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Integral formulae for special cases of Taylor's functional calculus

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Abstract. In this paper integral formulae, based on Taylor's functional calculus for several operators, are found. Special cases of these formulae include those of Vasilescu and Janas, and an integral formula for commuting operators with real spectra.

Introduction. Let X be a complex Banach space and Ω be an open subset in $\mathbb{C}^n$. Suppose that

$$a = (a_1, \ldots, a_n)$$

is a tuple of commuting bounded operators on X.

Assume Taylor's joint spectrum, $\mathrm{Sp}(a, X)$ (see [9]), is a subset of Ω and f is a holomorphic function on Ω. Then Taylor (see [10]) defines his analytic functional calculus for a in terms of an abstract Cauchy–Weil integral:

$$f(a)x = \frac{1}{(2\pi i)^n} \int_{\Omega} (R_{\alpha(z)} f(z)x) \wedge dz_1 \wedge \ldots \wedge dz_n\,,$$

where $x \in X$, $R_{\alpha(z)}$ is a homomorphism of cohomology, and

$$\alpha(z) : X \otimes \Lambda[s] \to X \otimes \Lambda[s]\,,$$

$$\alpha(z)(x \otimes \xi) = (z_1 - a_1)x \otimes s_1 \wedge \xi + \ldots + (z_n - a_n)x \otimes s_n \wedge \xi\,,$$

where $\Lambda[s]$ is an exterior algebra, over $\mathbb{C}$, generated by $s_1, \ldots, s_n$.

This means that, in contrast with the Dunford–Schwartz calculus, the functional calculus of Taylor's is rather inexplicit.

In this paper, we show that (see Theorem 3.8) if there exist C^∞ $L(X)$-valued functions, $(b_1, \ldots, b_n)$, defined on $\Omega \setminus F$ (where F is a compact subset of Ω) and such that

(1) $\beta(z) = b_1(z)s_1 + \ldots + b_n(z)s_n$,
(2) $(b_1(z), \ldots, b_n(z))$ is a commuting tuple,
(3) $\alpha(z) + \beta^T(z)$ is invertible on $X \otimes \Lambda[s]$,

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for all $z \in \Omega \setminus F$, then

$$f(a) = \frac{1}{(2\pi i)^n} \int_{\partial D} (\alpha + \beta^T)^{-1}(\overline{\partial}_z(\alpha + \beta^T)^{-1})^{n-1} f(z) s_1 \wedge \ldots \wedge s_n \, dz ,$$

where D is an open subset of Ω which contains F, with compact closure and a piecewise C^1 boundary.

We then show that under certain circumstances, which include the following cases:

(1) X is a Hilbert space,

(2) a has real spectra, i.e. $\mathrm{Sp}(a, X) \subset \mathbb{R}$,

(3) $\mathrm{Sp}_{(a)''}(a) \subseteq \Omega$,

we can define a tuple $(b_1, \ldots, b_n)$ on $\Omega \setminus \mathrm{Sp}(a, X)$ which satisfies the above conditions. We illustrate how the above integral formula reduces to some well known results, including the integral formulae for an analytic functional calculus, found in Vasilescu [11] and Janas [5], and Martinelli–Bochner's and Henkin's integral representations, and obtain some new results.

1. Algebraic notation. Taylor uses homological techniques to define his functional calculus (see [10]). This requires a considerable amount of algebraic notation and terminology, most of which (together with some new definitions) is reproduced here for convenience.

Let K represent a commutative ring with identity, and $\Lambda[s]$ denote the exterior algebra generated by the indeterminates $s = (s_1, \ldots, s_n)$ over K. Then

$$\Lambda[s] = \bigoplus_{p=0}^{n} \Lambda^p[s] ,$$

where $\Lambda^p[s]$ is the K-module of elements of degree p in $\Lambda[s]$ ([1]).

Moreover, if X is a K-module, then we will denote $X \otimes \Lambda[s]$ and $X \otimes \Lambda^p[s]$ by $\Lambda[s, X]$ and $\Lambda^p[s, X]$ respectively, and write $x \otimes s_{j_1} \wedge \ldots \wedge s_{j_p} \in \Lambda^p[s, X]$ as $x s_{j_1} \wedge \ldots \wedge s_{j_p}$. So, if $t = (t_1, \ldots, t_m)$ are indeterminates distinct from s, then

$$\Lambda[s \cup t, X] = \bigoplus_{p=0}^{n} \Lambda^p[s, \Lambda[t, X]] .$$

Now we will introduce an operator, $\rfloor$, which is not mentioned by Taylor. It is a linear operator defined on $\Lambda[s]$ and has the following properties:

(1) $s_i \rfloor s_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$

(2) If $\sigma, \omega \in \Lambda[s]$ then $\sigma \rfloor \omega = \omega \rfloor \sigma$.

