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Interpolation by elementary operators

by

BOJAN MAGAJNA (Ljubljana)

Abstract. Given two n-tuples $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ of bounded linear operators on a Hilbert space the question of when there exists an elementary operator E such that $Ea_j = b_j$ for all $j = 1, \ldots, n$, is studied. The analogous question for left multiplications (instead of elementary operators) is answered in any C^*-algebra $\mathcal{A}$, as a consequence of the characterization of closed left $\mathcal{A}$-submodules in $\mathcal{A}^n$.

1. Introduction. An *elementary operator* on a ring $\mathcal{A}$ is a map $E : \mathcal{A} \to \mathcal{A}$ of the form

$$(1.1) \qquad Ex = \sum_{i=1}^{m} u_i x v_i \quad (x \in \mathcal{A}),$$

where u_i, v_i are fixed elements of $\mathcal{A}$. In the case when $\mathcal{A}$ is the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ or, more generally a prime C^*-algebra, much attention has been devoted to elementary operators (see [1], [3], [11], [12]). Important special cases of elementary operators are left multiplications and inner derivations (defined by $x \mapsto ax - xa$, where a is a fixed element from $\mathcal{A}$). A bounded linear map φ on a Banach algebra $\mathcal{A}$ is called a *local elementary operator* if for each $x \in \mathcal{A}$ there exists an elementary operator E_x on $\mathcal{A}$ such that $\varphi x = E_x x$. In the same way one defines *local left multiplications* and *local* (*inner*) *derivations*. Johnson [5] and Šulman [14] proved that each local left multiplication on a semisimple complex Banach algebra is necessarily a left multiplication. Larson and Sourour proved in [8] that on the algebra of all bounded operators on a Banach space every local derivation is a derivation, and Kadison proved in [6] that the same holds for bounded local derivations on von Neumann algebras. Concerning general local elementary operators, not all of them are elementary (see [8] for a counterexample in $\mathcal{B}(\mathcal{H})$), but on a C^*-algebra each local elementary operator lies in the point-norm closure of the set of elemen-

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tary operators. The last statement is derived in [10] as a consequence of the following result:

Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{E}$ the algebra of all elementary operators on $\mathcal{A}$, and $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathcal{A}^n$ (where n is a positive integer). Then $\mathbf{b}$ belongs to the norm closure $\overline{\overline{\mathcal{E}\mathbf{a}}}$ of the set

$$\mathcal{E}\mathbf{a} \stackrel{\text{def}}{=} \{(Ea_1, \ldots, Ea_n) : E \in \mathcal{E}\}$$

if and only if for each $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ the element $\boldsymbol{\lambda} \cdot \mathbf{b} \stackrel{\text{def}}{=} \sum_{j=1}^n \lambda_j b_j$ is in the closed two-sided ideal generated by $\boldsymbol{\lambda} \cdot \mathbf{a}$.

In [10] it has also been observed that this statement cannot be generalized to all semisimple (or even primitive) Banach algebras. Motivated by the above quoted result we can now ask the following question.

PROBLEM. *Given a C^*-algebra $\mathcal{A}$ and $\mathbf{a}, \mathbf{b} \in \mathcal{A}^n$, what are the necessary and sufficient conditions for the existence of an elementary operator E on $\mathcal{A}$ such that $E\mathbf{a} = \mathbf{b}$?*

Here we shall study this question in the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$. One motivation for our study is that the answer could give us some information about approximation by elementary operators of mappings that preserve two-sided ideals. Another motivation is the range inclusion problem for elementary operators (see [1] or [3] for the formulation of this problem; we shall say a few words about that at the end of Section 3).

For each $a \in \mathcal{B}(\mathcal{H})$ and each $k = 0, 1, 2, \ldots$, the *singular number* $s_k(a)$ is defined as the distance of a to the set of all operators of rank at most k in $\mathcal{B}(\mathcal{H})$. If a is compact then $(s_k(a))$ is just the sequence of eigenvalues of $|a| = \sqrt{a^*a}$ arranged in nonincreasing order (each counted according to its multiplicity). Singular numbers play a very important role in the theory of ideals in $\mathcal{B}(\mathcal{H})$ (see [4] or [2, Section 1]). Note that in our notation the largest singular number of a is $s_0(a)$ (and not $s_1(a)$), so that $s_0(a) = \|a\|$.

It is well known (and easy to see) that $s_k(uav) \le \|u\| s_k(a) \|v\|$ and $s_{mk}(\sum_{i=1}^m x_i) \le \sum_{i=1}^m s_k(x_i)$ for arbitrary $u, v, a, x_i \in \mathcal{B}(\mathcal{H})$ and any positive integers k and m. Thus, if for two given n-tuples $\mathbf{a}, \mathbf{b} \in \mathcal{B}(\mathcal{H})$ there exists an operator E of the form (1.1) such that $E\mathbf{a} = \mathbf{b}$, then for each $\boldsymbol{\lambda} \in \mathbb{C}^n$ we have $E(\boldsymbol{\lambda} \cdot \mathbf{a}) = \boldsymbol{\lambda} \cdot \mathbf{b}$, and this implies that

$$s_{mk}(\boldsymbol{\lambda} \cdot \mathbf{b}) \le \sum_{i=1}^m \|u_i\| \|v_i\| s_k(\boldsymbol{\lambda} \cdot \mathbf{a}).$$

Thus, a necessary condition for the existence of an elementary operator E satisfying $E\mathbf{a} = \mathbf{b}$ is that there exists a constant κ and a positive integer m such that $s_{mk}(\boldsymbol{\lambda} \cdot \mathbf{b}) \le \kappa s_k(\boldsymbol{\lambda} \cdot \mathbf{a})$ for all $\boldsymbol{\lambda} \in \mathbb{C}^n$ and all $k = 0, 1, 2, \ldots$ The main result in Section 3 implies that a somewhat stronger variant of this

condition is sufficient for the existence of E. As a corollary, we give a simple necessary and sufficient condition for the existence of a *generalized elementary operator* F on $\mathcal{B}(\mathcal{H})$ satisfying $F\mathbf{a} = \mathbf{b}$. Here "generalized elementary" means an operator of the form

$$Fx = \sum_{i=1}^{\infty} u_i x v_i \quad (x \in \mathcal{B}(\mathcal{H})),$$

where $u_i, v_i \in \mathcal{B}(\mathcal{H})$ are such that the two series $\sum_{i=1}^\infty u_i u_i^*$ and $\sum_{i=1}^\infty v_i^* v_i$ are convergent in the norm topology.

A necessary algebraic condition for the existence of an elementary operator E satisfying $E\mathbf{a} = \mathbf{b}$ is obviously the fact that for each $\boldsymbol{\lambda} \in \mathbb{C}^n$ the linear combination $\boldsymbol{\lambda} \cdot \mathbf{b}$ is in the two-sided ideal generated by $\boldsymbol{\lambda} \cdot \mathbf{a}$. It can be shown by examples that this condition is weaker than the above condition expressed in terms of singular numbers. The concrete operators a and b in the following example were shown to us by P. Šemrl.

