}$$

are in the Sarason algebra CDA.

As the Shilov boundary of the algebra CDA can be identified with $\mathcal{M}(CD)$ [9], we can give the following reformulation of this theorem: The functions P_1, T are continuous on $\mathcal{M}(CD)$, and all discontinuity points of S_1 in $\mathcal{M}(CD)$ are in the zero set of T. For the function R_1 we have $R_1^2 = T\overline{B} \in CD \cap H^\infty = CDA \subset QDA$ and R_1 is an outer function. It can be proved that then $R_1 \in QDA$ by analogy with [4, p. 59], where the case of $D = H^\infty + C$ is studied. An open problem: is R_1 in CDA for every scaled NP-problem?

2. Proofs of the results.

The main idea is to apply the following lemma, attributed to Sarason in [1]; see also [2, p. 386]. We denote the distance in L^∞ by dist.

LEMMA 6. *If u is a unimodular function, $a \in \mathbb{D}$, b_a is the Blaschke factor with zero a and*

$$\operatorname{dist}(u, H^\infty) < 1, \quad \operatorname{dist}(u, b_a H^\infty) = 1,$$

then

$$\operatorname{dist}(\overline{u}, H^\infty) = \operatorname{dist}(u, H^\infty) < 1 \quad \text{and} \quad \overline{u} \in [H^\infty, u].$$

Proof. Making a conformal change of variables in $\mathbb{D}$ if necessary, we can consider only the case $a = 0$. The first assertion is proved in [1, 2]. It is also proved there that any $h \in H^\infty$ with $\|\overline{u} - h\|_\infty < 1$ is invertible in H^∞. Then $\|1 - hu\|_\infty < 1$, hu is invertible in $[H^\infty, u]$ and $\overline{u} = h(uh)^{-1} \in [H^\infty, u]$. ∎

Lemma 7 was proved by Adamyan, Arov and Krein (see [8, p. 310]).

LEMMA 7. *Let $g \in L^\infty$. If*

$$\operatorname{dist}(g, H^\infty) = 1, \quad \operatorname{dist}(g, H^\infty + C) < 1,$$

then there exists a unique $h \in L^\infty$ such that $g - h \in H^\infty$ and $g - h$ is unimodular.

Proof of Theorem 1. If the NP-problem is semiscaled but not scaled then a scaled problem can be obtained by removing some points $\{z_n\}_{n=1}^{N}$ from Z. Let V be a set so obtained for minimal N and $a = z_N$. Then for the function $g = f\overline{B}_V\overline{b}_a$ from (1) we have

$$\operatorname{dist}(g, H^\infty) = \operatorname{dist}(f, b_V b_a H^\infty) = 1, \tag{5}$$
$$\operatorname{dist}(g, H^\infty + C) \leq \operatorname{dist}(g, \overline{b}_a H^\infty) = \operatorname{dist}(f, BH^\infty) < 1. \tag{6}$$

Then by Lemma 7 the NP-problem $(V \cup \{a\}, f)$ has a unique solution, g is a unimodular function and so f is an inner function; let us write $f = I$. In the case of scaled problem let $V = Z$, $a \in \mathbb{D} \setminus Z$ and $g = I\overline{B}_V\overline{b}_a$. Because the NP-problem (V, I) is scaled, we have (5). As I is extremal, the NP-problem $(V \cup \{a\}, I)$ has a unique solution, so it is nonscaled, and so we have (6). In both cases Lemma 6 can be applied to the unimodular function $g = I\overline{B}_V$ and

$$\overline{g} \in [H^\infty, g], \quad \overline{I} = \overline{g}\overline{B}_V \in [H^\infty, g]\overline{B}_V \subset [H^\infty, \overline{B}_V]. \ ∎$$

Remark. The following result can be proved: for an NP-problem (Z, f), if $B = B_Z$ and $g = f\overline{B}$ then the NP-problem is scaled iff $\operatorname{dist}(g, H^\infty) < 1$, and semiscaled iff $\operatorname{dist}(g, H^\infty + C) < 1$. This result will not be used and so the proof is omitted.

