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The Słodkowski spectra and higher Shilov boundaries

by

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Abstract. We investigate relations between the spectra defined by Słodkowski [14] and higher Shilov boundaries of the Taylor spectrum. The results generalize the well-known relation between the approximate point spectrum and the usual Shilov boundary.

Let $A=(A_1,\ldots,A_n)$ be an n-tuple of commuting operators in a Banach space X. We recall the definitions of the Taylor and Słodkowski spectra ([16] and [14]). Let Λ be the exterior algebra with n generators $e_1,\ldots,e_n$. Denote by Λ^p $(0\le p\le n)$ the subset of Λ consisting of all elements of degree p and set $K^p=X\otimes\Lambda^p$. The *Koszul complex* $K(A)$ of the n-tuple $A=(A_1,\ldots,A_n)$ is the cochain complex

$$0\longrightarrow K^0\xrightarrow{d_A^0}K^1\xrightarrow{d_A^1}\ldots\xrightarrow{d_A^{n-1}}K^n\longrightarrow 0$$

where the operators $d_A^p: K^p\to K^{p+1}$ $(0\le p\le n-1)$ are operators of "multiplication" by $A_1e_1+\ldots+A_ne_n$. More precisely,

$$d_A^p(xe_{i_1}\wedge\ldots\wedge e_{i_p})=\sum_{j=1}^n(A_jx)e_j\wedge e_{i_1}\wedge\ldots\wedge e_{i_p}$$
$$=\sum_{s=0}^p(-1)^s\sum_{i_s<j<i_{s+1}}(A_jx)e_{i_1}\wedge\ldots\wedge e_{i_{s-1}}\wedge e_j\wedge e_{i_s}\wedge\ldots\wedge e_{i_p}$$

for all $x\in X$ and $1\le i_1<\ldots<i_p\le n$.

The Słodkowski spectra $\sigma_{\pi,k}$ and $\sigma_{\delta,k}$ $(k=0,\ldots,n)$ are defined as follows:

Let $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{C}^n$. Then λ does not belong to $\sigma_{\pi,k}$ if and only if the Koszul complex $K(A-\lambda)$ is exact at $K^0,\ldots,K^k$ and $d^k_{A-\lambda}$ has closed range. Similarly, $\lambda\notin\sigma_{\delta,k}(A)$ if and only if $K(A-\lambda)$ is exact at $K^n,\ldots,K^{n-k}$. Clearly

$$\sigma_{\pi,0}(A)\subset\sigma_{\pi,1}(A)\subset\ldots\subset\sigma_{\pi,n}(A)=\sigma_{\mathrm{T}}(A)$$

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and

$$\sigma_{\delta,0}(A) \subset \sigma_{\delta,1}(A) \subset \ldots \subset \sigma_{\delta,n}(A) = \sigma_{\mathrm{T}}(A)$$

where $\sigma_{\mathrm{T}}(A)$ denotes the *Taylor spectrum* of A. Further, $\sigma_{\pi,0}(A) = \sigma_\pi(A)$ is the *approximate point spectrum* of A, i.e. $\lambda \in \sigma_\pi(A)$ if and only if

$$\inf\Big\{\sum_{i=1}^{n} \|(A_i - \lambda_i)x\| : x \in X,\ \|x\| = 1\Big\} = 0$$

and $\sigma_{\delta,0}(A) = \sigma_\delta(A)$ is the *defect spectrum*,

$$\lambda \in \sigma_\delta(A) \quad \text{if and only if} \quad \sum_{i=1}^{N}(A_i - \lambda_i)X \neq X\,.$$

The sets $\sigma_{\pi,k}, \sigma_{\delta,k}$ $(k = 0, \ldots, n)$ are non-empty compact subsets of $\mathbb{C}^n$ and the spectra $\sigma_{\pi,k}, \sigma_{\delta,k}$ possess the spectral mapping property for the Taylor functional calculus, i.e.

$$\sigma_{\pi,k}(f(A)) = f(\sigma_{\pi,k}(A)) \quad \text{and} \quad \sigma_{\delta,k}(f(A)) = f(\sigma_{\delta,k}(A))$$

for every $k = 0, \ldots, n$ and for every m-tuple $f = (f_1, \ldots, f_m)$ of functions analytic in a neighbourhood of $\sigma_{\mathrm{T}}(A)$ (see [11]).

