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On the joint spectral radii of commuting Banach algebra elements

by

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Abstract. Some inequalities are proved between the geometric joint spectral radius (cf. [3]) and the joint spectral radius as defined in [7] of finite commuting families of Banach algebra elements.

Let A be a complex Banach algebra with the unit denoted by 1. Let $a = (a_1, \ldots, a_n)$ be an n-tuple of pairwise commuting elements of A. The symbol $\sigma(a)$ will stand for the *Harte spectrum* of a, i.e. $(\lambda_1, \ldots, \lambda_n) \notin \sigma(a)$ if there exist elements $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ in A such that $\sum_{j=1}^{n} u_j(a_j - \lambda_j) = 1$ and $\sum_{j=1}^{n} (a_j - \lambda_j)v_j = 1$ (here we write for simplicity $a_j - \lambda_j$ instead of $a_j - \lambda_j 1$). We shall also need the *left approximate point spectrum* of a, i.e. the set

$$\tau_l(a) = \Big\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \inf_{\|b\|=1} \sum_{j=1}^{n} \|(a_j - \lambda_j)b\| = 0\Big\}.$$

The *geometric (joint) spectral radius* of a is defined (cf. [3]) to be the number

$$r(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}$$

where

$$|\lambda| = |(\lambda_1, \ldots, \lambda_n)| = \Big(\sum_{j=1}^{n} |\lambda_j|^2\Big)^{1/2}.$$

As was shown in [3] (cf. also [8]) $r(a)$ does not depend upon the choice of a joint spectrum of a. In particular, the Harte spectrum $\sigma(a)$ can be replaced by the left approximate point spectrum of a in the above formula without changing the value of $r(a)$.

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In the case when A is a C^*-algebra the following formula was proved in [6]:

$$r(a)=\lim_{s\to\infty}\Big\|\sum_{\alpha\in\mathbb{Z}_+^n,|\alpha|=s}\frac{s!}{\alpha!}a^{*\alpha}a^{\alpha}\Big\|^{1/(2s)}$$

where $\mathbb{Z}_+^n$ is the set of all multiindices $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_j\ge 0$ $(j=1,\ldots,n)$, and, as usual, $|\alpha|=\alpha_1+\ldots+\alpha_n$, $\alpha!=\alpha_1!\ldots\alpha_n!$, $a^\alpha=a_1^{\alpha_1}\ldots a_n^{\alpha_n}$, and $a^*=(a_1^*,\ldots,a_n^*)$. In fact, the proof of this result in [6] was done for a commuting n-tuple of Hilbert space operators but it goes exactly in the same way for arbitrary commuting C^*-algebra elements.

Another possible definition of a joint spectral radius of a commuting n-tuple of normed algebra elements is given in [7]. Namely, let the *joint spectral radius* of $a=(a_1,\ldots,a_n)$ be the number

$$\widehat{r}(a)=\lim_{s\to\infty}\max_{\alpha\in\mathbb{Z}_+^n,|\alpha|=s}\|a^\alpha\|^{1/s}.$$

It is an immediate consequence of this definition that equivalent algebra norms give the same joint spectral radius. Notice also that for a single element a of a Banach algebra A we have

$$r(a)=\widehat{r}(a)=\text{the spectral radius of } a.$$

In this paper we show how the above mentioned notions of joint spectral radii are related to each other. More precisely, we prove the following:

THEOREM 1. *Let $a=(a_1,\ldots,a_n)$ be a mutually commuting n-tuple of elements of a complex unital Banach algebra. Then*

$$\frac{1}{\sqrt{n}}r(a)\le\widehat{r}(a)\le r(a).$$

Before proceeding to the proof let us define one more notion. Let $a=(a_1,\ldots,a_n)$ be an n-tuple of mutually commuting elements of a normed algebra. Set

$$r_*(a)=\lim_{s\to\infty}\max_{\alpha\in\mathbb{Z}_+^n,|\alpha|=s}r(a^\alpha)^{1/s},$$

where $r(a^\alpha)=r(a_1^{\alpha_1}\ldots a_n^{\alpha_n})$ is the usual spectral radius of a^α, i.e. $r(a^\alpha)=\lim_{k\to\infty}\|a^{\alpha k}\|^{1/k}$. Notice that the above limit exists since the elements a_j $(j=1,\ldots,n)$ are commuting.

