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Pointwise multipliers for functions of weighted bounded mean oscillation

by

EIICHI NAKAI (Yuki)

Abstract. For $w : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ and $1 \le p < \infty$, let $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ be the set of locally integrable functions f on $\mathbb{R}^n$ for which

$$\sup_I \left(\frac{1}{w(I)} \int_I |f(x) - f_I|^p \, dx\right)^{1/p} < \infty$$

where $I = I(a,r)$ is the cube with center a whose edges have length r and are parallel to the coordinate axes, $w(I) = w(a,r)$ and f_I is the average of f over I. If w satisfies appropriate conditions, then the following are equivalent:

(1) $$fg \in \mathrm{bmo}_{w,p}(\mathbb{R}^n) \quad \text{whenever} \quad f \in \mathrm{bmo}_{w,p}(\mathbb{R}^n),$$

(2) $$g \in L^\infty(\mathbb{R}^n) \quad \text{and} \quad \sup_I \left(\frac{1}{w^*(I)} \int_I |g(x) - g_I|^p \, dx\right)^{1/p} < \infty,$$

where $w^* = w/\Psi$, $\Psi = \Psi_1 + \Psi_2$ and

$$\Psi_1(a,r) = \left(\int_1^{\max(2,|a|,r)} \frac{w(O,t)^{1/p}}{t^{n/p+1}} \, dt\right)^p,$$

$$\Psi_2(a,r) = \left(\int_r^{\max(2,|a|,r)} \frac{w(a,t)^{1/p}}{t^{n/p+1}} \, dt\right)^p.$$

1. Introduction. The purpose of this paper is to characterize the set of pointwise multipliers on $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$, which is the function space defined using the mean oscillation in L^p-sense ($1 \le p < \infty$) and a weight function $w(x,r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$.

To define $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$, let $I(a,r)$ be the cube $\{x \in \mathbb{R}^n : |x_i - a_i| \le r/2,\ i = 1,\ldots,n\}$ whose edges have length r and are parallel to the coordinate axes. For a function f and for a cube $I = I(a,r)$, we denote the mean

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value and the mean oscillation of f on I by

$$f_I = M(f,I) = M(f,a,r) = \frac{1}{|I|}\int_I f(x)\,dx$$

and

$$\mathrm{MO}(f,I) = \mathrm{MO}(f,a,r) = \frac{1}{|I|}\int_I |f(x)-f_I|\,dx$$

respectively, where $|I|$ is the Lebesgue measure of I, and we denote the weighted mean oscillation of f on I by

$$\mathrm{MO}_{w,p}(f,I) = \mathrm{MO}_{w,p}(f,a,r) = \left(\frac{1}{w(I)}\int_I |f(x)-f_I|^p\,dx\right)^{1/p}$$

where $w(I) = w(a,r)$ and $1 \le p < \infty$.

Now we define

$$\mathrm{bmo}_{w,p}(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_I \mathrm{MO}_{w,p}(f,I) < \infty\},$$

$$\|f\|_{\mathrm{BMO}_{w,p}} = \sup_I \mathrm{MO}_{w,p}(f,I), \quad \|f\|_{\mathrm{bmo}_{w,p}} = \|f\|_{\mathrm{BMO}_{w,p}} + |M(f,O,1)|.$$

A function g on $\mathbb{R}^n$ is called a *pointwise multiplier* on $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ if the pointwise product fg belongs to $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ for all $f \in \mathrm{bmo}_{w,p}(\mathbb{R}^n)$. $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ is a Banach space under the norm $\|f\|_{\mathrm{bmo}_{w,p}}$. Therefore, the closed graph theorem shows that every pointwise multiplier on $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ is a bounded operator. Usually, $\mathrm{bmo}_{w,p}$ is denoted by $\mathrm{BMO}_{w,p}$ and equipped with the seminorm $\|f\|_{\mathrm{BMO}_{w,p}}$. Then $\mathrm{BMO}_{w,p}$ modulo constants is a Banach space. But pointwise multipliers are defined on function spaces or on the spaces modulo null-functions. To consider pointwise multipliers, the space $\mathrm{bmo}_{w,p}$ is therefore more suitable than $\mathrm{BMO}_{w,p}$.

Our main result is the following.