EXAMPLE 1.1. Let a and b be any compact operators on $\mathcal{H}$ and set $\mathbf{a} = (a, ab)$, $\mathbf{b} = (a, 0)$. We claim that $\boldsymbol{\lambda} \cdot \mathbf{b} \in \langle \boldsymbol{\lambda} \cdot \mathbf{a} \rangle$ for each $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, where $\langle x \rangle$ is the two-sided ideal generated by x. If $\lambda_1 = 0$ the claim is obvious, so we have to prove that $a \in \langle a + \lambda ab \rangle$ for each $\lambda \in \mathbb{C}$. Since b is compact, $1 + \lambda b$ is Fredholm, hence there exists $x \in \mathcal{B}(\mathcal{H})$ such that $f \stackrel{\text{def}}{=} 1 - (1 + \lambda b)x$ is of finite rank. Since every two-sided ideal in $\mathcal{B}(\mathcal{H})$ contains all finite rank operators, we have

$$\langle a(1 + \lambda b) \rangle \supseteq \langle a(1 + \lambda b)x \rangle = \langle a(1 - f) \rangle = \langle a \rangle.$$

Let now a and b be the diagonal operators (relative to some orthonormal basis of $\mathcal{H}$) defined by

$$a = \operatorname{diag}\left(1, \frac{1}{2}, \ldots, \frac{1}{i}, \ldots\right)$$

and

$$b = \operatorname{diag}\left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots, \frac{1}{n}, \ldots\right),$$

where in b the term $1/n$ ($n \ge 2$) stands in positions $i = (n-1)! + 1, \ldots, n!$. We claim that there is no positive integer m and constant κ such that $s_{mk}(\boldsymbol{\lambda} \cdot \mathbf{b}) \le \kappa s_k(\boldsymbol{\lambda} \cdot \mathbf{a})$ for all $\boldsymbol{\lambda} \in \mathbb{C}^2$ and $k = 0, 1, 2, \ldots$ To see this, consider the situation when $\boldsymbol{\lambda} = (1, -n)$ and $k = (n-1)!$, where n is a large positive integer. Since all the diagonal terms of the operator $\boldsymbol{\lambda} \cdot \mathbf{a} = a(1 - nb)$ in positions $i = (n-1)! + 1, \ldots, n!$ are 0, while the terms after (and including) position $n! + 1$ are dominated by $1/(n! + 1)$, it follows that

$$s_k(a(1 - nb)) \le \frac{1}{n! + 1}.$$

On the other hand,

$$s_{mk}(\lambda\cdot \mathbf{b}) = s_{mk}(a) = \frac{1}{mk+1} = \frac{1}{m(n-1)!+1}.$$

It follows that the ratio

$$\frac{s_{mk}(\lambda\cdot \mathbf{b})}{s_k(\lambda\cdot \mathbf{a})} \geq \frac{n!+1}{m(n-1)!+1}$$

tends to ∞ as n increases. ∎

The above example suggests in particular that the interpolation problem for elementary operators on $\mathcal{B}(\mathcal{H})$ cannot be solved purely algebraically in terms of two-sided ideals of $\mathcal{B}(\mathcal{H})$. In contrast, we show in the next section that the analogous question for left multiplications (instead of elementary operators) can be answered in a purely algebraic language for each C^*-algebra.

2. Interpolation by left multiplications. Let $\mathcal{A}$ be C^*-algebra, n a positive integer and for each $\mathbf{x}\in\mathcal{A}^n$ let $\mathcal{I}(\mathbf{x})$ be a closed left ideal in $\mathcal{A}$. Then the set

$$\mathcal{M}_{\mathcal{I}(\mathbf{x})} \stackrel{\text{def}}{=} \{\mathbf{a}\in\mathcal{A}^n : \mathbf{a}\cdot\mathbf{x}\in\mathcal{I}(\mathbf{x})\}$$

is clearly a closed left $\mathcal{A}$-submodule of $\mathcal{A}^n$. The following proposition shows that each closed left submodule of $\mathcal{A}^n$ is the intersection of submodules of the form $\mathcal{M}_{\mathcal{I}(\mathbf{x})}$. Our result about interpolation by left mutiplications will be a consequence of this proposition.

Let us denote by $M_{m,n}(\mathcal{A})$ the set of all $m\times n$ matrices with entries in $\mathcal{A}$ and let $M_n(\mathcal{A}) = M_{n,n}(\mathcal{A})$.

PROPOSITION 2.1. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{M}$ a closed left submodule of $\mathcal{A}^n$ and $\mathbf{b} = (b_1,\ldots,b_n)\in\mathcal{A}^n$. Then $\mathbf{b}\in\mathcal{M}$ if and only if for each $\mathbf{x}\in\mathcal{A}^n$ the element $\mathbf{b}\cdot\mathbf{x}$ is contained in the closure of the left ideal $\mathcal{M}\cdot\mathbf{x}$.*

Proof. If $\mathbf{b}\in\mathcal{M}$ then clearly $\mathbf{b}\cdot\mathbf{x}\in\mathcal{M}\cdot\mathbf{x}$ for each $\mathbf{x}\in\mathcal{A}^n$. So assume that $\mathbf{b}\notin\mathcal{M}$ and let us then prove that there exists $\mathbf{x}\in\mathcal{A}^n$ such that $\mathbf{b}\cdot\mathbf{x}$ is not in the closed left ideal $\overline{\overline{\mathcal{M}\cdot\mathbf{x}}}$ (where the two bars denote closure in the norm topology). We can assume that $\mathcal{A}$ has a unit, for the general case can be easily deduced from the unital one by an application of approximate units. Let us identify $\mathcal{A}^n$ with $M_{1,n}(\mathcal{A})$, and identify $M_{1,n}(\mathcal{A})$ with the set of all elements in $M_n(\mathcal{A})$ that have only the first row different from 0. (In particular, $\mathbf{b}$ is identified with the $n\times n$ matrix which has the first row $(b_1,\ldots,b_n)$ and the remaining rows 0.) Similarly, identify $M_{n,1}(\mathcal{A})$ with the set of all matrices in $M_n(\mathcal{A})$ that have only the first column diferent from 0. Note that $\mathcal{J}\stackrel{\text{def}}{=}\overline{\overline{M_{n,1}(\mathcal{A})\mathcal{M}}}$ is a closed left ideal in $M_n(\mathcal{A})$ (in fact, since we have assumed that $\mathcal{A}$ has a unit, the ideal $M_{n,1}(\mathcal{A})\mathcal{M}$ is already closed). If $\mathbf{b}\in\mathcal{J}$, then we would have $M_{1,n}(\mathcal{A})\mathbf{b}\subseteq M_{1,n}(\mathcal{A})\mathcal{J}\subseteq \overline{\overline{M_{1,n}(\mathcal{A})M_{n,1}(\mathcal{A})\mathcal{M}}} = \overline{\overline{\mathcal{A}\mathcal{M}}} = \mathcal{M}$, but, since $\mathbf{b}\in M_{1,n}(\mathcal{A})\mathbf{b}$, this would contradict the assumption that $\mathbf{b}\notin\mathcal{M}$. It follows that $\mathbf{b}\notin\mathcal{J}$, hence there exists a pure state ω on the C^*-algebra $M_n(\mathcal{A})$ such that $\omega(\mathcal{J}) = 0$ and $\omega(\mathbf{b}^*\mathbf{b})\neq 0$ (see [7, p. 733]). Let ψ be the irreducible representation on a Hilbert space $\mathcal{K}$ that belongs to ω by the GNS construction and denote by ξ the corresponding cyclic vector. Then $\psi(\mathcal{M})\xi = 0$ (since $\mathcal{M}^*\mathcal{M}\subseteq\mathcal{J}$) and $\psi(\mathbf{b})\xi\neq 0$. Using the matrix units in $M_n(\mathcal{A})$ it is easy to see that ψ is (unitarily equivalent to a representation) of the form $\pi_n \equiv \pi\otimes 1_n$, where π is a representation of $\mathcal{A}$ on some Hilbert space $\mathcal{H}$ and π_n is defined by $\pi_n([x_{ij}]) = [\pi(x_{ij})]$ for each matrix $[x_{ij}]\in M_n(\mathcal{A})$ (in particular, $\mathcal{K}$ is identified with $\mathcal{H}^n$, see [7, Exercise 11.5.8] for a more general result). Let $\eta_1\in\mathcal{H}$ be any nonzero vector and write $\eta = (\eta_1, 0,\ldots,0)\in\mathcal{H}^n$. Since π_n is irreducible, there exists $x = [x_{ij}]\in M_n(\mathcal{A})$ such that $\xi = \pi_n(x)\eta$, hence we have $\pi_n(\mathcal{M}x)\eta = 0$ and $\pi_n(\mathbf{b}x)\eta\neq 0$, or