Denote by ∂K the topological boundary of a subset $K \subset \mathbb{C}^n$. It is well-known that $\partial\sigma_{\mathrm{T}}(A_1) \subset \sigma_\pi(A_1) \cap \sigma_\delta(A_1)$ for every Banach space operator A_1. Also

$$\partial\sigma_{\mathrm{T}}(A_1, A_2) \subset \sigma_\pi(A_1, A_2) \cup \sigma_\delta(A_1, A_2)$$

for every pair of commuting Banach space operators A_1, A_2 (see [5], [7] and [19]).

The following lemma is a generalization of these facts.

Lemma 1. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of mutually commuting operators in a Banach space X. Then*

(i) $\partial\sigma_{\mathrm{T}}(A) \subset \sigma_{\pi,n-1}(A)$,
(ii) $\partial\sigma_{\mathrm{T}}(A) \subset \sigma_{\delta,n-1}(A)$,
(iii) $\partial\sigma_{\mathrm{T}}(A) \subset \sigma_{\pi,k}(A) \cup \sigma_{\delta,n-k-2}$ $(k = 0, 1, \ldots, n-2)$.

Proof. (i) Let $\lambda \in \partial\sigma_{\mathrm{T}}(A)$ and $\lambda \notin \sigma_{\pi,n-1}(A)$. Then the Koszul complex $K(A-\lambda)$ is exact at $K^0, \ldots, K^{n-1}$ and all operators d_A^i $(i = 0, \ldots, n-1)$ have closed ranges. So $K(A-\lambda)$ is a semi-Fredholm complex (see Definition 2.1 of [1]) and

$$\operatorname{ind} K(A-\lambda) = (-1)^n \dim(K^n | d_A^{n-1} K^{n-1})\,.$$

As $\lambda \in \sigma_{\mathrm{T}}(A)$, the Koszul complex $K(A-\lambda)$ is not exact at K^n so that $\operatorname{ind} K(A-\lambda) \neq 0$. On the other hand, there exists a sequence $\{\lambda^{(s)}\}_{s=1}^\infty$ converging to λ such that $\lambda^{(s)} \notin \sigma_{\mathrm{T}}(A)$ so that $\operatorname{ind} K(A-\lambda^{(s)}) = 0$ for all s. This contradicts the stability of the index (see [1], Theorem 1.4).

The remaining inclusions can be proved analogously.

Higher Shilov boundaries of a uniform algebra were defined in [2] and [13]; for further results see [18] and [6].

We modify the definition slightly as we need Shilov boundaries of a compact subset $K \subset \mathbb{C}^n$ rather than the Shilov boundaries of a uniform algebra $\mathcal{A}$, which are subsets of the maximal ideal space of $\mathcal{A}$. This modified version is frequently used as the definition of the classical Shilov boundary (see e.g. [3], p. 112).

Let K be a nonempty compact subset of $\mathbb{C}^n$. Denote by $C(K)$ the algebra of all continuous functions on K. For a subset $M \subset K$ and a function $f \in C(K)$ set $\|f\|_M = \sup\{|f(z)| : z \in M\}$.

Let $\mathcal{A}$ be a subalgebra of $C(K)$ which contains constant functions and separates points of K.

The *Shilov boundary* $S_0(K, \mathcal{A})$ is the smallest closed subset F of K such that $\|f\|_F = \|f\|_K$ for every $f \in \mathcal{A}$. It is well-known that $\lambda \in K$ belongs to $S_0(K, \mathcal{A})$ if and only if, for every open neighbourhood U of λ, there exists $f \in \mathcal{A}$ such that $\|f\|_{K\cap U} > \|f\|_{K-U}$.