([1]) For convenience, we will define $\Lambda^{-1}[s] = \Lambda^{n+1}[s] = 0$.

(3) If $\sigma \in \Lambda^p[s]$, $\omega \in \Lambda[s]$ and $\xi \in \Lambda^q[s]$, then

$$\sigma \rfloor (\xi \wedge \omega) = (\sigma \rfloor \xi) \wedge \omega + (-1)^{pq} \xi \wedge (\sigma \rfloor \omega) .$$

Let A be an algebra of endomorphisms on X. Then $\Lambda[s, A]$ can act on $\Lambda[s \cup t, X]$ via two operations, $\alpha\psi$ and $\alpha^T\psi$, as follows ([2]). Let

$$\alpha = \sum_{j_1, \ldots, j_p} a_{j_1 \ldots j_p} s_{j_1} \wedge \ldots \wedge s_{j_p} \in \Lambda^p[s, A] ,$$

$$\psi = \sum_{k_1, \ldots, l_r} x_{k_1 \ldots k_q l_1 \ldots l_r} s_{k_1} \wedge \ldots \wedge t_{l_r} \in \Lambda^q[s \cup t, X].$$

Then we define $\alpha\psi$ and $\alpha^T\psi$ as follows:

$$\alpha\psi = \sum_{j_1, \ldots, l_r} (a_{j_1 \ldots j_p} x_{k_1 \ldots l_r}) s_{j_1} \wedge \ldots \wedge s_{j_p} \wedge s_{k_1} \wedge \ldots \wedge t_{l_r} ,$$

$$\alpha^T\psi = \sum_{j_1, \ldots, l_r} (a_{j_1 \ldots j_p} x_{k_1 \ldots l_r})(s_{j_1} \wedge \ldots \wedge s_{j_p}) \rfloor (s_{k_1} \wedge \ldots \wedge s_{k_q}) \wedge t_{l_1} \ldots \wedge t_{l_r} .$$

If $a = (a_1, \ldots, a_n)$ is a commuting tuple in A ($a_i a_j = a_j a_i$, for $1 \le i, j \le n$) and $\alpha = a_1 s_1 + \ldots + a_n s_n$ then the following sequence is called a *complex* (since $\alpha^2 = 0$), and will be denoted by $F(X, \alpha)$ ([3]):

$$0 \to \Lambda^0[s, X] \xrightarrow{\alpha} \Lambda^1[s, X] \xrightarrow{\alpha} \ldots \xrightarrow{\alpha} \Lambda^{n-1}[s, X] \xrightarrow{\alpha} \Lambda^n[s, X] \to 0 .$$

Moreover, the cohomology of $F(X, \alpha)$ is the graded module $H(X, \alpha) = \{H^p(X, \alpha)\}_{p=0}^n$, where

$$H^p(X, \alpha) = \mathrm{Ker}\{\alpha : \Lambda^p[s, X] \to \Lambda^{p+1}[s, X]\} / \mathrm{Im}\{\alpha : \Lambda^{p-1}[s, X] \to \Lambda^p[s, X]\} .$$

DEFINITION 1.1. α is called *non-singular* on X if the complex $F(X, \alpha)$ is exact, i.e., if $H^p(X, \alpha) = 0$ for each p.

DEFINITION 1.2. Suppose $F(X, \alpha)$ and $F(Y, \beta)$ are complexes of K-modules, where $\alpha = a_1 s_1 + \ldots + a_n s_n$, $\beta = b_1 t_1 + \ldots + b_m t_m$; $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$ are commuting tuples; and $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_m)$ are indeterminates. Then we say $f : F(X, \alpha) \to F(Y, \beta)$ is a *morphism (of complexes) of degree* r if for all p the following diagram of homomorphisms is commutative:

$$\begin{array}{ccc} \Lambda^{p-1}[s, X] & \xrightarrow{f} & \Lambda^{p+r-1}[s, Y] \\ \downarrow \alpha & & \downarrow \beta \\ \Lambda^p[s, X] & \xrightarrow{f} & \Lambda^{p+r}[s, Y]. \end{array}$$

([2]) The latter operation is a generalisation of δ_p, defined in [9].

([3]) Note also that $(\alpha^T)^2 = 0$ and that we can define a similar complex for α^T.

If $f : F(X, \alpha) \to F(Y, \beta)$ is a morphism then it induces a homomorphism $f_* : H(X, \alpha) \to H(Y, \beta)$ of cohomology (cf. [6]).