$$\sum_{j=1}^n \pi(a_j x_{j1})\eta_1 = 0 \quad\text{and}\quad \sum_{j=1}^n \pi(b_j x_{j1})\eta_1 \neq 0$$

for each $\mathbf{a} = (a_1,\ldots,a_n)\in\mathcal{M}$. Now the set $\mathcal{L}\stackrel{\text{def}}{=}\{z\in\mathcal{A} : \pi(z)\eta_1 = 0\}$ is clearly a closed left ideal in $\mathcal{A}$ and, with $\mathbf{x} = (x_{11},\ldots,x_{n1})$, we have $\mathcal{M}\cdot\mathbf{x}\subseteq\mathcal{L}$, $\mathbf{b}\cdot\mathbf{x}\notin\mathcal{L}$. This proves that $\mathbf{b}\cdot\mathbf{x}\notin\overline{\overline{\mathcal{M}\cdot\mathbf{x}}}$. ∎

COROLLARY 2.2. *Let $\mathcal{A}$ be a C^*-algebra and $\mathbf{a} = (a_1,\ldots,a_n)$, $\mathbf{b} = (b_1,\ldots,b_n)\in\mathcal{A}^n$. There exists $c\in\mathcal{A}$ such that $c\mathbf{a} = \mathbf{b}$ if and only if for each $\mathbf{x}\in\mathcal{A}^n$ the element $\mathbf{b}\cdot\mathbf{x}$ is in the left ideal generated by $\mathbf{a}\cdot\mathbf{x}$ in $\mathcal{A}$.*

Proof. We may assume that $\mathcal{A}\subseteq\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Put

$$d \stackrel{\text{def}}{=} \Big(\sum_{j=1}^n a_j a_j^*\Big)^{1/2}$$

and for each $j = 1,\ldots,n$ define the contraction $u_j\in\mathcal{B}(\mathcal{H})$ by

$$u_j d\xi = a_j^*\xi \quad (\xi\in\mathcal{H}) \quad\text{and}\quad u_j((d\mathcal{H})^\perp) = 0.$$

Note that $u_j d = a_j^*$. By hypothesis there exists $c\in\mathcal{A}$ such that

$$c\sum_{j=1}^n a_j a_j^* = \sum_{j=1}^n b_j a_j^*,$$

which can be written as

$$\Big(cd - \sum_{j=1}^n b_j u_j\Big)d = 0.$$

This means that $cd - \sum_{j=1}^{n} b_j u_j$ annihilates $\overline{d\mathcal{H}}$; but, by the definition we also have $u_j((d\mathcal{H})^{\perp}) = 0$ for each j, and obviously $d((d\mathcal{H})^{\perp}) = 0$, hence $cd - \sum_{j=1}^{n} b_j u_j = 0$, or

$$cd = \sum_{j=1}^{n} b_j u_j \,. \tag{2.1}$$

By Proposition 2.1 (applied to the closed submodule of $\mathcal{A}^n$ generated by a) there exists a sequence $(c_k) \subseteq \mathcal{A}$ such that

$$\lim_{k\to\infty} c_k a_j = b_j \quad (j = 1, \ldots, n)\,. \tag{2.2}$$

It is easy to see that $\sum_{j=1}^{n} u_j^* u_j$ is the range projection of d, hence

$$\sum_{j=1}^{n} a_j u_j = \sum_{j=1}^{n} (u_j d)^* u_j = d\,. \tag{2.3}$$

If we multiply, for each j, the relation (2.2) by u_j from the right and then add the resulting identities and use (2.3) we obtain

$$\lim_{k\to\infty} c_k d = \sum_{j=1}^{n} b_j u_j \,,$$

hence (2.1) implies that

$$\lim_{k\to\infty} (c - c_k) d = 0\,. \tag{2.4}$$

Since

$$\|(c - c_k)a_j\|^2 = \|(c - c_k)a_j a_j^*(c - c_k)^*\| \le \|(c - c_k)d\|^2\,,$$

we conclude from (2.4) and (2.2) that

$$ca_j = \lim_{k\to\infty} c_k a_j = b_j \quad (j = 1, \ldots, n)\,. \ \blacksquare$$

Note that the requirement that $\boldsymbol{\lambda}\cdot\mathbf{b} \in \mathcal{A}(\boldsymbol{\lambda}\cdot\mathbf{a})$ for each $\boldsymbol{\lambda} \in \mathbb{C}^n$ is much weaker than the condition in Corollary 2.2. To see this, consider, for example, $\mathbf{a} = (1, q)$ and $\mathbf{b} = (1, 0)$, where q is a nonzero quasinilpotent element in $\mathcal{A}$.

We do not know to which Banach algebras Corollary 2.2 can be extended, but there are finite-dimensional commutative complex algebras for which it does not hold.