The following lemma seems to be of independent interest.

LEMMA. *For every n-tuple $a=(a_1,\ldots,a_n)$ of pairwise commuting elements of a normed algebra A, $\widehat{r}(a)=r_*(a)$.*

Proof. Since $r(a^\alpha)\le\|a^\alpha\|$ we obviously have $r_*(a)\le\widehat{r}(a)$. To prove the opposite inequality it is enough to show that $r_*(a)<1$ implies $\widehat{r}(a)\le 1$.

Indeed, take $\varepsilon>0$ and $b_j=a_j/(r_*(a)+\varepsilon)$, $j=1,\ldots,n$. Then $r_*(b)=r_*(b_1,\ldots,b_n)<1$ and thus $\widehat{r}(b)\le 1$. This implies $\widehat{r}(a)\le r_*(a)+\varepsilon$ for all ε and finally $\widehat{r}(a)\le r_*(a)$ as claimed.

Now assume that $r_*(a)<1$. So there exists an integer s_0 such that for all $s\ge s_0$ and all $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|=s$ we have $r(a^\alpha)<1$. This implies $\|a^{\alpha k}\|^{1/k}<1$ for all $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|=s_0$ and $k\ge k_0$ (k_0 is the same for all α's with $|\alpha|=s_0$). In particular,

$$\|a_j^{s_0k}\|<1\quad\text{ for } j=1,\ldots,n \text{ and } k\ge k_0.$$

Let

$$K_j=\max\{1,\|a_j\|,\|a_j^2\|,\ldots,\|a_j^{s_0k_0-1}\|\}\quad\text{and}\quad K=\max_{1\le j\le n}K_j.$$

Now take $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}_+^n$. Then $\alpha_j=m_j(s_0k_0)+l_j$, where $m_j\ge 0$ and $0\le l_j<s_0k_0$ $(j=1,\ldots,n)$. Thus we get

$$\begin{aligned}\|a^\alpha\|&=\|a_1^{\alpha_1}\ldots a_n^{\alpha_n}\|=\|a_1^{m_1s_0k_0+l_1}\ldots a_n^{m_ns_0k_0+l_n}\|\\&\le\|a_1^{s_0m_1k_0}\|\ldots\|a_n^{s_0m_nk_0}\|\|a_1^{l_1}\|\ldots\|a_n^{l_n}\|\le K^n\end{aligned}$$

(here it is assumed that $\|a_j^0\|=1$ if the algebra A has no unit) and so the set $\{a_1^{\alpha_1}\ldots a_n^{\alpha_n}:(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}_+^n\}$ is bounded. By the lemma of Rota and Strang ([7], cf. also [2], p. 18, Thm. 1) there exists an equivalent algebra norm N on A such that $N(a^\alpha)\le 1$ for all $\alpha\in\mathbb{Z}_+^n$. Since $\widehat{r}(a)$ does not depend upon the choice of a particular algebra norm on A equivalent to the given one we obtain $\widehat{r}(a)\le 1$ and the proof is complete.

Remark. Berger and Wang ([1], Thm. IV) showed that the assertion of the lemma is true for every bounded family of $n\times n$ matrices. However, it is not true in the general case of an arbitrary Banach algebra even for a family of mutually commuting elements. This was observed by the referee who supplied the following example:

EXAMPLE. Let A be a commutative Banach algebra generated by countably many elements $x_1,x_2,\ldots$ satisfying $x_i^2=0$ $(i=1,2,\ldots)$. The elements of A are of the form $y=\sum_{n\in\mathbb{N}}\sum_{i_1<\ldots<i_n}\alpha_{i_1\ldots i_n}x_{i_1}\ldots x_{i_n}$ with the norm $\|y\|=\sum_{n\in\mathbb{N}}\sum_{i_1<\ldots<i_n}|\alpha_{i_1\ldots i_n}|$. Obviously, the set $M=\{x_i:i=1,2,\ldots\}$ is bounded with $\widehat{r}(M)=1$ and $r_*(M)=0$.