THEOREM. *Let $1 \le p < \infty$. Assume that there exists a constant $A > 0$ such that for any $a, b \in \mathbb{R}^n$, $r > 0$, $s \ge 1$,*

$$A^{-1} \le w(a,r)/w(a,2r) \le A, \tag{1.1}$$

$$\left(\int_0^r \frac{w(a,t)^{1/p}}{t}\,dt\right)^p \le Aw(a,r), \tag{1.2}$$

$$|a-b| \le r \Rightarrow A^{-1} \le w(a,r)/w(b,r) \le A, \tag{1.3}$$

$$w(a,sr) \le As^{n+p}w(a,r). \tag{1.4}$$

Then a function g is a pointwise multiplier on $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ if and only if $g \in \mathrm{bmo}_{w^*,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $w^* = w/\Psi$, $\Psi = \Psi_1 + \Psi_2$ and

$$\Psi_1(a,r) = \left(\int_1^{\max(2,|a|,r)} \frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt\right)^p, \tag{1.5}$$

$$\Psi_2(a,r) = \left(\int_r^{\max(2,|a|,r)} \frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\right)^p. \tag{1.6}$$

Moreover, the operator norm of g is comparable to $\|g\|_{\mathrm{BMO}_{w^*,p}} + \|g\|_\infty$.

Janson [4] has characterized pointwise multipliers on $\mathrm{bmo}_{w,1}(\mathbb{T}^n)$ on the n-dimensional torus $\mathbb{T}^n$, where $w(a,r) = r^n\phi(r)$, ϕ is nondecreasing and there is a constant $A > 0$ such that $\phi(r)/r \le A\phi(r')/r'$ for $r \ge r'$. In this case, the Ψ in our theorem is

$$\Psi(r) = \int_r^1 \frac{\phi(t)}{t}\,dt.$$

Nakai and Yabuta [8] have extended Janson's result to the case of $\mathbb{R}^n$. In this case,

$$\Psi(a,r) = \int_1^{2+|a|} \frac{\phi(t)}{t}\,dt + \left|\int_1^r \frac{\phi(t)}{t}\,dt\right|.$$

Our result is a generalization of these.

Next, we state corollaries for the Morrey spaces and for the space of functions of bounded mean oscillation with a Muckenhoupt weight.

COROLLARY 1.1. *For $w(x,r) = r^\alpha$, $0 < \alpha < n$, $1 \le p < \infty$, g is a pointwise multiplier on $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ if and only if g is bounded and in $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$.*

COROLLARY 1.2. *For $w(x,r) = r^\alpha$, $0 < \alpha < n, 1 \le p < \infty$, on the n-dimensional torus $\mathbb{T}^n$, g is a pointwise multiplier on $\mathrm{bmo}_{w,p}(\mathbb{T}^n)$ if and only if g is bounded.*

In order to state the next corollaries, we recall the definitions of the classes A_p of weights (see Muckenhoupt [6] and [7]). A locally integrable and nonnegative function u is said to belong to A_p, $1 < p < \infty$, if there is a constant C such that

$$\left(\frac{1}{|I|}\int_I u(x)\,dx\right)\left(\frac{1}{|I|}\int_I u(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C$$

for any I, and is said to belong to A_1 if there is a constant C such that

$$\frac{1}{|I|}\int_I u(x)\,dx \le C \operatorname*{ess\,inf}_I u$$

for any I.

COROLLARY 1.3. *Let* $1 \le p < \infty$, $0 < \alpha \le \min(p,(n+p)/n)$, $1 \le q \le (n+p)/(n\alpha)$ *and*

$$w(I) = \Big(\int_I u(x)\,dx\Big)^{\alpha}, \quad u \in A_q.$$

Then g *is a pointwise multiplier on* $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ *if and only if* $g \in \mathrm{bmo}_{w^*,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *where* $w^* = w/\Psi$, $\Psi = \Psi_1 + \Psi_2$ *and*

$$(1.7)\quad \Psi_1(a,r) = \Big(\int_{I(O,\max(2,|a|,r))\setminus I(O,1)} u(x)^{\alpha/p}|x|^{-n(1-\alpha/p+1/p)}\,dx\Big)^p,$$

$$(1.8)\quad \Psi_2(a,r) = \Big(\int_{I(a,\max(2,|a|,r))\setminus I(a,r)} u(x)^{\alpha/p}|x-a|^{-n(1-\alpha/p+1/p)}\,dx\Big)^p.$$

COROLLARY 1.4. *Let* $1 \le p < \infty$, $0 < \alpha < 1$, $1 \le q \le 1/\alpha$ *and*

$$w(I) = \Big(\int_I u(x)\,dx\Big)^{\alpha}, \quad u \in A_q.$$

Then g *is a pointwise multiplier on* $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ *if and only if* $g \in \mathrm{bmo}_{w^*,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *where*

$$(1.9)\quad w^*(a,r) = \frac{w(a,r)}{1+r^{-n}w(a,r)}.$$

Sections 2 and 3 contain preliminaries and lemmas. In Section 4 we give proofs of the theorem and corollaries. The letter C will always denote a constant, not necessarily the same one.

The author would like to thank Professor Kôzô Yabuta for his many suggestions, and also to thank the referee for his criticism and helpful suggestions.

2. Preliminaries. In this section, we state some simple lemmas. The first three lemmas are shown by elementary calculations. (See for example Spanne [9].)