EXAMPLE 2.3. Let $\mathcal{A} \subseteq M_5(\mathbb{C})$ be generated by $\{1, a_1, a_2\}$, where 1 is the identity matrix and

$$a_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 1 & 0 \end{bmatrix}.$$

The matrices a_1, a_2 satisfy

$$a_1^3 = a_2^3 = a_1^2 a_2 = a_1 a_2^2 = 0\,, \qquad a_1 a_2 = a_2 a_1 = \tfrac{1}{2}(a_1^2 + a_2^2)$$

(in particular, $\mathcal{A}$ is commutative) and $\{1, a_1, a_1^2, a_2, a_2^2\}$ is a vector space basis for $\mathcal{A}$. Put $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (a_1^2, a_2^2)$. It can be proved by computation that for each $\mathbf{x} \in \mathcal{A}^2$ the element $\mathbf{b}\cdot\mathbf{x}$ is in the ideal generated by $\mathbf{a}\cdot\mathbf{x}$, but nevertheless there does not exist any $c \in \mathcal{A}$ satisfying $c\mathbf{a} = \mathbf{b}$. $\blacksquare$

3. Interpolation by elementary operators on $\mathcal{B}(\mathcal{H})$. Throughout this section let $\mathcal{H}$ denote a separable infinite-dimensional Hilbert space and $\mathcal{E}$ the algebra of all elementary operators on $\mathcal{B}(\mathcal{H})$. For each subspace $\mathcal{L} \subseteq \mathcal{B}(\mathcal{H})$, each $a \in \mathcal{B}(\mathcal{H})$ and $k = 0, 1, \ldots$ set

$$s_k^{\mathcal{L}}(a) = \inf_{x\in\mathcal{L}} s_k(a + x)$$

and

$$\operatorname{ann}_{\mathcal{E}}(\mathcal{L}) = \{E \in \mathcal{E} : Ex = 0 \ \forall x \in \mathcal{L}\}\,.$$

(Thus $\operatorname{ann}_{\mathcal{E}}(\mathcal{L})$ is the annihilator of $\mathcal{L}$ in $\mathcal{E}$.) Now the main result of this section can be formulated as follows.

THEOREM 3.1. *Let $a, b \in \mathcal{B}(\mathcal{H})$ and let $\mathcal{L}$ be a finite-dimensional subspace of $\mathcal{B}(\mathcal{H})$. Then $b \in \operatorname{ann}_{\mathcal{E}}(\mathcal{L})a$ if and only if there exists a positive integer m and a constant κ such that*

$$s_{mk}(b) \le \kappa s_k^{\mathcal{L}}(a) \tag{3.1}$$

for all $k = 0, 1, 2, \ldots$

Let $n = 1 + \dim\mathcal{L}$, choose a basis $\{a_2, \ldots, a_n\}$ for $\mathcal{L}$ and put

$$\mathbf{a} = (a_1, a_2, \ldots, a_n)\,,\ \mathbf{b} = (b, 0, \ldots, 0) \in \mathcal{B}(\mathcal{H})^n\,,$$

where $a_1 = a$. Then (3.1) can be written in the form

$$s_{mk}(\boldsymbol{\lambda}\cdot\mathbf{b}) \le \kappa s_k(\boldsymbol{\lambda}\cdot\mathbf{a}) \quad (\boldsymbol{\lambda} \in \mathbb{C}^n\,,\ k = 0, 1, 2, \ldots)\,. \tag{3.1$'$}$$

It has already been observed in the introduction that (3.1′) is necessary for the existence of an elementary operator E satisfying $E\mathbf{a} = \mathbf{b}$. Before proving sufficiency we make some comments.

Suppose that we have two general n-tuples $\mathbf{a}, \mathbf{b} \in \mathcal{B}(\mathcal{H})^n$ such that there exists a positive integer m and a constant κ satisfying

$$\max_{\boldsymbol{\lambda}\in\mathbf{S}_n} s_{mk}(\boldsymbol{\lambda}\cdot\mathbf{b}) \le \kappa \min_{\boldsymbol{\lambda}\in\mathbf{S}_n} s_k(\boldsymbol{\lambda}\cdot\mathbf{a}) \quad \text{for all } k = 0, 1, 2, \ldots, \tag{3.1$''$}$$

where $\mathbf{S}_n$ is the unit sphere in $\mathbb{C}^n$ (that is, $\mathbf{S}_n = \{\lambda \in \mathbb{C}^n : \|\lambda\| = 1\}$). Then it is easy to verify that (3.1) is satisfied for each b_i (in place of b) and $\operatorname{span}\{a_j : j \neq i,\ j = 1, \ldots, n\}$ (in place of $\mathcal{L}$), hence by Theorem 3.1 for each i there exists $E_i \in \mathcal{E}$ satisfying $E_i a_i = b_i$ and $E_i a_j = 0$ for $j \neq i$. With $E = E_1 + \ldots + E_n$ we then have $E\mathbf{a} = \mathbf{b}$. Thus, (3.1″) (which is in general stronger than (3.1′)) is always sufficient for the existence of an elementary operator E satisfying $E\mathbf{a} = \mathbf{b}$. (3.1″) is satisfied, for example, if the components of $\mathbf{a}$ are linearly independent modulo the ideal $\mathcal{K}(\mathcal{H})$ of compact operators, since in this case the singular numbers $s_k(\lambda \cdot \mathbf{a})$ are uniformly bounded below by some positive constant when λ runs over $\mathbf{S}_n$ and $k = 0, 1, 2, \ldots$ (In the case when the components of $\mathbf{a}$ are linearly independent modulo $\mathcal{K}(\mathcal{H})$ a stronger result than Theorem 3.1 is proved in [9].)

If (3.1′) holds and the components of $\mathbf{b}$ are linearly independent modulo $\mathcal{K}(\mathcal{H})$, then the same must hold for the components of $\mathbf{a}$, hence by the previous paragraph there exists an elementary operator E such that $E\mathbf{a} = \mathbf{b}$. On the other hand, if the components of $\mathbf{b}$ are compact operators, this argument breaks down, but under some additional hypothesis about $\mathbf{b}$ (for example, if the components b_i of $\mathbf{b}$ are commuting normal operators) the proof of Theorem 3.1 given below can be adapted to show that (3.1′) is again sufficient for the existence of an elementary operator E satisfying $E\mathbf{a} = \mathbf{b}$. However, the problem of a general n-tuple $\mathbf{b}$ is open and seems to be very difficult.

Throughout the rest of the paper we denote by $\mathcal{P}_n(\mathcal{H})$ the set of all orthogonal projections of rank n on $\mathcal{H}$ ($n = 0, 1, 2, \ldots$), and by $\mathbf{S}_n$ the unit sphere in $\mathbb{C}^n$. To prove Theorem 3.1 we need a couple of lemmas.

LEMMA 3.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the space of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let $\mathcal{S}$ be an n-dimensional subspace of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then there exists an n-dimensional subspace $\mathcal{M} \subseteq \mathcal{H}_1$ such that*

$$\|a|\mathcal{M}\| \geq \gamma \|a\|$$

for every $a \in \mathcal{S}$, where γ is a positive constant (which depends only on n but not on $\mathcal{S}$; in fact, one can choose for γ any positive constant less than $\frac{1}{2}(2^n - 1)^{-1}$). Moreover, if $\mathcal{H}_1 = \mathcal{H}_2$, then there exists a projection $p \in \mathcal{B}(\mathcal{H}_1)$ of rank n such that

$$\|pap\| \geq \gamma \|a\|$$

for every $a \in \mathcal{S}$.