Proof of Theorem 1. First we prove that $(1/\sqrt{n})\,r(a)\le\widehat{r}(a)$. Take $\lambda=(\lambda_1,\ldots,\lambda_n)\in\tau_l(a)$. Then there exists a sequence (b_k) in A such that $\|b_k\|=1$ for all k and $\|(a_j-\lambda_j)b_k\|\to 0$ as $k\to\infty$ $(j=1,\ldots,n)$. Thus for all $\alpha\in\mathbb{Z}_+^n$, $|\alpha|=s$, we obtain

$$\|a_1^{\alpha_1}\ldots a_n^{\alpha_n}\|\ge\|(a_1^{\alpha_1}\ldots a_n^{\alpha_n})b_k\|\to|\lambda_1^{\alpha_1}\ldots\lambda_n^{\alpha_n}|$$

as $k \to \infty$. This implies $\|a_1^{\alpha_1} \ldots a_n^{\alpha_n}\| \geq |\lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n}|$. Notice that

$$\max_{\alpha \in \mathbb{Z}_+^n, |\alpha| = s} |\lambda_1|^{2\alpha_1} \ldots |\lambda_n|^{2\alpha_n} \geq \frac{1}{n^s} \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = s} \frac{s!}{\alpha!} |\lambda_1|^{2\alpha_1} \ldots |\lambda_n|^{2\alpha_n} = \frac{1}{n^s} |\lambda|^{2s}.$$

Hence we get

$$\max_{\alpha \in \mathbb{Z}_+^n, |\alpha| = s} \|a_1^{\alpha_1} \ldots a_n^{\alpha_n}\| \geq \frac{1}{\sqrt{n^s}} |\lambda|^s.$$

As $\lambda \in \tau_l(a)$ was arbitrary the above implies

$$\max_{\alpha \in \mathbb{Z}_+^n, |\alpha| = s} \|a_1^{\alpha_1} \ldots a_n^{\alpha_n}\|^{1/s} \geq \frac{1}{\sqrt{n}} r(a),$$

which finally gives $\widehat{r}(a) \geq (1/\sqrt{n})\, r(a)$ as claimed.

Now we prove $\widehat{r}(a) \leq r(a)$. In view of the lemma it is enough to show that $r_*(a) \leq r(a)$. Let $\alpha \in \mathbb{Z}_+^n$, $|\alpha| = s$. By the spectral mapping property of the Harte spectrum σ ([4], Thm. 4.3) we have

$$\sigma(a^\alpha) = p\sigma(a),$$

where $p(z) = z^\alpha$, i.e. $p(z_1, \ldots, z_n) = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. So suppose $\lambda \in \sigma(a^\alpha)$. Then there exists $\mu = (\mu_1, \ldots, \mu_n) \in \sigma(a_1, \ldots, a_n)$ such that $\lambda = \mu_1^{\alpha_1} \ldots \mu_n^{\alpha_n}$. Hence

$$|\lambda|^2 = |\mu_1|^{2\alpha_1} \ldots |\mu_n|^{2\alpha_n} \leq \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = s} \frac{s!}{\alpha!} |\mu_1|^{2\alpha_1} \ldots |\mu_n|^{2\alpha_n} = |\mu|^{2s}$$

and therefore $|\lambda| \leq |\mu|^s \leq r(a)^s$ for all $\lambda \in \sigma(a^\alpha)$. Thus $r(a^\alpha) \leq r(a)^s$ and consequently

$$\max_{\alpha \in \mathbb{Z}_+^n, |\alpha| = s} r(a^\alpha) \leq r(a)^s.$$

This finally gives $r_*(a) \leq r(a)$ and concludes the proof.

Remarks. 1. In the case when A is a C^*-algebra it is possible to prove $\widehat{r}(a) \leq r(a)$ directly without using the lemma.

Namely, by Theorem 1 of [6] we have

$$r(a) = \lim_{s \to \infty} \left\| \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = s} \frac{s!}{\alpha!} a^{*\alpha} a^\alpha \right\|^{1/(2s)}.$$

Elements $a^{*\alpha} a^\alpha$ of a C^*-algebra A are positive and so is the sum within the norm signs. Therefore (see [5], p. 269, Thm. 7.77(VII))

$$\|a^\alpha\|^2 = \|a^{*\alpha} a^\alpha\| \leq \frac{s!}{\alpha!} \|a^{*\alpha} a^\alpha\| \leq \left\| \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = s} \frac{s!}{\alpha!} a^{*\alpha} a^\alpha \right\|.$$

This implies

$$\lim_{s \to \infty} \max_{\alpha \in \mathbb{Z}_+^n, |\alpha| = s} \|a^\alpha\|^{1/s} \leq \lim_{s \to \infty} \left\| \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = s} \frac{s!}{\alpha!} a^{*\alpha} a^\alpha \right\|^{1/(2s)},$$

which was to be proved.