LEMMA 2.1. $\Big(\int_I |f(x)-f_I|^p\,dx\Big)^{1/p} \le 2\inf_c \Big(\int_I |f(x)-c|^p\,dx\Big)^{1/p}.$

LEMMA 2.2. *If* $|F(z_1)-F(z_2)| \le C|z_1-z_2|$, *then*

$$\mathrm{MO}_{w,p}(F(f(\cdot)),I) \le 2C\,\mathrm{MO}_{w,p}(f,I).$$

LEMMA 2.3. *If* $I_1 \subset I_2$, *then*

$$(2.1)\quad |M(f,I_1)-M(f,I_2)| \le \frac{|I_2|}{|I_1|}\mathrm{MO}(f,I_2)$$

and

$$(2.2)\quad \mathrm{MO}(f,I_1) \le 2\frac{|I_2|}{|I_1|}\mathrm{MO}(f,I_2).$$

LEMMA 2.4. *There is a constant* $C > 0$ *such that*

$$(2.3)\quad |M(f,a,r)-M(f,a,s)| \le C\int_r^{2s}\frac{\mathrm{MO}(f,a,t)}{t}\,dt \quad \text{for } 0<r<s$$

where C *is independent of* f,a,r *and* s.

Proof. By (2.2), we have

$$(2.4)\quad \mathrm{MO}(f,a,r) = (\log 2)^{-1}\int_r^{2r}\frac{\mathrm{MO}(f,a,r)}{t}\,dt \le C\int_r^{2r}\frac{\mathrm{MO}(f,a,t)}{t}\,dt.$$

If $2^{-k-1}s \le r < 2^{-k}s$, then

$$\begin{aligned}|M(f,a,r)-M(f,a,s)| &\le |M(f,a,r)-M(f,a,2^{-k}s)| \\ &\quad + \sum_{j=0}^{k-1}|M(f,a,2^{-j-1}s)-M(f,a,2^{-j}s)| \\ &\le 2^n\sum_{j=0}^{k}\mathrm{MO}(f,a,2^{-j}s) \le C\sum_{j=0}^{k}\int_{2^{-j}s}^{2^{-j+1}s}\frac{\mathrm{MO}(f,a,t)}{t}\,dt\end{aligned}$$

by (2.1) and (2.4). This proves (2.3).

LEMMA 2.5. *Let* $1 \le p < \infty$. *There is a constant* $C > 0$ *such that*

$$\int_{|x-a|<r}\Big(\int_{|x-a|}^{r}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\Big)^p dx \le Cw(a,r)$$

where C *is independent of* a *and* r.

Proof. We denote the volume of the unit ball by σ_n. Then we have

$$\int\limits_{|x-a|<r}\Big(\int\limits_{|x-a|}^{r}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\Big)^p dx=\int\limits_0^r\Big(\int\limits_{\varrho}^{r}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\Big)^p\sigma_n\varrho^{n-1}\,d\varrho$$
$$\le\Big(\int\limits_0^r\Big(\int\limits_0^t\sigma_n\varrho^{n-1}\,d\varrho\Big)^{1/p}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\Big)^p$$
$$=\frac{\sigma_n}{n}\Big(\int\limits_0^r\frac{w(a,t)^{1/p}}{t}\,dt\Big)^p\le Cw(a,r)$$

by Minkowski's inequality and (1.2).

LEMMA 2.6. *Let $1\le p<\infty$. There is a constant $C>0$ such that*

$$\int\limits_r^{2s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\le C\int\limits_r^{s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\quad \text{for } 0<2r\le s$$

where C is independent of a,r and s.

Proof. By a change of variable and (1.1), we have

$$\int\limits_s^{2s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt=\int\limits_{s/2}^{s}\frac{w(a,2t)^{1/p}}{(2t)^{n/p+1}}2\,dt\le\Big(\frac{A}{2^n}\Big)^{1/p}\int\limits_{s/2}^{s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt$$
$$\le\Big(\frac{A}{2^n}\Big)^{1/p}\int\limits_r^{s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt.$$

Therefore

$$\int\limits_r^{2s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\le\Big(1+\Big(\frac{A}{2^n}\Big)^{1/p}\Big)\int\limits_r^{s}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt.$$

3. Lemmas. In this section we show some lemmas needed to prove the theorem. Let $1\le p<\infty$. First, for $a\in\mathbb{R}^n$ and $r>0$, we define

$$W(a,r)=\int\limits_r^1\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt. \tag{3.1}$$

LEMMA 3.1. *For $a\in\mathbb{R}^n$, let*

$$f_a(x)=W(a,|x-a|).$$

Then $\|f_a\|_{\mathrm{BMO}_{w,p}}\le C$ independently of a.