Proof. First note that it suffices to prove the last statement of the lemma. Indeed, let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ (orthogonal sum) and identify $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with the subspace of $\mathcal{B}(\mathcal{H})$ consisting of all operators that map $\mathcal{H}_1$ into $\mathcal{H}_2$ and $\mathcal{H}_2$ to 0. Suppose that we have found a projection $p \in \mathcal{P}_n(\mathcal{H})$ satisfying $\|pap\| \geq \gamma \|a\|$ for all $a \in \mathcal{S}$ and denote by $\mathcal{N}$ the range of p. Then $\|a|\mathcal{N}\| \geq \gamma \|a\|$ for each $a \in \mathcal{S}$, and it is clear that this inequality also holds if $\mathcal{N}$ is replaced by the orthogonal projection $\mathcal{N}_1$ of $\mathcal{N}$ to the space $\mathcal{H}_1$, hence the same inequality is also satisfied if $\mathcal{N}$ is replaced by any n-dimensional subspace $\mathcal{M}$ of $\mathcal{H}_1$ which contains $\mathcal{N}_1$.

So, let us prove the last statement of the lemma. To simplify the notation, we shall write $\mathcal{H}$ instead of $\mathcal{H}_1$. It is slightly easier to work with the numerical radius $w(a)$ instead of the norm. Let $\varepsilon > 0$. We shall prove by induction on n that there exists $p \in \mathcal{P}_n(\mathcal{H})$ such that

$$w(pap) \geq (\gamma_n - \varepsilon) w(a) \tag{3.2}$$

for each $a \in \mathcal{S}$, where $\gamma_n = (2^n - 1)^{-1}$. Since $\|pap\| \geq w(pap)$ and $w(a) \geq \|a\|/2$ (see [13, p. 98]), this will prove the lemma. If $n = 1$, choose any unit vector $\xi \in \mathcal{H}$ satisfying $\langle a\xi, \xi\rangle \geq (1 - \varepsilon) w(a)$ and let p be the projection onto $\mathbb{C}\xi$; it is then clear that (3.2) is satisfied with $\gamma_1 = 1$. So, let now $n > 1$, assume inductively that the lemma holds for all subspaces of dimension less than n and let $\mathcal{S}$ be any n-dimensional subspace of $\mathcal{B}(\mathcal{H})$. Choose a basis $\mathbf{a} = (a_1, \ldots, a_n)$ for $\mathcal{S}$ so that

$$w(\lambda \cdot \mathbf{a}) \leq w(a_1) \tag{3.3}$$

for all $\lambda \in \mathbf{S}_n$. (To see that such a basis exists, choose first any basis $\widetilde{\mathbf{a}} = (\widetilde{a}_1, \ldots, \widetilde{a}_n)$ for $\mathcal{S}$, then choose $\lambda_0 \in \mathbf{S}_n$ so that

$$w(\lambda_0 \cdot \widetilde{\mathbf{a}}) = \max\{w(\lambda \cdot \widetilde{\mathbf{a}}) : \lambda \in \mathbf{S}_n\},$$

let $[\mu_{ij}]$ be an $n \times n$ unitary matrix with the first row equal to λ_0 and put $a_i = \sum_{j=1}^n \mu_{ij} \widetilde{a}_j$ for each $i = 1, \ldots, n$.)

CLAIM. *If (ξ_k) is any sequence of unit vectors in $\mathcal{H}$ such that*

$$\lim_{k\to\infty} |\langle a_1 \xi_k, \xi_k\rangle| = w(a_1), \tag{3.4}$$

then

$$\lim_{k\to\infty} \langle a_j \xi_k, \xi_k\rangle = 0 \quad \textit{for } j = 2, \ldots, n. \tag{3.5}$$

To prove this claim, fix $k \in \{0, 1, 2, \ldots\}$, $j \in \{2, \ldots, n\}$, and define $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{S}_n$ by

$$\lambda_1 = \sqrt{1 - t^2} \exp(-\sqrt{-1} \arg \langle a_1 \xi_k, \xi_k\rangle),$$
$$\lambda_j = t \exp(-\sqrt{-1} \arg \langle a_j \xi_k, \xi_k\rangle)$$

and $\lambda_l = 0$ for $l \neq 1, j$, where $t \in (0, 1)$. From (3.3) we have $|\langle(\lambda \cdot \mathbf{a})\xi_k, \xi_k\rangle| \leq w(a_1)$, hence

$$\sqrt{1 - t^2} |\langle a_1 \xi_k, \xi_k\rangle| + t |\langle a_j \xi_k, \xi_k\rangle| \leq w(a_1),$$

or

$$|\langle a_j\xi_k,\xi_k\rangle| \le \frac{1}{t}(w(a_1)-|\langle a_1\xi_k,\xi_k\rangle|)+\frac{1}{t}(1-\sqrt{1-t^2})|\langle a_1\xi_k,\xi_k\rangle|\,.$$

Choosing $t=t_k$, where (t_k) is a sequence converging to 0 slowly enough to insure that the sequence $t_k^{-1}(w(a_1)-|\langle a_1\xi_k,\xi_k\rangle|)$ also converges to 0, the claim follows from the last inequality.

Let $\mathcal{T}$ be the span of $\{a_2,\ldots,a_n\}$. By the inductive hypothesis, for each $k=1,2,\ldots$, there exists a projection $q_k\in\mathcal{P}_{n-1}(\mathcal{H})$ such that

$$w(q_kbq_k)\ge\Big(\gamma_{n-1}-\frac{1}{k}\Big)w(b) \tag{3.6}$$

for every $b\in\mathcal{T}$, where $\gamma_{n-1}=(2^{n-1}-1)^{-1}$. Let (ξ_k) be any sequence of unit vectors in $\mathcal{H}$ satisfying (3.4) and for each k choose $p_k\in\mathcal{P}_n(\mathcal{H})$ so that the range of p_k contains the range of q_k and ξ_k. We shall now show that for sufficiently large k the projection $p\overset{\text{def}}{=}p_k$ satisfies (3.2) and this will prove the lemma. Let $a=\boldsymbol{\lambda}\cdot\mathbf{a}$ be the expansion of a in the basis $\mathbf{a}$ of $\mathcal{S}$; by homogeneity of (3.2) we may assume that $\|\boldsymbol{\lambda}\|=1$. Put

$$\delta=\min\{w(\boldsymbol{\lambda}\cdot\mathbf{a}):\boldsymbol{\lambda}\in\mathbf{S}_n\}$$

and note that $\delta>0$ by compactness of $\mathbf{S}_n$, since $a_1,\ldots,a_n$ are linearly independent. By (3.4) and (3.5) there exists a positive integer k_0 such that

$$\sum_{j=2}^{n}|\langle a_j\xi_k,\xi_k\rangle|<\frac{\varepsilon\delta}{2(n-1)}\quad\text{and}\quad|\langle a_1\xi_k,\xi_k\rangle|w(a_1)^{-1}>1-\frac{\varepsilon}{2\gamma_n} \tag{3.7}$$