Let $(a_1, \ldots, a_n, d_1, \ldots, d_m)$ be a commuting tuple of elements in B, a K-module, which leave the submodule $B_0 \subset B$ invariant. Set $\alpha = a_1 s_1 + \ldots + a_n s_n$ and $\delta = d_1 t_1 + \ldots + d_m t_m$, where $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_m)$ are indeterminates.

DEFINITION 1.3. (B, B_0, α, δ) is called a *Cauchy–Weil system* if α is non-singular on the quotient module B/B_0.

We would now like to define the *Cauchy–Weil resolvent homomorphism*, R_α. To do this we need to define three homomorphisms, s_*, i_*, and π_*, each one induced by a morphism of complexes.

$s_* : H(B, \delta) \to H(B, \alpha + \delta)$ is induced by the morphism $s : F(B, \delta) \to F(B, \alpha + \delta)$, which, for each $0 \le p \le m$, is defined as follows:

$$s : \Lambda^p[t, B] \to \Lambda^{p+n}[s \cup t, B], \qquad f \mapsto f \wedge s_1 \wedge \ldots \wedge s_n .$$

The inclusion map $\underline{i} : B_0 \to B$ induces the inclusion morphism $i : F(B_0, \alpha + \delta) \to F(B, \alpha + \delta)$, which in turn induces the homomorphism $i_* : H(B_0, \alpha + \delta) \to H(B, \alpha + \delta)$.

Note that since, by hypothesis, α is non-singular on B/B_0, $\alpha + \delta$ is also non-singular on B/B_0 (by [9], Lemma 1.3). Hence (by [9], Lemma 1.2), i_* is an isomorphism.

Finally, the homomorphism $\pi_* : H(B_0, \alpha + \delta) \to H(B_0, \delta)$ is induced by the morphism $\pi : F(B_0, \alpha + \delta) \to F(B_0, \delta)$, which has the following properties:

(1) $\pi(s_i) = 0$, for each $1 \le i \le n$,
(2) $\pi(t_j) = t_j$, for each $1 \le j \le m$,
(3) $\pi(x s_{j_1} \wedge \ldots \wedge s_{j_p} \wedge t_{k_1} \wedge \ldots \wedge t_{k_q}) = x\pi(s_{j_1}) \wedge \ldots \wedge \pi(t_{k_q})$.

DEFINITION 1.4. If (B, B_0, α, δ) is a Cauchy–Weil system, and s_*, i_*, and π_* are defined as above, then $R_\alpha = (-1)^n \pi_* i_*^{-1} s_*$ is called the *Cauchy–Weil resolvent homomorphism* for this system.

Since the integral kernel of Taylor's Cauchy–Weil integral is a Cauchy–Weil resolvent homomorphism, it is important to know how to evaluate $R_\alpha[f]$, where $f \in \Lambda^p[s \cup t, B]$ and [ ] represents cohomology class. This means we need to know how to evaluate $i_*^{-1}[f]$.

Let r be a morphism defined by the exact sequence

$$(1) \qquad 0 \to F(B_0, \alpha + \delta) \xrightarrow{i} F(B, \alpha + \delta) \xrightarrow{r} F(B/B_0, \alpha + \delta) \to 0$$

and consider the following commutative diagram:

$$(2)\qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & \Lambda^{p-1}[s \cup t, B_0] & \xrightarrow{\alpha+\delta} & \Lambda^{p}[s \cup t, B_0] & \xrightarrow{\alpha+\delta} & \Lambda^{p+1}[s \cup t, B_0] & \longrightarrow & \ldots \\
 & & \downarrow i & & \downarrow i & & \downarrow i & & \\
\ldots & \longrightarrow & \Lambda^{p-1}[s \cup t, B] & \xrightarrow{\alpha+\delta} & \Lambda^{p}[s \cup t, B] & \xrightarrow{\alpha+\delta} & \Lambda^{p+1}[s \cup t, B] & \longrightarrow & \ldots \\
 & & \downarrow r & & \downarrow r & & \downarrow r & & \\
\ldots & \longrightarrow & \Lambda^{p-1}[s \cup t, B/B_0] & \xrightarrow{\alpha+\delta} & \Lambda^{p}[s \cup t, B/B_0] & \xrightarrow{\alpha+\delta} & \Lambda^{p+1}[s \cup t, B/B_0] & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If $[f] \in H^p[B, \alpha + \delta]$, then we could evaluate $i_*^{-1}[f]$ by a diagram chase:

$$\begin{array}{ccccccc}
 & & f - (\alpha+\delta)g & & & & \\
 & & \downarrow i & & & & \\
g & \xrightarrow{\alpha+\delta} & (\alpha+\delta)g & & f & \xrightarrow{\alpha+\delta} & 0 \\
\downarrow r & & \downarrow r & & \downarrow r & & \downarrow r \\
h & \xrightarrow{\alpha+\delta} & r(f) & = & r(f) & \xrightarrow{\alpha+\delta} & 0
\end{array}$$