Let $r \geq 1$. For $f = (f_1, \ldots, f_r) \in \mathcal{A}^r$ we denote by V_f the zero set of f, i.e. $V_f = \{z \in K : f_1(z) = \ldots = f_r(z) = 0\}$.

The higher Shilov boundaries $S_r(K, \mathcal{A})$ $(r = 1, 2, \ldots)$ are defined by

$$S_r(K, \mathcal{A}) = \overline{\bigcup_{f \in \mathcal{A}^r} S_0(V_f, \mathcal{A}|V_f)}$$

where $\mathcal{A}|V_f$ is the algebra of all restrictions $\{g|V_f : g \in \mathcal{A}\}$.

Denote by $\mathcal{A}_K \subset C(K)$ the algebra of all restrictions to K of functions analytic in an open neighbourhood of K. It is easy to see that $S_r(K, \mathcal{A}_K)$ are nonempty compact sets and

$$S_0(K, \mathcal{A}_K) \subset \ldots \subset S_{n-1}(K, \mathcal{A}_K) \subset S_n(K, \mathcal{A}_K) = K\,.$$

The meaning of higher Shilov boundaries can be illustrated by the following example (see [13]):

Example. Let K be the closed unit polydisc in $\mathbb{C}^n$,

$$K = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| \leq 1 \ (i = 1, \ldots, n)\}\,.$$

Then

$$S_r(K, \mathcal{A}_K) = \{z \in K : \text{at least } n-r \text{ coordinates of } z \text{ are of modulus } 1\}\,.$$

Theorem 2. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of mutually commuting operators in a Banach space X. Set $K = \sigma_{\mathrm{T}}(A)$. Then*

(i) $S_r(K, \mathcal{A}_K) \subset \sigma_{\pi,r}(A)$ $(r = 0, 1, \ldots, n-1)$,
(ii) $S_r(K, \mathcal{A}_K) \subset \sigma_{\delta,r}(A)$ $(r = 0, 1, \ldots, n-1)$,
(iii) $S_r(K, \mathcal{A}_K) \subset \sigma_{\pi,k}(A) \cup \sigma_{\delta,r-k-1}(A)$ $(r = 0 \leq k < r \leq n-1)$.

Proof. We prove only (iii), as the remaining inclusions are quite analogous.

Let $0 \le k < r \le n-1$. Let $f = (f_1, \dots, f_r) \in \mathcal{A}^n$, $\lambda \in S_0(V_f, \mathcal{A}|V_f)$ and let U be an open neighbourhood of λ. Then there exists a function $g \in \mathcal{A}_K$ such that

$$\sup\{|g(Z)| : z \in V_f \cap U\} > \sup\{|g(Z)| : z \in V_f - U\}.$$

Choose $z_0 \in V_f \subset K$ such that $|g(z_0)| = \max\{|g(z)| : z \in V_f\}$. Clearly $z_0 \in U$. Write $B = (g(A), f_1(A), \dots, f_r(A)) \in B(X)^{r+1}$ (see [16]). By the spectral mapping property [17] we have

$$(g(z_0), 0, \dots, 0) = (g(z_0), f_1(z_0), \dots, f_r(z_0)) \in \sigma_{\mathrm{T}}(B).$$

Further,

$$\begin{aligned}\max\{|u| : u \in \mathbb{C},\ (u, 0, \dots, 0) \in \sigma_{\mathrm{T}}(B)\} &= \max\{|g(z)| : z \in \sigma_{\mathrm{T}}(A),\ f_1(z) = \dots = f_r(z) = 0\} \\ &= \max\{|g(z)| : z \in V_f\} = |g(z_0)|.\end{aligned}$$