for all $k\ge k_0$. We may assume that ε is so small that $1-\varepsilon/(2\gamma_n)>0$. Let us now estimate $w(p_k(\boldsymbol{\lambda}\cdot\mathbf{a})p_k)$, where $k\ge k_0$. If $|\lambda_1|\ge\gamma_nw(\boldsymbol{\lambda}\cdot\mathbf{a})w(a_1)^{-1}$, then we have

$$\begin{aligned}w(p_k(\boldsymbol{\lambda}\cdot\mathbf{a})p_k)&\ge|\langle(\boldsymbol{\lambda}\cdot\mathbf{a})\xi_k,\xi_k\rangle|\\&\ge|\lambda_1|\,|\langle a_1\xi_k,\xi_k\rangle|-\sum_{j=2}^{n}|\langle a_j\xi_k,\xi_k\rangle|\\&\ge\gamma_nw(\boldsymbol{\lambda}\cdot\mathbf{a})w(a_1)^{-1}|\langle a_1\xi_k,\xi_k\rangle|-\sum_{j=2}^{n}|\langle a_j\xi_k,\xi_k\rangle|\\&\ge\gamma_nw(\boldsymbol{\lambda}\cdot\mathbf{a})\Big(1-\frac{\varepsilon}{2\gamma_n}\Big)-\frac{\varepsilon\delta}{2}\qquad(\text{by }(3.7))\\&\ge(\gamma_n-\varepsilon)w(\boldsymbol{\lambda}\cdot\mathbf{a})\,.\end{aligned}$$

If $|\lambda_1|<\gamma_nw(\boldsymbol{\lambda}\cdot\mathbf{a})w(a_1)^{-1}$, then we compute (using (3.6))

$$\begin{aligned}w(p_k(\boldsymbol{\lambda}\cdot\mathbf{a})p_k)&\ge w(q_k(\boldsymbol{\lambda}\cdot\mathbf{a})q_k)\\&\ge w\Big(q_k\Big(\sum_{j=2}^{n}\lambda_ja_j\Big)q_k\Big)-|\lambda_1|w(q_ka_1q_k)\\&\ge\Big(\gamma_{n-1}-\frac{1}{k}\Big)w\Big(\sum_{j=2}^{n}\lambda_ja_j\Big)-|\lambda_1|w(a_1)\\&\ge\Big(\gamma_{n-1}-\frac{1}{k}\Big)[w(\boldsymbol{\lambda}\cdot\mathbf{a})-|\lambda_1|w(a_1)]-|\lambda_1|w(a_1)\\&\ge\Big(\gamma_{n-1}-\frac{1}{k}\Big)[w(\boldsymbol{\lambda}\cdot\mathbf{a})-\gamma_nw(\boldsymbol{\lambda}\cdot\mathbf{a})]-\gamma_nw(\boldsymbol{\lambda}\cdot\mathbf{a})\\&=\Big[\gamma_{n-1}(1-\gamma_n)-\gamma_n-\frac{1}{k}(1-\gamma_n)\Big]w(\boldsymbol{\lambda}\cdot\mathbf{a})\,.\end{aligned}$$

(Here we have assumed that k is so large that $\gamma_{n-1}-1/k>0$.) Since $\gamma_{n-1}(1-\gamma_n)-\gamma_n=\gamma_n$, we see that for any sufficiently large k we have $w(p_k(\boldsymbol{\lambda}\cdot\mathbf{a})p_k)\ge(\gamma_n-\varepsilon)w(\boldsymbol{\lambda}\cdot\mathbf{a})$ for all $\boldsymbol{\lambda}\in\mathbf{S}_n$. This proves (3.2). ∎

Our next lemma is just a corollary to the previous one. In the proof we shall use the following fact concerning singular numbers: $s_k(a)\le\|a|\mathcal{K}\|$ for each subspace $\mathcal{K}$ in $\mathcal{H}$ of codimension at most k, where $a\in\mathcal{B}(\mathcal{H})$ and $k=0,1,2,\ldots$

LEMMA 3.3. *Let $\mathcal{S}$ be an n-dimensional subspace of $\mathcal{B}(\mathcal{H})$. There exists an orthogonal sequence of n-dimensional subspaces $\mathcal{M}_k\subseteq\mathcal{H}$ $(k=0,1,2,\ldots)$ such that the spaces $\mathcal{S}\mathcal{M}_i$ and $\mathcal{S}\mathcal{M}_k$ are orthogonal for $i\ne k$ and*

$$\|a|\mathcal{M}_k\|\ge\gamma s_{rk}(a)$$

for all k and all $a\in\mathcal{S}$, where $r=n(n^2+1)$ and γ is the constant from Lemma 3.2.

Proof. By Lemma 3.2 there exists an n-dimensional subspace $\mathcal{M}_0$ in $\mathcal{H}$ such that $\|a|\mathcal{M}_0\|\ge\gamma\|a\|=\gamma s_0(a)$. Assume inductively that for some $i>0$ the subspaces $\mathcal{M}_0,\mathcal{M}_1,\ldots,\mathcal{M}_{i-1}$ have already been found so that the requirements of the lemma are satisfied and set

$$\mathcal{H}_i\overset{\text{def}}{=}\Big(\sum_{k=0}^{i-1}\mathcal{S}^*\mathcal{S}\mathcal{M}_k+\mathcal{M}_k\Big)^{\perp}.$$

Since $\operatorname{codim}(\mathcal{H}_i)\le i(n^2n+n)=ir$, we have $\|a|\mathcal{H}_i\|\ge s_{ri}(a)$ for each $a\in\mathcal{S}$. By Lemma 3.2 there exists an n-dimensional subspace $\mathcal{M}_i\subseteq\mathcal{H}_i$ such that $\|a|\mathcal{M}_i\|\ge\gamma\|a|\mathcal{H}_i\|$ for all $a\in\mathcal{S}$. Thus, $\|a|\mathcal{M}_i\|\ge\gamma s_{ri}(a)$, and by construction $\mathcal{S}\mathcal{M}_i\perp\mathcal{S}\mathcal{M}_k$ and $\mathcal{M}_i\perp\mathcal{M}_k$ for $k<i$, so the lemma follows by induction. ∎

If $a \in \mathcal{B}(\mathcal{H})$ and n is a positive integer, we denote by $a^{(n)}$ the direct sum of n copies of a. For a subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ we write $\mathcal{S}^{(n)} = \{a^{(n)} : a \in \mathcal{S}\}$.