Since $[f] \in H^p[B, \alpha + \delta]$, we have $f \in \Lambda^p[s \cup t, B]$ and $(\alpha + \delta)f = 0$. This gives $(\alpha + \delta)r(f) = 0$, since diagram (2) commutes. Since (B, B_0, α, δ) is a Cauchy–Weil system, the bottom sequence of diagram (2) is exact; this means there exists $h \in \Lambda^{p-1}[s \cup t, B/B_0]$ such that $f = (\alpha + \delta)h$. However, diagram (1) is exact, hence there exists $g \in \Lambda^{p-1}[s \cup t, B]$ such that $h = r(g)$. So (since diagram (2) commutes) we have $r(\alpha + \delta)g = r(f)$. Hence, $r(f - (\alpha + \delta)g) = 0$, and since diagram (1) is exact and i is an inclusion, we have $f - (\alpha + \delta)g \in \Lambda^p[s \cup t, B_0]$. So

$$i_*[f - (\alpha + \delta)g] = [f - (\alpha + \delta)g] = [f] .$$

This means

$$i_*^{-1}[f] = [f - (\alpha + \delta)g] ,$$

where

$$(\alpha + \delta)r(g) = r(f) ,$$
$$(\alpha + \delta)r(f) = 0 .$$

Thus, the question of an explicit expression for i_*^{-1} is closely tied to the finding of an explicit expression for $(\alpha + \delta)^{-1}$.

2. Algebraic results. Let A be an algebra of endomorphisms on X, a K-module, and $\alpha = a_1 s_1 + \ldots + a_n s_n \in \Lambda[s, A]$, where $(a_1, \ldots, a_n)$ is a commuting tuple, and $s = (s_1, \ldots, s_n)$.

DEFINITION 2.1. We say $\beta = b_1 s_1 + \ldots + b_n s_n \in \Lambda[s, A]$ *splits* α (over $\Lambda[s, X]$) if

(1) $(b_1, \ldots, b_n)$ is a commuting tuple, i.e. $(\beta^T)^2 = 0$,
(2) $\alpha + \beta^T$ is invertible on $\Lambda[s, X]$.

Let $t = (t_1, \ldots, t_m)$ be indeterminates distinct from s.

PROPOSITION 2.2. *If β splits α over $\Lambda[s, X]$, then β splits α over $\Lambda[s \cup t, X]$.*

Proof. Since $\alpha + \beta^T$ is invertible on $\Lambda[s, X]$, $\alpha + \beta^T$ is invertible on $\Lambda^p[t, \Lambda[s, X]]$, for each p. Hence, as

$$\Lambda[s \cup t, X] = \bigoplus_{p=0}^{m} \Lambda^p[t, \Lambda[s, X]],$$

$\alpha + \beta^T$ is invertible on $\Lambda[s \cup t, X]$.

LEMMA 2.3. *Let β split α, and $x \in \operatorname{Ker}(\alpha)$; then $x = \alpha(\alpha + \beta^T)^{-1} x$. Moreover, $F(X, \alpha)$ is exact.*

Proof. Let $y = (\alpha + \beta^T)^{-1} x$. Then since $(\beta^T)^2 = 0$ and $\alpha x = 0$, we have

$$\begin{aligned}(\alpha + \beta^T)^2 y &= \alpha(\alpha + \beta^T) y + \beta^T(\alpha + \beta^T) y \\ &= \alpha x + \beta^T \alpha y = \beta^T \alpha y .\end{aligned}$$

Hence, since $\alpha^2 = (\beta^T)^2 = 0$, we have

$$\begin{aligned}x &= (\alpha + \beta^T) y = (\alpha + \beta^T)(\alpha + \beta^T)^{-2} \beta^T \alpha y \\ &= (\alpha + \beta^T)^{-2} \alpha \beta^T \alpha y = (\alpha + \beta^T)^{-2} (\alpha + \beta^T)^2 \alpha y = \alpha(\alpha + \beta^T)^{-1} x .\end{aligned}$$

However, x was an arbitrary element of $\operatorname{Ker}(\alpha)$, so $\operatorname{Ker}(\alpha) \subseteq \operatorname{Im}(\alpha)$, and, since $\alpha^2 = 0$, $F(X, \alpha)$ is exact.