Proof. We show

$$\mathrm{MO}_{w,p}(f_a,b,r)\le C\quad\text{independently of } a,b, \text{ and } r. \tag{3.2}$$

Case 1: $|a-b|<\sqrt{n}\,r$. Since $I(b,r)\subset\{|x-a|\le 2\sqrt{n}\,r\}$, we have

$$\int\limits_{I(b,r)}|f_a(x)-W(a,2\sqrt{n}\,r)|^p\,dx\le\int\limits_{|x-a|\le2\sqrt{n}\,r}\Big(\int\limits_{|x-a|}^{2\sqrt{n}\,r}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\Big)^p dx$$
$$\le Cw(a,2\sqrt{n}\,r)\le Cw(b,2\sqrt{n}\,r)\le Cw(b,r),$$

by Lemma 2.5, (1.1) and (1.3). This inequality and Lemma 2.1 show (3.2).

Case 2: $|a-b|\ge\sqrt{n}\,r$. It follows from (1.3) and (1.4) that

$$w(a,|a-b|)\le Aw(b,|a-b|)\le A^2\Big(\frac{|a-b|}{r}\Big)^{n+p}w(b,r). \tag{3.3}$$

If $x\in I(b,r)$, then $|x-a|$ is comparable to $|a-b|$. Therefore, for $|x-a|\le t\le|a-b|$ or for $|x-a|\ge t\ge|a-b|$,

$$\frac{w(a,t)}{t^{n+p}}\le C\frac{w(a,|a-b|)}{|a-b|^{n+p}}. \tag{3.4}$$

By (3.3) and (3.4), we have

$$\int\limits_{I(b,r)}|f_a(x)-W(a,|a-b|)|^p\,dx=\int\limits_{I(b,r)}\Big|\int\limits_{|x-a|}^{|a-b|}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\Big|^p dx$$
$$\le C\frac{w(b,r)}{r^{n+p}}\int\limits_{I(b,r)}\big||a-b|-|x-a|\big|^p\,dx$$
$$\le C\frac{w(b,r)}{r^{n+p}}\int\limits_{I(b,r)}|x-b|^p\,dx\le Cw(b,r).$$

This inequality and Lemma 2.1 show (3.2).

LEMMA 3.2. *Suppose Ψ is defined by (1.5) and (1.6). Then there is a constant $C>0$ such that*

$$|M(f,a,r)|\le C\|f\|_{\mathrm{bmo}_{w,p}}\Psi(a,r)^{1/p}$$

where C is independent of $f\in\mathrm{bmo}_{w,p}(\mathbb{R}^n)$, a and r.

Proof. We show

$$|M(f,a,r)-M(f,O,1)|\le C\|f\|_{\mathrm{BMO}_{w,p}}\Psi(a,r)^{1/p} \tag{3.5}$$

by using (2.1), (2.3), (2.4) and Lemma 2.6.

Case 1: $\max(r,1)\le|a|/2$. Since $I(a,r)\subset I(a,|a|/2)\subset I(O,3|a|)$ and

$I(O,1)\subset I(O,3|a|)$, we have

$$|M(f,a,r)-M(f,a,|a|/2)|\le C_1\int\limits_r^{|a|}\frac{\mathrm{MO}(f,a,t)}{t}\,dt$$
$$\le C_1\|f\|_{\mathrm{BMO}_{w,p}}\int\limits_r^{|a|}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt$$

and

$$|M(f,a,|a|/2)-M(f,O,3|a|)|+|M(f,O,3|a|)-M(f,O,1)|$$
$$\le 6^n\mathrm{MO}(f,O,3|a|)+C_2\int\limits_1^{6|a|}\frac{\mathrm{MO}(f,O,t)}{t}\,dt\le C_3\int\limits_1^{6|a|}\frac{\mathrm{MO}(f,O,t)}{t}\,dt$$
$$\le C_3\|f\|_{\mathrm{BMO}_{w,p}}\int\limits_1^{6|a|}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt\le C_4\|f\|_{\mathrm{BMO}_{w,p}}\int\limits_1^{|a|}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt.$$

Hence (3.5) follows.

Case 2: $\max(|a|/2,1)\le r$. Since $I(a,r),I(O,1)\subset I(O,5r)$, we have

$$|M(f,a,r)-M(f,O,1)|$$
$$\le|M(f,a,r)-M(f,O,5r)|+|M(f,O,5r)-M(f,O,1)|$$
$$\le 5^n\mathrm{MO}(f,O,5r)+C_5\int\limits_1^{10r}\frac{\mathrm{MO}(f,O,t)}{t}\,dt\le C_6\int\limits_1^{10r}\frac{\mathrm{MO}(f,O,t)}{t}\,dt$$
$$\le C_6\|f\|_{\mathrm{BMO}_{w,p}}\int\limits_1^{10r}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt\le C\|f\|_{\mathrm{BMO}_{w,p}}\int\limits_1^{\max(2,r)}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt.$$

Hence (3.5) follows.