Proof of Theorem 3.1. The necessity of the conditions (3.1) has already been observed, so we only prove the sufficiency. If b is not compact, then the singular numbers of b are bounded below by some positive constant, hence the conditions (3.1) are then also satisfied if b is replaced by the identity operator 1. If there exists an elementary operator F satisfying $Fa = 1$ and $F\mathcal{L} = 0$, then the operator E defined by $Ez = bFz$ $(z \in \mathcal{B}(\mathcal{H}))$ satisfies $Ea = b$ and $E\mathcal{L} = 0$. So we can assume without loss of generality that b is either the identity operator or a compact operator. Assume also that b is not of finite rank; if it is, the proof below requires only some trivial notational changes. Let $(\eta_k)_{k=0}^{\infty}$ be an orthonormal basis of $\mathcal{K} \overset{\text{def}}{=} (\ker b)^{\perp}$ such that each η_k is the eigenvector of $|b|$ corresponding to the eigenvalue $s_k(b)$. Then

$$\|b\eta_k\| = s_k(b) \quad (k = 0, 1, 2, \ldots). \tag{3.8}$$

By Lemma 3.3 applied to $\mathcal{S} \overset{\text{def}}{=} \mathcal{L} + \mathbb{C}a$ there exists an orthogonal sequence of n-dimensional subspaces $\mathcal{M}_k$ $(k = 0, 1, 2, \ldots)$ in $\mathcal{H}$ such that $\mathcal{S}\mathcal{M}_i \perp \mathcal{S}\mathcal{M}_k$ if $i \neq k$ and

$$\|y|\mathcal{M}_k\| \geq \gamma s_{rk}(y)$$

for all $y \in \mathcal{S}$ and $k = 0, 1, 2, \ldots$ For each $k = 0, 1, 2, \ldots$ choose an orthonormal basis $\{\xi_{1k}, \ldots, \xi_{nk}\}$ in $\mathcal{M}_k$ and put

$$\xi_k = \frac{1}{\sqrt{n}}(\xi_{1k}, \ldots, \xi_{nk}).$$

Then (ξ_k) is an orthonormal sequence in $\mathcal{H}^n$, $\mathcal{S}^{(n)}\xi_k \perp \mathcal{S}^{(n)}\xi_i$ if $i \neq k$, and

$$\|y^{(n)}\xi_k\| \geq \frac{\gamma}{\sqrt{n}} s_{rk}(y) \quad (y \in \mathcal{S},\ k = 0, 1, 2, \ldots),$$

since the Hilbert–Schmidt norm dominates the operator norm. From this relation and the hypothesis (3.1) it now follows that

$$\|(a + x)^{(n)}\xi_k\| \geq \gamma\kappa^{-1}n^{-1/2}s_{rmk}(b) \quad (x \in \mathcal{L},\ k = 0, 1, 2, \ldots).$$

Since the singular numbers are arranged in nonincreasing order, the last inequality can be rewritten as

$$\|(a + x)^{(n)}\xi_{[k/(rm)]}\| \geq \gamma\kappa^{-1}n^{-1/2}s_k(b) \quad (x \in \mathcal{L},\ k = 0, 1, 2, \ldots), \tag{3.9}$$

where $[k/(rm)]$ is the greatest integer less than or equal to $k/(rm)$. Put $l = rmn$ and for each k let $\widetilde{\xi}_k \in \mathcal{H}^l = (\mathcal{H}^n)^{rm}$ be defined by

$$\widetilde{\xi}_k = (0, \ldots, 0, \xi_{[k/(rm)]}, 0, \ldots, 0),$$

where there are rm components and the nonzero component is in position $t(k) + 1$, where $t(k)$ is the remainder of the division of k by rm. Then $(\widetilde{\xi}_k)$ is an orthonormal sequence of vectors and $(\mathcal{S}^{(l)}\widetilde{\xi}_k)$ is an orthogonal sequence of subspaces $\mathcal{H}^l$, and it follows from (3.9) and (3.8) that

$$\|b\eta_k\| \leq \mu\|(a + x)^{(l)}\widetilde{\xi}_k\| \quad (x \in \mathcal{L},\ k = 0, 1, 2, \ldots), \tag{3.10}$$

where μ is a positive constant. By the last inequality the correspondence

$$(\lambda a + x)^{(l)}\widetilde{\xi}_k \mapsto \lambda b\eta_k \quad (\lambda \in \mathbb{C},\ x \in \mathcal{L})$$

is a well defined linear operator u_k from $\mathcal{S}^{(l)}\widetilde{\xi}_k$ to $\mathbb{C}b\eta_k$ with norm at most μ. Therefore the orthogonal sum of the operators u_k can be extended to a bounded operator $u : \mathcal{H}^l \to \mathcal{H}$, which satisfies

$$u((\lambda a + x)^{(l)})\widetilde{\xi}_k = \lambda b\eta_k \quad (\lambda \in \mathbb{C},\ x \in \mathcal{L},\ k = 0, 1, 2, \ldots).$$

Finally, let $v : \mathcal{H} \to \mathcal{H}^l$ be the partial isometry defined by $v\eta_k = \widetilde{\xi}_k$ for $k = 0, 1, 2, \ldots$ and $v|\ker b = 0$. Then $u((\lambda a + x)^{(l)})v\eta_k = \lambda b\eta_k$ for each $\lambda \in \mathbb{C}$, $x \in \mathcal{L}$ and $k = 0, 1, 2, \ldots$, hence (since (η_k) is an orthonormal basis for $(\ker b)^{\perp}$)

$$u((\lambda a + x)^{(l)})v = \lambda b \quad (\lambda \in \mathbb{C},\ x \in \mathcal{L}).$$

From the last identity we see that the elementary operator

$$Ez \overset{\text{def}}{=} uz^{(l)}v \quad (z \in \mathcal{B}(\mathcal{H}))$$

satisfies $Ea = b$ and $E\mathcal{L} = 0$. ∎

The notion of a generalized elementary operator which occurs in the following corollary has been defined in the introduction.

COROLLARY 3.4. *Let* $\mathbf{a}, \mathbf{b} \in \mathcal{B}(\mathcal{H})^n$. *There exists a generalized elementary operator* F *on* $\mathcal{B}(\mathcal{H})$ *satisfying* $F\mathbf{a} = \mathbf{b}$ *if and only if for each* $\boldsymbol{\lambda} \in \mathbb{C}^n$ *the condition* $\boldsymbol{\lambda} \cdot \mathbf{a} = 0$ *implies that* $\boldsymbol{\lambda} \cdot \mathbf{b} = 0$, *and* $\boldsymbol{\lambda} \cdot \mathbf{a}$ *being compact implies that* $\boldsymbol{\lambda} \cdot \mathbf{b}$ *is compact.*

Proof. Only the sufficiency is nontrivial. We may assume that $a_1, \ldots, a_n$ are linearly independent and that for some $m \leq n$ the components $a_1, \ldots, a_m$ are linearly independent modulo the ideal of all compact operators while $a_{m+1}, \ldots, a_n$ are compact. (The general case can be easily reduced to this situation. If all a_j's are compact let $m = 0$.) It suffices to prove that for each $i = 1, \ldots, n$ there exists a generalized elementary operator F_i satisfying $F_i a_i = b_i$ and $F_i a_j = 0$ for $j \neq i$, $j = 1, \ldots, n$ (for then $F \overset{\text{def}}{=} \sum F_i$ satisfies $F\mathbf{a} = \mathbf{b}$). First consider the case $i \leq m$. For each $x \in \mathcal{B}(\mathcal{H})$ denote by $\dot{x}$ the coset of x in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Since $\dot{a}_1, \ldots, \dot{a}_m$ are linearly independent and $\dot{a}_{m+1} = 0, \ldots, \dot{a}_n = 0$, we have

$$\delta \overset{\text{def}}{=} \min\Big\{\Big\|\dot{a}_i + \sum_{j \neq i}\lambda_j\dot{a}_j\Big\| : (\ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots) \in \mathbb{C}^{n-1}\Big\} > 0.$$

Since

$$s_k\Big(a_i + \sum_{j\neq i} \lambda_j a_j\Big) \geq \delta\,,$$

we now have, with a sufficiently large $\kappa > 0$,

$$s_k(b_i) \leq \kappa s_k\Big(a_i + \sum_{j\neq i} \lambda_j a_j\Big)$$

for all $\lambda_j \in \mathbb{C}$ ($j = 1, \ldots, n$, $j \neq i$). Then by Theorem 3.1 there exists an elementary operator F_i such that $F_i a_i = b_i$ and $F_i a_j = 0$ for $j \neq i$, $j = 1, \ldots, n$.