LEMMA 2.4. *Let β split α over $\Lambda[s, X]$, $y \in \Lambda[s \cup t, X]$, and $\alpha y = 0$. If $y \in \Lambda^p[s, \Lambda[t, X]]$ then there exist $x \in \Lambda^{p-1}[s, \Lambda[t, X]]$ such that $\alpha x = y$.*

Proof. Since β splits α over $\Lambda[s, X]$, by Proposition 2.2, β splits α over $\Lambda[s \cup t, X]$. Hence, as $\alpha y = 0$, by Lemma 2.3, if $x = (\alpha + \beta^T)^{-1} y$ then $\alpha x = y$. Therefore

$$\alpha x = y \quad \text{and} \quad (\alpha + \beta^T) x = y .$$

Hence

$$\alpha x = y \quad \text{and} \quad \beta^T x = 0 .$$

Let $x = x_0 + \ldots + x_n$, where $x_p \in \Lambda^p[s, \Lambda[t, X]]$, for each p. It follows that

$$(1) \qquad \alpha x_k = 0 \ \text{ for } k \neq p - 1, \qquad \beta^T x_k = 0 \ \text{ for } 0 \le k \le n .$$

However, since $\alpha^2 = (\beta^T)^2 = 0$,

$$(\alpha + \beta^T)^2 = \alpha \beta^T + \beta^T \alpha ,$$

and so

$$(\alpha + \beta^T)^{-2} (\alpha \beta^T + \beta^T \alpha) x_k = x_k ,$$

for each k. So, by (1), it follows that $x = x_{p-1}$.

The following result was suggested by results in Vasilescu [11].

THEOREM 2.5. *Let $\delta = d_1 t_1 + \ldots + d_m t_m$, $(d_1, \ldots, d_m)$ be a commuting tuple in A, β split α, $\alpha\delta + \delta\alpha = 0$, and $(\alpha + \delta) y = 0$. Let $y = y_0 + \ldots + y_n$, where $y_p \in \Lambda^p[s, \Lambda[t, X]]$, for each p. Then*

$$x = \sum_{k=0}^{n-1} \sum_{m=0}^{n-k-1} (-1)^m (\alpha + \beta^T)^{-1} (\delta(\alpha + \beta^T)^{-1})^m y_{k+m+1}$$

is a solution of $(\alpha + \delta) x = y$.

Proof. If we let $x_n = 0$ and for $0 \le p \le n - 1$,

$$(2.6) \qquad x_p = \sum_{m=0}^{n-p-1} (-1)^m (\alpha + \beta^T)^{-1} (\delta(\alpha + \beta^T)^{-1})^m y_{p+m+1}$$

then all we need to show is the following:

(1) $x_p \in \Lambda^p[s, \Lambda[t, X]]$,
(2) $\alpha x_p + \delta x_{p+1} = y_{p+1}$,
(3) $\delta x_0 = y_0$,

for each $0 \le p \le n - 1$.

First, however, we will prove

$$(2.7) \qquad x_p = (\alpha + \beta^T)^{-1} (y_{p+1} - \delta x_{p+1}), \quad \text{for } 0 \le p \le n - 1 .$$

Let $p = n - 1$. Since, by definition, $x_n = 0$ and $x_{n-1} = (\alpha + \beta^T)^{-1} y_n$, we have

$$x_{n-1} = (\alpha + \beta^T)^{-1} (y_n - \delta x_n) .$$

So, the result holds for $p = n - 1$. Now, assume the result holds for $p = k$. Then we have

$$\begin{aligned}(\alpha + \beta^T)^{-1} &(y_k - \delta x_k) \\ &= (\alpha + \beta^T)^{-1} y_k \\ &\quad + \sum_{m=0}^{n-k-1} (-1)^{m+1} (\alpha+\beta^T)^{-1} (\delta(\alpha+\beta^T)^{-1})^{m+1} y_{k+1+m} \quad \text{by (1)} \\ &= \sum_{m=0}^{n-k} (-1)^m (\alpha + \beta^T)^{-1} (\delta(\alpha + \beta^T)^{-1})^m y_{k+m} = x_{k-1}\end{aligned}$$

and so, by induction, (2.7) follows.