To prove Corollary 2 we need the following characterization of a singly generated Douglas algebra due to Izuchi:

THEOREM 8 [7, Cor. 2.5]. *Let $f \in L^\infty$ and let $D = [H^\infty, f]$ be the Douglas algebra generated by f. Then*

$$D = \{g \in L^\infty : N(g) \subset N(f)\}.$$

Proof of Corollary 2. From Theorems 1 and 8 we have $\bar{I} \in [H^\infty, \overline{B}]$ and $N(\bar{I}) \subset N(\overline{B})$. By applying Lemma 6 to the unimodular function $\overline{g} = \bar{I}B_V$ we have $\overline{B}_V \in I[H^\infty, \bar{I}] \subset [H^\infty, \bar{I}]$ and so $N(\overline{B}) \subset N(\bar{I})$.

Let us denote the fiber over a point $t \in \mathbb{T}$ by $X_t = \{x \in \mathcal{M}(H^\infty) : x(z) = t\}$. The function B can be continuously extended to t iff $B|_{X_t} = c$ for some c in $\mathbb{T}$. Then for every f in $[H^\infty, \overline{B}]$ we have $f|_{X_t} \in H^\infty|_{X_t}$ and so by Theorem 1, $\bar{I}|_{X_t} \in H^\infty|_{X_t}$; then the function $I|_{X_t}$ is invertible in the algebra $H^\infty|_{X_t}$. If the inner function I is not continuous at t then it is easy to prove that there exists a sequence $\{z_n\}_{n=1}^\infty$ in $\mathbb{D}$ such that $\lim_{n\to\infty} z_n = t$ and $\lim_{n\to\infty} I(z_n) = 0$ (see for example [9, p. 63]); so $I(m) = 0$ for some m in X_t, contradicting the invertibility of $I|_{X_t}$. So I is continuous at t. ∎

Corollary 3 follows directly from the next result:

THEOREM 9 [11]. (i) *Each class of inner functions described in Corollary* 3(ii)–(v) *is equal to the class of all invertible inner functions in some Douglas algebra.*

(ii) *For an inner function I, every invertible inner function in the Douglas algebra $[H^\infty, \bar{I}]$ is a Carleson Blaschke product iff $C(I) = 1$.*

The proof of the first part given in [11] consists in constructing the corresponding Douglas algebras. For the case (v) such a Douglas algebra was investigated by Sarason: it is generated by H^∞ and the functions on $\mathbb{T}$ with only one discontinuity point [2, p. 396].

Proof of Theorem 4. Let I be an extremal solution of the NP-problem (Z, f) and $u = I\overline{B}$. Then $\alpha = \operatorname{dist}(u, H^\infty) \le \|f\|_\infty < C$. As in the proof of Theorem 1 we use Lemma 6 to get $\operatorname{dist}(\overline{u}, H^\infty) = \alpha$ and then $\operatorname{dist}(B, IH^\infty) = \|B - Ig\|_\infty = \alpha$ for some g in H^∞. By the Corona Theorem, the unit disc $\mathbb{D}$ is dense in $\mathcal{M}(H^\infty)$ and so for all maximal ideals m we have $|B(m) - I(m)g(m)| \le \alpha$. But $\|g\|_\infty = \|Ig\|_\infty \le \|B\|_\infty + \alpha < 2$, and thus for all maximal ideals $m \in \mathcal{MP}$ we have $|I(m)| \ge (|B(m)| - \alpha)/\|g\|_\infty > (C - \alpha)/2 > 0$ and so $I \in$ Carl.