Case 3: $\max(|a|/2,r)\le 1$. Since $I(a,r),I(O,1)\subset I(a,5)$, we have

$$|M(f,a,r)-M(f,O,1)|$$
$$\le|M(f,a,r)-M(f,a,5)|+|M(f,a,5)-M(f,O,1)|$$
$$\le C_7\int\limits_r^{10}\frac{\mathrm{MO}(f,a,t)}{t}\,dt+5^n\mathrm{MO}(f,a,5)\le C_8\int\limits_r^{10}\frac{\mathrm{MO}(f,a,t)}{t}\,dt$$
$$\le C_8\|f\|_{\mathrm{BMO}_{w,p}}\int\limits_r^{10}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt\le C\|f\|_{\mathrm{BMO}_{w,p}}\int\limits_r^{2}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt.$$

Hence (3.5) follows.

The next two lemmas show that the estimate in Lemma 3.2 is sharp.

LEMMA 3.3. *Let*

$$f(x)=\max(-W(O,2),-W(O,|x|))=\int\limits_1^{\max(2,|x|)}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt.$$

Then $f\in\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ and there is a constant $C>0$ such that

$$M(f,a,r)\ge C\Psi_1(a,r)^{1/p}\tag{3.6}$$

where C is independent of $I(a,r)$.

Proof. It follows from Lemmas 3.1 and 2.2 that $f\in\mathrm{bmo}_{w,p}(\mathbb{R}^n)$. Next we show (3.6), by using Lemma 2.6 and the fact that $W(O,r)$ is decreasing with respect to r.

Case 1: $4|a|\le r$. Since $\{|x|\le r/4\}\subset I(a,r)$, we have

$$M(f,a,r)\ge r^{-n}\int\limits_{r/8\le|x|\le r/4}f(x)\,dx$$
$$\ge r^{-n}\int\limits_{r/8\le|x|\le r/4}\max(-W(O,2),-W(O,r/8))\,dx$$
$$=C\int\limits_1^{\max(2,r/8)}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt\ge C'\int\limits_1^{8\max(2,r/8)}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt.$$

This proves (3.6).

Case 2: $4|a|\ge r$. Since $I(a,r/(4\sqrt{n}))\subset\{|x|\ge|a|/2\}$, we have

$$M(f,a,r)\ge r^{-n}\int\limits_{I(a,r/(4\sqrt{n}))}f(x)\,dx$$
$$\ge r^{-n}\int\limits_{I(a,r/(4\sqrt{n}))}\max(-W(O,2),-W(O,|a|/2))\,dx$$
$$=C\int\limits_1^{\max(2,|a|/2)}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt\ge C'\int\limits_1^{8\max(2,|a|/2)}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt.$$

This proves (3.6).

LEMMA 3.4. *For any $I(a,r)$ there is an $f\in\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathrm{bmo}_{w,p}}\le C_1\quad\text{and}\tag{3.7}$$
$$M(f,a,r)\ge C_2\Psi_2(a,r)^{1/p}\tag{3.8}$$

where $C_1>0$ and $C_2>0$ are independent of $I(a,r)$ and f.

Proof. Case 1: $\max(r,1)\le|a|/(2\sqrt{n})$. For $I(a,r)$, let

$$f(x)=W(a,|x-a|)-M(W(a,|x-a|),O,1).$$

Then $M(f,O,1)=0$, so Lemmas 3.1 and 2.2 show (3.7). To prove (3.8), we note that $W(a,r)$ is decreasing with respect to r. Since $|x-a|\ge|a|-|x|\ge|a|-\sqrt{n}/2\ge|a|/2$ for $x\in I(O,1)$, we have

$$M(W(a,|x-a|),O,1)\le W(a,|a|/2).$$

Since $|x-a|\le\sqrt{n}\,r/2$ for $x\in I(a,r)$,

$$M(W(a,|x-a|),a,r)\ge W(a,\sqrt{n}\,r/2).$$

Therefore, by a change of variable, (1.1) and Lemma 2.6, we have

$$M(f,a,r)\ge W(a,\sqrt{n}\,r/2)-W(a,|a|/2)=\int_{\sqrt{n}r/2}^{|a|/2}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt$$

$$\ge C\int_{r}^{|a|/\sqrt{n}}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\ge C'\int_{r}^{|a|}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt.$$

This proves (3.8).