We will now prove the following for each p:

$$(2.8)\qquad \begin{aligned} &(1)\quad x_p \in \Lambda^p[s, \Lambda[t, X]],\\ &(2)\quad \alpha x_p + \delta x_{p+1} = y_{p+1},\\ &(3)\quad \alpha(y_p - \delta x_p) = 0. \end{aligned}$$

Let $p = n-1$. Then, by Lemma 2.4 and (2.7), since $\alpha y_n = 0$, and β splits α over $\Lambda[s, X]$, we have

$$x_{n-1} = (\alpha + \beta^T)^{-1} y_n \in \Lambda^{n-1}[s, \Lambda[t, X]],$$

and since $x_n = 0$,

$$\alpha x_{n-1} + \delta x_n = \alpha(\alpha + \beta^T)^{-1} y_n = y_n \quad \text{by Lemma 2.3.}$$

Also, since $\alpha\delta + \delta\alpha = 0$,

$$\begin{aligned} \alpha(y_{n-1} - \delta x_{n-1}) &= \alpha y_{n-1} + \delta\alpha x_{n-1}\\ &= \alpha y_{n-1} + \delta y_n = 0 \quad \text{since } (\alpha+\delta)y = 0. \end{aligned}$$

Now assume the result holds for $p = k+1$. Hence we have

$$\alpha(y_{k+1} - \delta x_{k+1}) = 0 \quad \text{and} \quad y_{k+1} - \delta x_{k+1} \in \Lambda^{k+1}[s, \Lambda[t, X]].$$

So, since β splits α, by Lemma 2.4 and (2.7), we have

$$x_k = (\alpha + \beta^T)^{-1}(y_{k+1} - \delta x_{k+1}) \in \Lambda^k[s, \Lambda[t, X]]$$

and

$$\alpha x_k = y_{k+1} - \delta x_{k+1}.$$

Hence, since $\alpha\delta + \delta\alpha = 0$, it follows that

$$\begin{aligned} \alpha(y_k - \delta x_k) &= \alpha y_k + \delta\alpha x_k\\ &= \alpha y_k + \delta y_{k+1} - \delta^2 x_{k+1} = 0 \quad \text{since } \delta^2 = (\alpha+\delta)y = 0. \end{aligned}$$

So, by induction (2.8) is proved.

It only remains to prove that $\delta x_0 = y_0$.

Since $\alpha^2 = 0$, and β splits α, by Lemma 2.3, $F(\Lambda[t, X], \alpha)$ is exact. However, by (2.8),

$$\alpha(y_0 - \delta x_0) = 0 \quad \text{and} \quad y_0 - \delta x_0 \in \Lambda^0[t, X].$$

So, $y_0 - \delta x_0 = 0$.

3. Cauchy–Weil integral. We now need to define the function spaces used in the definition of Taylor's functional calculus.

Let $C_0^\infty(\Omega)$ denote the set of infinitely differentiable functions with compact support on an open subset, Ω, of $\mathbb{C}^m$.

If X is a Banach space, then $C(\Omega, X)$ and $C^\infty(\Omega, X)$ represent, respectively, the set of continuous X-valued functions defined on Ω, and the set of infinitely differentiable X-valued functions defined on Ω.

Let $B^{(0,\ldots,0)}(\Omega, X) = C(\Omega, X)$, where $(0,\ldots,0)$ has length m. Moreover, if $f, g \in B^{(j_1,\ldots,j_m)}(\Omega, X)$ we shall say $f \in B^{(j_1,\ldots,j_i+1,\ldots,j_m)}(\Omega, X)$ and $(\partial/\partial z_i) f = g$ provided (4)

$$\int_\Omega \left(\frac{\partial}{\partial \overline{z}_i}\psi\right) f\, dz \wedge d\overline{z} = -\int_\Omega \psi g\, dz \wedge d\overline{z}$$

for every $\psi \in C_0^\infty(\Omega)$, where

$$dz = dz_1 \wedge \ldots \wedge dz_m, \qquad d\overline{z} = d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_m.$$

Let

$$B(\Omega, X) = \bigcap_{j_1,\ldots,j_m} B^{(j_1,\ldots,j_m)}(\Omega, X),$$

and let $B_0(\Omega, X)$ be the set consisting of elements of $B(\Omega, X)$ with compact support.

Proposition 3.1. *$f\phi \in B(\Omega, X)$, for every $f \in B(\Omega, X)$ and $\phi \in C^\infty(\Omega, X)$.*

Proof. Let $f \in B^{(0,\ldots,0)}(\Omega, X) = C(\Omega, X)$, and $\phi \in C^\infty(\Omega, X)$; then $f\phi \in B^{(0,\ldots,0)}(\Omega, X)$.

Now, suppose $f\phi \in B^{(j_1,\ldots,j_m)}(\Omega, X)$, for every $f \in B(\Omega, X)$, and $\phi \in C^\infty(\Omega, X)$; and that $1 \le i \le m$.