Thus

$$(g(z_0), 0, \dots, 0) \in \partial\sigma_{\mathrm{T}}(B) \subset \sigma_{\pi,k}(B) \cup \sigma_{\delta,r-k-1}(B).$$

By the spectral mapping property for $\sigma_{\pi,k}$ and $\sigma_{\delta,r-k-1}$ (see [11]) there exists z_1 in $\sigma_{\pi,k}(A) \cup \sigma_{\delta,r-k-1}(A)$ such that $g(z_1) = g(z_0)$ and $f_1(z_1) = \dots = f_r(z_1) = 0$. So $z_1 \in V_f$ and $z_1 \in U$. Thus $[\sigma_{\pi,k}(A) \cup \sigma_{\delta,r-k-1}(A)] \cap U \ne \emptyset$ for every neighbourhood U of λ. From the compactness of $\sigma_{\pi,k}(A) \cup \sigma_{\delta,r-k-1}(A)$ we conclude that $\lambda \in \sigma_{\pi,k}(A) \cup \sigma_{\delta,r-k-1}(A)$. Hence $S_r(K, \mathcal{A}_K) \subset \sigma_{\pi,k}(A) \cup \sigma_{\delta,r-k-1}(A)$.

Let $A = (A_1, \dots, A_n) \in B(X)^n$ be an n-tuple of mutually commuting operators. Denote by $\sigma_{\mathrm{H}}(A)$ the *Harte spectrum* of A, i.e. $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ does not belong to $\sigma_{\mathrm{H}}(A)$ if and only if there exist operators $L_1, \dots, L_n$, $R_1, \dots, R_n \in B(X)$ such that

$$\sum_{i=1}^{n} L_i(A_i - \lambda_i) = I = \sum_{i=1}^{n} (A_i - \lambda_i) R_i.$$

COROLLARY 3. *Let $A = (A_1, \dots, A_n)$ be an n-tuple of mutually commuting operators in a Banach space X. Set $K = \sigma_{\mathrm{T}}(A)$. Then $S_1(K, \mathcal{A}_K) \subset \sigma_{\mathrm{H}}(A)$.*

Proof. By Theorem 2(iii) for $s = 0$ we have

$$S_1(K, \mathcal{A}_K) \subset \sigma_\pi(A) \cup \sigma_\delta(A)$$

and both $\sigma_\pi(A)$ and $\sigma_\delta(A)$ are contained in $\sigma_{\mathrm{H}}(A)$.

Remarks. (a) Denote by $\mathcal{P}_K$ the algebra of all polynomials on K. As clearly $S_r(K, \mathcal{P}_K) \subset S_r(K, \mathcal{A}_K)$ for every r, we can replace the algebra $\mathcal{A}_K$ in Theorem 2 and Corollary 3 by $\mathcal{P}_K$ (the results are, however, in general weaker).

(b) Denote by $[A]$ the smallest closed subalgebra of $B(X)$ containing $A_1, \dots, A_n$ and the identity operator I. Denote further by L the spectrum of $(A_1, \dots, A_n)$ in the commutative Banach algebra $[A]$. It is well-known that $S_0(L, \mathcal{P}_L) \subset \sigma_\pi(A)$ (see [15]). (Actually, $S_0(L, \mathcal{A}_L) = S_0(L, \mathcal{P}_L)$ as L is a polynomially convex set and so, by the Oka–Weyl approximation theorem, any function $f \in \mathcal{A}_L$ can be uniformly approximated by polynomials.) However, the inclusion $S_r(L, \mathcal{P}_L) \subset \sigma_{\pi,r}(A)$ is no longer true for $r \ge 1$. For an example see [7], Remark 3.4(c).