Case 2: $\max(1,|a|/(2\sqrt{n}))\le r$. For $I(a,r)$, let

$$f(x)=\max(W(O,1/(8\sqrt{n}))-W(O,|x|),0),$$

which is independent of $I(a,r)$. There is a cube $I(b,r/4)\subset I(a,r)\cap\{|x|\ge r/4\}$. Since $1/(8\sqrt{n})\le r/(8\sqrt{n})\le r/4\le|x|$ for $x\in I(b,r/4)$, we have

$$f(x)\ge W(O,r/(8\sqrt{n}))-W(O,r/4)=\int_{r/(8\sqrt{n})}^{r/4}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt$$

$$\ge C\int_{r}^{2\sqrt{n}r}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt\quad\text{for } x\in I(b,r/4)$$

and

$$M(f,a,r)\ge 4^{-n}M(f,b,r/4)\ge 4^{-n}C\int_{r}^{2\sqrt{n}r}\frac{w(O,t)^{1/p}}{t^{n/p+1}}\,dt.$$

For $r\le t\le 2\sqrt{n}\,r$ and for $|a|\le 2\sqrt{n}\,r$, $w(O,t)$ is comparable to $w(a,t)$. Then

$$M(f,a,r)\ge C'\int_{r}^{2\sqrt{n}r}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt.$$

This proves (3.8).

Case 3: $\max(r,|a|/(2\sqrt{n}))\le 1$. For $I(a,r)$, let

$$f(x)=\max(W(a,|x-a|)-W(a,n),0).$$

Then $\|f\|_{\mathrm{BMO}_{w,p}}$ is independent of I, and

$$|M(f,O,1)|\le\Big(\int_{I(O,1)}|f|^p\,dx\Big)^{1/p}$$

$$\le\Big(\int_{|x-a|\le n}\Big(\int_{|x-a|}^{n}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\Big)^p\,dx\Big)^{1/p}$$

$$\le Cw(a,n)^{1/p}\le C'w(O,n)^{1/p}.$$

This proves (3.7). Since $|x-a|\le\sqrt{n}\,r/2$ for $x\in I(a,r)$, we have

$$M(f,a,r)\ge W(a,\sqrt{n}\,r/2)-W(a,n)$$

$$=\int_{\sqrt{n}r/2}^{n}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\ge C\int_{r}^{2\sqrt{n}}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt.$$

This proves (3.8).

LEMMA 3.5. *Suppose $f\in\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ and $g\in L^\infty(\mathbb{R}^n)$. Then fg belongs to $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$ if and only if*

$$F(f,g)=\sup_I|f_I|\mathrm{MO}_{w,p}(g,I)<\infty.$$

In this case,

$$\big|\|fg\|_{\mathrm{BMO}_{w,p}}-F(f,g)\big|\le 2\|f\|_{\mathrm{BMO}_{w,p}}\|g\|_\infty. \tag{3.9}$$

Proof. For any cube I, we have

$$\big|\|(fg)(\cdot)-(fg)_I\|_{L^p(I)}-|f_I|\|g(\cdot)-g_I\|_{L^p(I)}\big|$$
$$\le\|(fg)(\cdot)-(fg)_I-f_Ig(\cdot)+f_Ig_I\|_{L^p(I)}$$
$$\le\|(f(\cdot)-f_I)g(\cdot)\|_{L^p(I)}+|(fg)_I-f_Ig_I||I|^{1/p}$$
$$=\|(f(\cdot)-f_I)g(\cdot)\|_{L^p(I)}+\Big|\frac{1}{|I|}\int_I((fg)(x)-f_Ig(x))\,dx\Big||I|^{1/p}$$
$$\le 2\Big(\int_I|(f(x)-f_I)g(x)|^p\,dx\Big)^{1/p}$$
$$\le 2w(I)^{1/p}\mathrm{MO}_{w,p}(f,I)\|g\|_\infty.$$

Hence

$$|\mathrm{MO}_{w,p}(fg,I)-|f_I|\mathrm{MO}_{w,p}(g,I)|\le 2\mathrm{MO}_{w,p}(f,I)\|g\|_\infty,$$

which shows (3.9).

4. Proofs of the theorem and corollaries. We write $\Psi(I)=\Psi(a,r)$ for $I=I(a,r)$.

Proof of Theorem. Suppose $g\in \mathrm{bmo}_{w^*,p}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. For any $f\in \mathrm{bmo}_{w,p}(\mathbb{R}^n)$ and for any I, by Lemma 3.2, we have

$$\begin{aligned}|f_I|\mathrm{MO}_{w,p}(g,I)&\le C\|f\|_{\mathrm{bmo}_{w,p}}\Psi(I)^{1/p}\mathrm{MO}_{w,p}(g,I)\\&\le C\|f\|_{\mathrm{bmo}_{w,p}}\|g\|_{\mathrm{BMO}_{w^*,p}}<\infty.\end{aligned}$$