Let $f \in B(\Omega, X)$ and $\phi \in C^\infty(\Omega, X)$. Then since $f \in B(\Omega, X)$, for each $\psi \in C_0^\infty(\Omega, X)$, we have

$$\begin{aligned} \int_\Omega \left(\frac{\partial\psi}{\partial\overline{z}_i}\right) f\phi\, dz \wedge d\overline{z} &= \int_\Omega \left(\frac{\partial(\psi\phi)}{\partial\overline{z}_i}\right) f\, dz \wedge d\overline{z} - \int_\Omega \left(\frac{\partial\phi}{\partial\overline{z}_i}\right) f\psi\, dz \wedge d\overline{z}\\ &= -\int_\Omega \psi\left(\phi\frac{\partial f}{\partial\overline{z}_i} + \frac{\partial\phi}{\partial\overline{z}_i} f\right) dz \wedge d\overline{z}. \end{aligned}$$

Moreover, since $f \in B(\Omega, X)$, $\partial f/\partial\overline{z}_i \in B(\Omega, X)$; and hence, by hypothesis $\phi\partial f/\partial\overline{z}_i + \partial\phi/\partial\overline{z}_i f \in B^{(j_1,\ldots,j_m)}(\Omega, X)$. Therefore, $f\phi \in B^{(j_1,\ldots,j_i+1,\ldots,j_m)}(\Omega, X)$; and by induction the result follows.

Let

$$\overline{\partial}_z^\alpha = \frac{\partial^{\alpha_1}}{\partial\overline{z}_1} \cdots \frac{\partial^{\alpha_m}}{\partial\overline{z}_m},$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$.

(4) This is well defined, see [10].

DEFINITION 3.2. We give $B(\Omega,X)$ the Fréchet space topology in which a sequence $\{f_p\}_{p=1}^\infty$ in $B(\Omega,X)$ converges to 0 in $B(\Omega,X)$ if for every compact set F in Ω, and for every $\alpha\ge 0$ (i.e. $\alpha_1\ge 0,\ldots,\alpha_m\ge 0$), we have

$$\lim_{p\to\infty}\sup_{z\in F}|\partial_z^\alpha f_p(z)|=0\,.$$

LEMMA 3.3. *If $f\in B(\Omega,X)$ then there exists a sequence, $\{f_p\}_{p=1}^\infty$, in $C^\infty(\Omega,X)$ which converges to f in $B(\Omega,X)$.*

Proof. Since $\Omega\subset\mathbb{C}^m$, there exists a sequence of compact subsets, $\{F_p\}_{p=1}^\infty$, of Ω such that, for each $p\ge 1$, $F_p\subset\operatorname{int}(F_{p+1})$, where $\operatorname{int}(F_{p+1})$ is the interior of F_{p+1}, and $\Omega=\bigcup_{p=1}^\infty F_p$. For each p, let $u_p\in C_0^\infty(\Omega)$ such that $u_p=1$ on F_p. Then, by Proposition 3.1, $u_pf\in B_0(\Omega,X)$. Let F be a compact subset of Ω; then there exists P such that $F\subset F_P$. Hence, if $p\ge P$ then $f=fu_p$ on F; and so fu_p converges to f in $B(\Omega,X)$.

So, now suppose $f\in B_0(\Omega,X)$ and let $\psi\in C_0^\infty(\mathbb{C}^m)$ such that

$$\int_{\mathbb{C}^m}\psi\,dz\wedge d\overline{z}=1\,.$$

Then, for each $p\ge 1$, we define for $\xi\in\Omega$

$$f_p(\xi)=\int_\Omega p^{2m}\psi(p(\xi-z))f(z)\,dz\wedge d\overline{z}\,.$$

It follows, by the properties of convolutions, that $f_p\in C^\infty(\Omega,X)$, and f_p converges to f in $B(\Omega,X)$.