Now consider the case $i > m$. Then a_i is compact, hence by the hypothesis b_i must be compact. By Theorem 3.1 there exists an elementary operator E such that Ea_i is an operator c of rank 1 and norm 1 and $Ea_j = 0$ for $j \neq i$. It now suffices to prove that there exists a generalized elementary operator G satisfying $Gc = b_i$, for then $F_i \stackrel{\text{def}}{=} GE$ satisfies $F_i a_i = b_i$ and $F_i a_j = 0$ for $j \neq i$. As a rank 1 operator, c has the form $c = \xi \otimes \eta$ for appropriate vectors $\xi, \eta \in \mathcal{H}$ (where $(\xi \otimes \eta)\zeta = \langle \zeta, \eta\rangle \xi$ for each $\zeta \in \mathcal{H}$), and, as any compact operator, b_i can be written in the form

$$b_i = \sum_{k=0}^{\infty} \beta_k \xi_k \otimes \eta_k\,,$$

where (ξ_k) and (η_k) are orthonormal sequences in $\mathcal{H}$ and (β_k) is a sequence of positive scalars tending to 0. With $u_k = \beta_k^{1/2}\xi_k \otimes \xi$ and $v_k = \beta_k^{1/2}\eta \otimes \eta_k$, it is easy to verify that the series $\sum_{k=0}^{\infty} u_k u_k^*$ and $\sum_{k=0}^{\infty} v_k^* v_k$ converge in the norm topology and that the operator

$$Gx \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} u_k x v_k \quad (x \in \mathcal{B}(\mathcal{H}))$$

satisfies $Gc = b_i$. ∎

Since the ideal of compact operators is the only proper closed nonzero two-sided ideal in $\mathcal{B}(\mathcal{H})$, the condition in Corollary 3.4 can also be formulated as follows: for each $\boldsymbol{\lambda} \in \mathbb{C}^n$ the element $\boldsymbol{\lambda} \cdot \mathbf{b}$ is in the closed two-sided ideal generated by $\boldsymbol{\lambda} \cdot \mathbf{a}$. Thus, in the special case of the C^*-algebra $\mathcal{B}(\mathcal{H})$, Corollary 3.4 can be regarded as an improvement of [10, Theorem 2.1]. Let us remark that Corollary 3.4 (with the formulation just indicated) cannot be extended to general C^*-algebras (consider, for example, commutative C^*-algebras), but perhaps it can be extended to factors.

As another consequence of Theorem 3.1 we mention the fact that the ideal $\mathcal{F}(\mathcal{H})$ of finite rank operators is strongly prime in the sense of [3]. (An ideal $\mathcal{J}$ in $\mathcal{B}(\mathcal{H})$ is called *prime* if for any $u, v \in \mathcal{B}(\mathcal{H})$ the inclusion $u\mathcal{B}(\mathcal{H})v \subseteq \mathcal{J}$ implies that $u \in \mathcal{J}$ or $v \in \mathcal{J}$. An ideal $\mathcal{J}$ is called

strongly prime if for any elementary operator E of the form (1.1) the inclusion $E(\mathcal{B}(\mathcal{H})) \subseteq \mathcal{J}$ implies that at least one of the two sets $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_m\}$ is linearly dependent modulo $\mathcal{J}$. Each strongly prime ideal is prime, but it is not known whether the converse is true in $\mathcal{B}(\mathcal{H})$. This result was proved in [3] as a consequence of [1, Proposition 5.2], but it can also be deduced from Theorem 3.1. Namely, if the range of an elementary operator E of the form (1.1) is contained in $\mathcal{F}(\mathcal{H})$ and the coefficients u_i are linearly independent modulo $\mathcal{F}(\mathcal{H})$, then let F be an elementary operator satisfying $c \stackrel{\text{def}}{=} Fu_1 \notin \mathcal{F}(\mathcal{H})$ and $Fu_j = 0$ for $j > 1$; a short computation then shows that $c\mathcal{B}(\mathcal{H})v_1 \subseteq \mathcal{F}(\mathcal{H})$ (see the beginning of the proof of Corollary 2.6 in [10]), hence $v_1 \in \mathcal{F}(\mathcal{H})$. It would be interesting to know whether Theorem 3.1 can be used to prove that some other prime ideals in $\mathcal{B}(\mathcal{H})$ are strongly prime.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LJUBLJANA
JADRANSKA 19
LJUBLJANA 61000, SLOVENIA

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On the joint spectral radii of commuting Banach algebra elements

by

ANDRZEJ SOŁTYSIAK (Poznań)

Abstract. Some inequalities are proved between the geometric joint spectral radius (cf. [3]) and the joint spectral radius as defined in [7] of finite commuting families of Banach algebra elements.

Let A be a complex Banach algebra with the unit denoted by 1. Let $a = (a_1, \ldots, a_n)$ be an n-tuple of pairwise commuting elements of A. The symbol $\sigma(a)$ will stand for the *Harte spectrum* of a, i.e. $(\lambda_1, \ldots, \lambda_n) \notin \sigma(a)$ if there exist elements $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ in A such that $\sum_{j=1}^n u_j(a_j - \lambda_j) = 1$ and $\sum_{j=1}^n (a_j - \lambda_j)v_j = 1$ (here we write for simplicity $a_j - \lambda_j$ instead of $a_j - \lambda_j 1$). We shall also need the *left approximate point spectrum* of a, i.e. the set

$$\tau_l(a) = \Big\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \inf_{\|b\|=1} \sum_{j=1}^n \|(a_j - \lambda_j)b\| = 0\Big\}.$$

The *geometric (joint) spectral radius* of a is defined (cf. [3]) to be the number

$$r(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}$$

where

$$|\lambda| = |(\lambda_1, \ldots, \lambda_n)| = \Big(\sum_{j=1}^n |\lambda_j|^2\Big)^{1/2}.$$

As was shown in [3] (cf. also [8]) $r(a)$ does not depend upon the choice of a joint spectrum of a. In particular, the Harte spectrum $\sigma(a)$ can be replaced by the left approximate point spectrum of a in the above formula without changing the value of $r(a)$.

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