Therefore, by Lemma 3.5, $fg\in \mathrm{bmo}_{w,p}(\mathbb{R}^n)$ and

$$\|fg\|_{\mathrm{BMO}_{w,p}}\le C\|f\|_{\mathrm{bmo}_{w,p}}\|g\|_{\mathrm{BMO}_{w^*,p}}+2\|f\|_{\mathrm{BMO}_{w,p}}\|g\|_\infty .$$

Since $|M(fg,O,1)|\le \|g\|_\infty(\mathrm{MO}(f,O,1)+|M(f,O,1)|)$, we have

$$\|fg\|_{\mathrm{bmo}_{w,p}}\le C(\|g\|_{\mathrm{BMO}_{w^*,p}}+\|g\|_\infty)\|f\|_{\mathrm{bmo}_{w,p}},$$

which shows that g is a pointwise multiplier on $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$, and

$$\|g\|_{\mathrm{Op}}\le C(\|g\|_{\mathrm{BMO}_{w^*,p}}+\|g\|_\infty)$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of g.

Conversely, suppose g is a pointwise multiplier on $\mathrm{bmo}_{w,p}(\mathbb{R}^n)$. First we show $g\in L^\infty(\mathbb{R}^n)$. For any cube $I=I(a,r)$ with $r<1$, we define $h(x)$ as follows:

$$h(x)=\max(W(a,|x-a|)-W(a,r),0)=\max\Big(\int_{|x-a|}^{r}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt,0\Big).$$

Then it follows from Lemmas 3.1 and 2.2 that $\|h\|_{\mathrm{BMO}_{w,p}}\le C$ independently of I. For $|a|>1+\sqrt{n}/2$, $M(h,O,1)=0$, since $I(O,1)$ and the support of h are disjoint. For $|a|\le 1+\sqrt{n}/2$, by Lemma 2.5,

$$\begin{aligned}|M(h,O,1)|&\le\Big(\int_{I(O,1)}|h|^p\,dx\Big)^{1/p}\\&\le\Big(\int_{|x-a|<1}\Big(\int_{|x-a|}^{1}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\Big)^p dx\Big)^{1/p}\\&\le Cw(a,1)^{1/p}\le Cw(O,1)^{1/p}.\end{aligned}$$

Hence $\|h\|_{\mathrm{bmo}_{w,p}}\le C$ independently of I. Now, if $|x-a|<r/2$, then

$$h(x)\ge\int_{r/2}^{r}\frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt\ge Cw(a,r)^{1/p}\int_{r/2}^{r}\frac{1}{t^{n/p+1}}\,dt=C\frac{w(a,r)^{1/p}}{r^{n/p}}.$$

Therefore, by considering the support of h, for $\sigma=M(gh,a,4r)$,

$$\begin{aligned}\int_{I(a,4r)}|gh(x)-\sigma|^p\,dx&\ge\int_{|x-a|<r/2}|gh(x)-\sigma|^p\,dx+\int_{I(a,4r)\setminus I(a,2r)}|\sigma|^p\,dx\\&\ge\int_{|x-a|<r/2}(|gh(x)-\sigma|^p+|\sigma|^p)\,dx\\&\ge\int_{|x-a|<r/2}2^{1-p}|gh(x)|^p\,dx\\&\ge C\frac{w(a,r)}{r^n}\int_{|x-a|<r/2}|g(x)|^p\,dx.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{r^n}\int_{|x-a|<r/2}|g(x)|^p\,dx&\le C\frac{1}{w(a,r)}\int_{I(a,4r)}|gh(x)-\sigma|^p\,dx\\&\le C(\|gh\|_{\mathrm{bmo}_{w,p}})^p\le C(\|g\|_{\mathrm{Op}})^p.\end{aligned}$$

Letting r tend to zero, we have

$$|g(a)|\le C\|g\|_{\mathrm{Op}}\quad\text{a.e.}\quad\text{and}\quad\|g\|_\infty\le C\|g\|_{\mathrm{Op}}.$$

Second, we show $g\in\mathrm{bmo}_{w^*,p}(\mathbb{R}^n)$. By Lemma 3.5, we have

$$\begin{aligned}\sup_I|f_I|\mathrm{MO}_{w,p}(g,I)&\le\|fg\|_{\mathrm{BMO}_{w,p}}+2\|f\|_{\mathrm{BMO}_{w,p}}\|g\|_\infty\\&\le(\|g\|_{\mathrm{Op}}+2\|g\|_\infty)\|f\|_{\mathrm{bmo}_{w,p}}\le C\|g\|_{\mathrm{Op}}\|f\|_{\mathrm{bmo}_{w,p}},\end{aligned}$$

for any $f\in\mathrm{bmo}_{w,p}(\mathbb{R}^n)$. Applying Lemmas 3.3 and 3.4, we have

$$\Psi_i(I)^{1/p}\mathrm{MO}_{w,p}(g,I)\le C\|g\|_{\mathrm{Op}}\quad\text{for any } I,\quad i=1,2,$$

which proves $g\in\mathrm{bmo}_{w^*,p}(\mathbb{R}^n)$ and $\|g\|_{\mathrm{BMO}_{w^*,p}}\le C\|g\|_{\mathrm{Op}}$. The proof is complete.