2. The following example shows that both inequalities in Theorem 1 may be strict.

Let A be the algebra of all 2×2 matrices with complex entries. Let $a = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 2 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 2 & -3 \\ 0 & 1 \end{smallmatrix}\right)$. Then $ab = ba$ and moreover $\sigma(a, b) = \{(1, 2), (2, 1)\}$. Hence $r(a, b) = \sqrt{5}$. It is easy to see that $r(a^\alpha b^\beta) = \max\{2^\alpha, 2^\beta\}$ for all $\alpha, \beta \geq 0$. Therefore, in view of the lemma (cf. also [1], Thm. IV),

$$\widehat{r}(a, b) = \lim_{s \to \infty} \max_{\alpha + \beta = s} (r(a^\alpha b^\beta))^{1/s} = 2.$$

Finally, we get

$$\frac{1}{\sqrt{2}} r(a, b) = \sqrt{\frac{5}{2}} < 2 = \widehat{r}(a, b) < r(a, b) = \sqrt{5}.$$

3. It is easy to give examples showing that both constants in the inequalities of Theorem 1 are the best possible.

Namely, let as before A be the algebra of all complex 2×2 matrices. If we take $a = 1$ (= the identity matrix) and $b = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then $b^2 = 1$, and $\sigma(a, b) = \{(1, 1), (1, -1)\}$. Therefore $r(a, b) = \sqrt{2}$. On the other hand, $\max_{\alpha + \beta = s} \|a^\alpha b^\beta\| = \|b\|$ for all $s \geq 1$, which gives

$$\widehat{r}(a, b) = \lim_{s \to \infty} \max_{\alpha + \beta = s} \|a^\alpha b^\beta\|^{1/s} = 1.$$

Hence we have

$$\frac{1}{\sqrt{2}} r(a, b) = 1 = \widehat{r}(a, b) < \sqrt{2} = r(a, b).$$

To see that the other constant is the best possible take $c = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $d = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. Then $c^2 = 0$, $cd = dc = c$, and $\sigma(c, d) = \{(0, 1)\}$. Thus $r(c, d) = 1$ and $\max_{\alpha + \beta = s} r(c^\alpha d^\beta) = 1$, which by the lemma gives $\widehat{r}(c, d) = 1$. Thus we obtain

$$\frac{1}{\sqrt{2}} r(c, d) = \frac{1}{\sqrt{2}} < 1 = \widehat{r}(c, d) = r(c, d).$$

4. One can take any other (than Euclidean) norm on each $\mathbb{C}^n$ and define the geometric spectral radius with respect to this norm. Chō and Żelazko showed in fact ([3], Cor. 10) that for many joint spectra (spectroids of class Σ_0 in the terminology of [3]) the geometric spectral radius defined in that way does not depend upon the particular spectrum.

Now observe that if we define the geometric spectral radius with respect to the l_p-norm by the formula

$$r_p(a) = \max\{|\lambda|_p : \lambda \in \sigma(a)\}$$

where

$$|\lambda|_p = |(\lambda_1, \ldots, \lambda_n)|_p = \Big(\sum_{j=1}^{n} |\lambda_j|^p\Big)^{1/p} \quad \text{for } 1 \le p < \infty, \text{ and}$$

$$|\lambda|_\infty = |(\lambda_1, \ldots, \lambda_n)|_\infty = \max_{1 \le j \le n} |\lambda_j|,$$

then reasoning analogously to the proof of Theorem 1 we get

THEOREM 2. *Let $a = (a_1, \ldots, a_n)$ be a pairwise commuting n-tuple of elements of a complex unital Banach algebra. Then*

$$\frac{1}{n^{1/p}} r_p(a) \le \widehat{r}(a) \le r_p(a)$$

for every $1 \le p < \infty$ and

$$r_\infty(a) = \widehat{r}(a).$$

Finally, notice that the last equality is the multivariable variant of the Beurling–Gelfand spectral radius formula.

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