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On the eigenvalue asymptotics of certain Schrödinger operators

by

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Abstract. Subelliptic estimates on nilpotent Lie groups and the Cwikel–Lieb–Rosenblum inequality are used to estimate the number of eigenvalues for Schrödinger operators with polynomial potentials.

Introduction. The Bohr–Sommerfeld quantization principle, according to which volume h^d in the phase space corresponds to one bound state of the quantum system, has been fully mathematically expressed in the form of the Cwikel–Lieb–Rosenblum inequality.

For the Schrödinger operator $-\Delta+V$, denote the dimension of the image of the spectral projector $P(-\infty,\lambda)$ by $N(\lambda,V)$ and let

$$\operatorname{Vol}(\lambda,V)=|\{(x,\xi):\xi^2+V(x)<\lambda\}|\,.$$

THE CWIKEL–LIEB–ROSENBLUM INEQUALITY [5]. *For $d\ge 3$, there exists a constant $C=C(d)$ such that for every potential V on $\mathbb{R}^d$ and for every λ, $N(\lambda,V)\le C\operatorname{Vol}(\lambda,V)$.*

If p is a positive polynomial on $\mathbb{R}^d$ such that for every $x\neq 0$ there exists y with $p(x+y)\neq p(y)$ then, as is proved in [1], $N(\lambda,p)<\infty$ for every $\lambda>0$, but there exists a wide range of polynomials (e.g. $p(x)=x_1^2x_2^2\ldots x_d^2$) for which $\operatorname{Vol}(\lambda,p)=\infty$. Let $n(\lambda,r,p)$ denote the maximal number of disjoint balls with radii $r>0$ contained in the domain $\{x:p(x)<\lambda\}$. The following theorem of Fefferman [2] gives a qualitative description of the eigenvalue asymptotics of $-\Delta+p$.

THE FEFFERMAN ESTIMATES. *There exists a constant C, depending on the dimension d, and a constant c, depending on d and the degree of p, such that for every $\lambda>0$,*

$$n(\lambda,C\lambda^{-1/2},p)\le N(\lambda,p)\le n(\lambda,c\lambda^{-1/2},p)\,.$$

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The aim of this paper is to estimate the number $n(\lambda, r, p)$ and thus to give a quantitative upper bound for $N(\lambda, p)$.

The estimates. Let G be a stratified nilpotent Lie group, i.e. there is a decomposition of the Lie algebra of G as a vector space sum $\mathfrak{g} = V_1 \oplus \ldots \oplus V_m$ such that $[V_1, V_j] = V_{j+1}$ for $1 \le j < m$ and $[V_1, V_m] = \{0\}$. Fix any linear basis $X_1, \ldots, X_n$ of V_1 and set $\mathcal{H} = -X_1^2 - \ldots - X_{n-1}^2 - X_n^2$. The following lemma is proved in [3].

LEMMA 1. *Let $I = (i_1, \ldots, i_k)$ be a k-tuple of integers with k arbitrary and $1 \le i_j \le n$ for $j = 1, \ldots, k$. Set $|I| = k$ and $X_I = X_{i_1} \ldots X_{i_k}$. For every natural k there exists a constant C such that*

$$\|(\mathcal{H}+1)^{k/2} f\| \ge C \sum_{|I| \le k} \|X_I f\|$$

for every $f \in C_c^\infty(G)$, where $\|\cdot\|$ denotes L^2-norm.

LEMMA 2. *Let p be a polynomial on $\mathbb{R}^d$. There exists a constant c such that if $\deg(p) \le N$ then*

$$((-\Delta + p^2)\phi, \phi) \ge c \int \sum_{|I| \le N} |\partial_I p(x)|^{2/(|I|+1)} |\phi(x)|^2 \, dx - \|\phi\|^2$$

for every $\phi \in C_c^\infty(\mathbb{R}^d)$. The constant c depends only on d and N.

Proof. Let $\mathfrak{g}$ be a free nilpotent Lie algebra with free generators $X_1, \ldots, X_{d+1}$ and with nilpotence class $N+1$. Let $\mathcal{H} = -X_1^2 - \ldots - X_d^2 - X_{d+1}^2$ be viewed as an operator on $L^2(\exp(\mathfrak{g}))$. By Lemma 1 there exists a constant c such that

$$\|(\mathcal{H}+1)^{|I|/2} f\| \ge c \|X_I f\| . \tag{1}$$

Let π be the representation of G defined by

$$X_j \mapsto \begin{cases} \partial_j, & j = 1, \ldots, d, \\ ip, & j = d+1. \end{cases}$$