Proof of Corollary 1.1. In this case, w satisfies (1.1) to (1.4). Since

$$\Psi_1(a,r)=\Big(\frac{p}{n-\alpha}(1-\max(2,|a|,r)^{-(n-\alpha)/p})\Big)^p\quad\text{and}$$

$$\Psi_2(a,r)=\Big(\frac{p}{n-\alpha}(r^{-(n-\alpha)/p}-\max(2,|a|,r)^{-(n-\alpha)/p})\Big)^p,$$

$\Psi_1(a,r)$ is comparable to 1, $\Psi_2(a,r)$ is comparable to $r^{-(n-\alpha)}$ for $r\le 1$, and $\Psi_2(a,r)$ is less than a constant for $r>1$. Therefore, $w^*(a,r)=(w/\Psi)(a,r)$ is comparable to r^n $(r\le 1)$, r^α $(r>1)$. On the other hand, if g is bounded, then $\mathrm{MO}_{w,p}(g,a,r)\le\mathrm{MO}_{w^*,p}(g,a,r)\le 2\|g\|_\infty$ for $r\le 1$.

Proof of Corollary 1.2. This corollary is obtained from Corollary 1.1, since, in this case, we can assume that w is defined only for $r < 1$.

In order to prove the last two corollaries, we state some basic properties of A_p weights. (See for example [3].)

LEMMA 4.1. *If u belongs to A_p, $1 \le p < \infty$, then there are constants $C > 0$ and $\delta > 0$ such that*

$$C^{-1}\left(\frac{|E|}{|I|}\right)^p \le \frac{\int_E u(x)\,dx}{\int_I u(x)\,dx} \le C\left(\frac{|E|}{|I|}\right)^\delta$$

for any I and for any measurable set $E \subset I$.

LEMMA 4.2. *If u belongs to A_p, $1 \le p < \infty$, then for $0 < \alpha \le 1$ there is a constant $C > 0$ such that*

$$\frac{1}{|I|}\int_I u(x)^\alpha\,dx \le \left(\frac{1}{|I|}\int_I u(x)\,dx\right)^\alpha \le C\frac{1}{|I|}\int_I u(x)^\alpha\,dx\,.$$

LEMMA 4.3. *If u belongs to A_p, $1 \le p < \infty$, then for $\beta > 0$ and for $0 < \gamma \le 1$ there is a constant $C > 0$ such that*

$$C^{-1} \le \frac{\int_r^{2r}(\int_{I(a,t)} u(x)\,dx)^\gamma t^{-n\beta-1}\,dt}{\int_{I(a,2r)\setminus I(a,r)} u(x)^\gamma |x-a|^{-n(1-\gamma+\beta)}\,dx} \le C\,,$$

for any $a \in \mathbb{R}^n$ and $r > 0$.

LEMMA 4.4. *If u belongs to A_p, $1 \le p < \infty$, then for $p' \ge p$ and $p' > 1$ there is a constant $C > 0$ such that*

$$C^{-1} \le \frac{\int_{I(a,R)\setminus I(a,r)} u(x)|x-a|^{-np'}\,dx}{r^{-np'}\int_{I(a,r)} u(x)\,dx} \le C\,,$$

for any $a \in \mathbb{R}^n$ and $0 < 2r \le R$.

Proof of Corollary 1.3. By Lemma 4.1, w satisfies (1.1) to (1.4). It follows from Lemma 4.3 that

$$\int_r^R \frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt$$

is comparable to

$$\int_{I(a,R)\setminus I(a,r)} u(x)^{\alpha/p}|x-a|^{-n(1-\alpha/p+1/p)}\,dx\,,$$

for $0 < 2r \le R$. Therefore, we have (1.7) and (1.8).

Proof of Corollary 1.4. If u belongs to A_q, then $u^{\alpha/p}$ belongs to $A_{(q-1)\alpha/p+1}$. Therefore, by Lemmas 4.3, 4.4 and 4.2, the following are comparable:

$$\int_r^R \frac{w(a,t)^{1/p}}{t^{n/p+1}}\,dt, \qquad \int_{I(a,R)\setminus I(a,r)} u(x)^{\alpha/p}|x-a|^{-n(1-\alpha/p+1/p)}\,dx,$$

$$r^{-n(1-\alpha/p+1/p)}\int_{I(a,r)} u(x)^{\alpha/p}\,dx, \qquad r^{-n/p}w(a,r)^{1/p}\,,$$

for $0 < 2r \le R$. This shows (1.9).

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