(c) In general, the inclusion $S_r(K, \mathcal{A}_K) \subset \sigma_{\mathrm{H}}(A)$ is not satisfied for $r \ge 2$. Let H be a separable Hilbert space and $U_+ \in B(H)$ a unilateral shift. Consider operators $A_1, A_2 \in B(H \otimes H)$, $A_1 = U_+ \otimes I$, $A_2 = I \otimes U_+^*$ (see [7], Remark 3.4(a)). Clearly $A_1(U_+^* \otimes I) = I_{H \otimes H} = (I \otimes U_+)A_2$ so that $(0,0) \notin \sigma_{\mathrm{H}}(A_1, A_2)$. On the other hand, it is easy to verify that $(0,0) \in K = S_2(K, \mathcal{A}_K)$ where $K = \sigma_{\mathrm{T}}(A_1, A_2)$.

The preceding results have a natural analogue for the essential spectrum.

Let $A = (A_1, \dots, A_n) \in B(X)^n$ be a commuting n-tuple and let $\lambda \in \mathbb{C}^n$. Then $\lambda \notin \sigma_{\pi e,k}(A)$ $(0 \le k \le n)$ if and only if the Koszul complex $K(A - \lambda)$ is Fredholm at $K^0, \dots, K^k$ (i.e. $\dim \operatorname{Ker} d^0_{A-\lambda} < \infty$ and $\dim(\operatorname{Ker} d^i_{A-\lambda} / \operatorname{Im} d^{i-1}_{A-\lambda}) < \infty$ for all $i = 1, \dots, k$) and $d^k_{A-\lambda}$ has closed range.

Further, $\lambda \notin \sigma_{\delta e,k}(A)$ if and only if $K(A-\lambda)$ is Fredholm at $K^n, \dots, K^{n-k}$.

Again $\sigma_{\pi e,n}(A) = \sigma_{\delta e,n}(A) = \sigma_{\mathrm{Te}}(A)$ where $\sigma_{\mathrm{Te}}(A)$ is the essential Taylor spectrum.

By using the construction of Sadovskiĭ [12] (see also [4]) it is possible to reduce problems involving the essential spectrum to the non-essential case.

Let X be a Banach space. Denote by $\ell^\infty(X)$ the space of all bounded sequences in X with sup norm and let $m(X)$ be the closed subspace of $\ell^\infty(X)$ consisting of all sequences relatively compact in X. Define $\widetilde{X} = \ell^\infty(X)/m(X)$.

Let X and Y be Banach spaces and let $T : X \to Y$ be an operator. Define $T^\infty : \ell^\infty(X) \to \ell^\infty(Y)$ by $T^\infty(\{x_i\}_{i=1}^\infty) = \{Tx_i\}_{i=1}^\infty$. It is easy to see that $T^\infty m(X) \subset m(Y)$ so that we can define naturally the operator $\widetilde{T} : \widetilde{X} \to \widetilde{Y}$.

By [10], $\widetilde{T}$ is injective $\Leftrightarrow$ T is upper semi-Fredholm (i.e. $\dim \operatorname{Ker} T < \infty$ and T has closed range), and $\widetilde{T}$ is surjective $\Leftrightarrow$ T is lower semi-Fredholm (i.e. $\operatorname{codim} TX < \infty$). Also [8], [9], for a commuting n-tuple $A \in B(X)^n$, $\sigma_{\pi e,k}(A) = \sigma_{\pi,k}(\widetilde{A})$, $\sigma_{\delta e,k}(A) = \sigma_{\delta,k}(\widetilde{A})$. Thus we have

COROLLARY 4. *Let $A \in B(X)^n$ be a commuting n-tuple and $K = \sigma_{\mathrm{Te}}(A)$. Then*

$$S_r(K, \mathcal{A}_K) \subset \sigma_{\pi e,r}(A) \cap \sigma_{\delta e,r}(A) \quad (r = 0, \ldots, n-1)$$

and

$$S_r(K, \mathcal{A}_K) \subset \sigma_{\pi e,k}(A) \cup \sigma_{\delta e,r-k-1}(A) \quad (0 \le k < r = 0 \le n-1)\,.$$

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