Let now $C < 1$. If $f|_Z = C$, $f \in \mathbb{U}$, then for $g = \frac{f - C}{1 - \overline{C}f}$ we have $g|_Z = 0$, $g \in \mathbb{U}$ and thus $g = Bw$ for some w in H^∞. So the Nevanlinna parametrization (2) for our NP-problem $f|_Z = C$ is

$$E = \left\{\frac{C + Bw}{1 + \overline{C}Bw} : w \in \mathbb{U}\right\}.$$

Let $m \in \mathcal{MP}$ be such that $|B(m)| = C(B)$ ($\mathcal{MP}$ is compact). Then we can take a constant function w such that $C + B(m)w = 0$ and for the corresponding extremal function I we have $I(m) = 0$, so $I \notin$ Carl. ∎

Proof of Theorem 5. Let I_w be the extremal function with Nevanlinna parameter $w \in \mathbb{T}$. Then

$$I_w = \frac{P_1 + Q_1 w}{1 + S_1 w} = P_1 - \frac{Tw}{1 + S_1 w}. \tag{7}$$

Let N be a CD-level set, i.e. a maximal subset of X where the functions from the symmetric algebra CD are constant. Then by Theorem 1 all functions I_w are constant on N, and it is sufficient to prove that so are all functions (4).

If $m \in \mathcal{M}(H^\infty)$ and $w \in \mathbb{T}$, $w \ne -\overline{S_1(m)}$, then we can multiply (7) by $1 + S_1 w$, substitute m and then divide by $1 + S_1(m)w$, to get

$$I_w(m) = P_1(m) - \frac{Tw}{1 + S_1(m)w}. \tag{8}$$

For x_1, x_2 in N, let

$$X_{12} = \{w \in \mathbb{T} : w \ne -\overline{S_1(x_1)},\ w \ne -\overline{S_1(x_2)}\}.$$

Then the fractional-linear transforms corresponding by formula (8) to the points x_1, x_2 are equal on X_{12} and so $P_1(x_1) = P_1(x_2)$, $T(x_1) = T(x_2)$, and if the last number is nonzero then $S_1(x_1) = S_1(x_2)$. Hence the functions P_1, TS_1^k $(k \ge 0)$ are constant on N for any CD-level set N and so they are in CD; by Nevanlinna's theorem they are in H^∞, and therefore in CDA. ∎

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Range inclusion results for derivations on noncommutative Banach algebras

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Abstract. Let A be a Banach algebra, and let $D : A \to A$ be a (possibly unbounded) derivation. We are interested in two problems concerning the range of D:

1. When does D map into the (Jacobson) radical of A?
2. If $[a, Da] = 0$ for some $a \in A$, is Da necessarily quasinilpotent?

We prove that derivations satisfying certain polynomial identities map into the radical. As an application, we show that if $[a,[a,[a, Da]]]$ lies in the prime radical of A for all $a \in A$, then D maps into the radical. This generalizes a result by M. Mathieu and the author which asserts that every centralizing derivation on a Banach algebra maps into the radical. As far as the second question is concerned, we are unable to settle it, but we obtain a reduction of the problem and can prove the quasinilpotency of Da under commutativity assumptions slightly stronger than $[a, Da] = 0$.

Introduction. The interest in range inclusion results for derivations on Banach algebras goes back to I. M. Singer's and J. Wermer's paper [S–W] from 1955, in which they proved that every bounded derivation on a commutative Banach algebra maps into the (Jacobson) radical. In a footnote they conjectured that the boundedness requirement for the derivation was superfluous. It took more than thirty years until this conjecture was finally proved by M. P. Thomas ([Tho 1]).

The simple-minded attempt to extend these results to noncommutative Banach algebras obviously fails, even for bounded derivations: Let A be a noncommutative, semisimple Banach algebra, and fix some $a \in A$ which does not lie in the center $Z(A)$ of A. Then $A \ni x \mapsto [a, x] := ax - xa$ is a bounded derivation, which is nonzero, and therefore does not map into the radical. There are, however, various meaningful generalizations of the bounded Singer–Wermer theorem to the noncommutative setting (see [Yoo], [M–M] and [Vuk 1], for instance). All these results require at some point the

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