Then (1) implies that

$$\|\pi_{(\mathcal{H}+1)^{|I|/2}} \phi\| \ge c_I \|\pi_{X_I} \phi\|$$

for $\phi \in C_c^\infty(\mathbb{R}^d)$. Hence, if we put $Y = [X_{i_1}, \ldots [X_{i_k}, X_{d+1}] \ldots]$, with $1 \le i_j \le d$, we obtain $\|\pi_{(\mathcal{H}+1)^{|I|/2}} \phi\| \ge c \|\pi_Y \phi\|$, and thus

$$((-\Delta + p^2 + 1)^{|I|+1} \phi, \phi) \ge c(|\partial_I p|^2 \phi, \phi) .$$

This implies (see [4]) that

$$((-\Delta + p^2 + 1)\phi, \phi) \ge c \int |\partial_I p(x)|^{2/(|I|+1)} |\phi(x)|^2 \, dx ,$$

which completes the proof.

PROPOSITION 1. *For $d \ge 3$ and any natural number N there exist constants $C = C(d)$ and $c = c(d, N)$ such that, for every polynomial p on $\mathbb{R}^d$, if $\deg(p) \le N$ then*

$$N(\lambda, p^2) \le C \left| \left\{ (x, \xi) : \xi^2 + p(x)^2 + c \sum_{|I| \le N} |\partial_I p(x)|^{2/(|I|+1)} \le 2(\lambda + 1) \right\} \right|$$

for every $\lambda > 0$.

Proof. Let c be the constant from the previous lemma. For $\lambda > 0$ we define

$$\Upsilon_\lambda(x) = \begin{cases} 0 & \text{if } \sum_{|I| \le N} |\partial_I p(x)|^{2/(|I|+1)} > (2/c)(\lambda + 1), \\ (c/2) \sum_{|I| \le N} |\partial_I p(x)|^{2/(|I|+1)} - \lambda - 1 & \text{for the remaining } x. \end{cases}$$

Hence, using Lemma 2 we obtain

$$\begin{aligned} ((-\Delta + p^2)\phi, \phi) &= ((-2^{-1}\Delta + 2^{-1}p^2 - \Upsilon_\lambda)\phi, \phi) \\ &\quad + ((-2^{-1}\Delta + 2^{-1}p^2 + \Upsilon_\lambda)\phi, \phi) \\ &\ge \lambda \|\phi\|^2 + ((-2^{-1}\Delta + 2^{-1}p^2 + \Upsilon_\lambda)\phi, \phi) \end{aligned}$$

for $\phi \in C_c^\infty$. By the minimax principle

$$N(\lambda, p^2) \le N(0, -2^{-1}\Delta + 2^{-1}p^2 + \Upsilon_\lambda) ,$$

and the Cwikel–Lieb–Rosenblum inequality finishes the proof.

PROPOSITION 2. *For $d \ge 3$ and any natural number N there exist constants $C = C(d)$, $c_1(d)$ and $c_2(d, N)$ such that for every polynomial p on $\mathbb{R}^d$, if $\deg(p) \le N$ then*

$$n(\lambda, r, |p|) \le C \left| \left\{ (x, \xi) : \xi^2 + c_1 \frac{p(x)^2}{\lambda^2 r^2} + c_2 \sum_{|I| \le N} \left| \frac{\partial_I p(x)}{\lambda r} \right|^{\frac{2}{|I|+1}} \le \frac{c_1}{r^2} + 2 \right\} \right|$$

for any λ, $r > 0$.

Proof. Noticing that

$$n(\lambda, r, |p|) = n(cr^{-2}, r, cr^{-2}\lambda^{-2}p^2) ,$$

we can use the lower bound in the Fefferman estimates and Proposition 1.

The following theorem is an easy consequence of the upper bound in the Fefferman estimates and of Proposition 2.

THEOREM. *For $d \ge 3$ and any natural number N there exist constants $C = C(d)$, $c_1 = c_1(d, N)$ and $c_2 = c_2(d, N)$ such that for every positive polynomial p on $\mathbb{R}^d$, if $\deg(p) \le N$ then*

$$N(\lambda,p) \le C\Big|\Big\{(x,\xi):\xi^2+p(x)+c_1\sum_{|I|\le N}\lambda^{\frac{2|I|+1}{2|I|+2}}|\partial_I p(x)|^{\frac{1}{|I|+1}}\le c_2(\lambda+1)\Big\}\Big|$$

for any $\lambda>0$.

Proof. In order to replace $p(x)^2$ by $p(x)$ we use the inequality

$$\sum_{i=1}^m a_i^2 \le \Big(\sum_{i=1}^m |a_i|\Big)^2.$$

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[6] D. Beck, *Introduction to Dynamical Systems*, Vol. 2, Progr. Math. 54, Birkhäuser, Basel 1978.

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