PROPOSITION 3.4. *Let X and Y be Banach spaces, and Ω an open subset in $\mathbb{C}^m$. Suppose $L\in C^\infty(\Omega,L(X,Y))$ and $f\in B(\Omega,X)$. Then $Lf\in B(\Omega,Y)$.*

Proof. Since $f\in B(\Omega,X)$, by the last proposition there exists a sequence $\{f_p\}_{p=1}^\infty$ in $C^\infty(\Omega,X)$ which converges to f in $B(\Omega,X)$. Then for each compact subset, F, of Ω, and each α, we have

$$\begin{aligned}&\lim_{p\to\infty}\sup_{z\in F}\|\overline{\partial}_z^\alpha Lf(z)-\overline{\partial}_z^\alpha Lf_p(z)\|\\&\le\sum_{\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}\sup_{z\in F}\|\overline{\partial}_z^\beta L(z)\|\lim_{p\to\infty}\sup_{z\in F}\|\overline{\partial}_z^{\alpha-\beta}f(z)-\overline{\partial}_z^{\alpha-\beta}f_p(z)\|=0\,.\end{aligned}$$

LEMMA 3.5 ([10], Lemma 3.3). *The quotient space $B(\Omega,X)/B_0(\Omega,X)$ is the inductive limit, denoted by $\varinjlim$, of the system $\{B(V,X):V\subset\Omega,\ \Omega\setminus V$ compact$\}$, where $\{V:V\subset\Omega,\ \Omega\setminus V$ compact$\}$ is directed downward by inclusion and for $V_1\subset V_2$ we map $B(V_2,X)$ into $B(V_1,X)$ by restriction.*

Let

$$r_*:B(\Omega,X)\to\varinjlim\{B(V,X):V\subset\Omega,\ \Omega\setminus V\text{ compact}\}\,,$$

where r_* is induced by the restrictions $r:B(\Omega,X)\to B(V,X)$. Then by the last lemma, r_* is surjective.

Let X be a Banach space, $(a_1,\ldots,a_n)$ be a commuting tuple in $L(X)$, and $(s_1,\ldots,s_n)$ a tuple of indeterminates. Then we will write, for $z\in\mathbb{C}^n$,

$$\alpha(z)=(z_1-a_1)s_1+\ldots+(z_n-a_n)s_n.$$

LEMMA 3.6. *Suppose Ω is an open subset in $\mathbb{C}^n$, F a compact subset of Ω, and $\beta\in C^\infty(\Omega\setminus F,L(\Lambda[s,X]))$, where $\beta(z)$ splits $\alpha(z)$ over $\Lambda[s,X]$, for each $z\in\Omega\setminus F$. Then β splits α over $\Lambda[s,B(\Omega,X)/B_0(\Omega,X)]$.*

Proof. Take any $f\in\Lambda[s,B(\Omega,X)]$, and let

$$h(z)=(\alpha(z)+\beta^T(z))^{-1}f(z)\qquad\forall z\in\Omega\setminus F\,.$$

Then, since $\alpha,\beta\in C^\infty(\Omega\setminus F,L(\Lambda[s,X]))$ and $f\in\Lambda[s,B(\Omega,X)]$, by Proposition 3.4, $h\in B(\Omega\setminus F,X)$. Moreover, $(\alpha+\beta^T)r_*h=r_*f$. So, since r_* is surjective so is $\alpha+\beta^T$.

Now suppose there exists $k\in B(\Omega,X)$ such that

$$(\alpha+\beta^T)r_*k=0\,.$$

Therefore

$$(\alpha(z)+\beta^T(z))k(z)=0\qquad\forall z\in\Omega\setminus F\,,$$

and hence

$$k(z)=0\qquad\forall z\in\Omega\setminus F\,.$$

Thus $r_*k=0$, and since r_* is surjective, $\alpha+\beta^T$ is injective.

Corresponding to the coordinates in $\mathbb{C}^n$ we choose the tuple of indeterminates $d\overline{z}=(d\overline{z}_1,\ldots,d\overline{z}_n)$. Also, we consider $\partial/\partial\overline{z}_1,\ldots,\partial/\partial\overline{z}_n$, on $B(\Omega,X)$, and define

$$\overline{\partial}_z=\frac{\partial}{\partial\overline{z}_1}d\overline{z}_1+\ldots+\frac{\partial}{\partial\overline{z}_n}d\overline{z}_n\in\Lambda[d\overline{z},B(\Omega,X)]\,.$$

One can show ([5]) that $(B(\Omega,X),B_0(\Omega,X),\alpha,\overline{\partial}_z)$ is a Cauchy–Weil system.

Let Ω be an open subset of $\mathbb{C}^n$, and $A(\Omega,X)$ represent the set of X-valued analytic functions defined on Ω.

DEFINITION 3.7. If $f\in A(\Omega,X)$ then

$$f(a)=\frac{1}{(2\pi i)^n}\int_\Omega R_\alpha f(z)\,dz\,,$$

where R_α is the Cauchy–Weil resolvent homomorphism for $(B(\Omega,X),B_0(\Omega,X),\alpha,\overline{\partial}_z)$ and $a=(a_1,\ldots,a_n)$.

([5]) See [10], Lemma 3.4.

THEOREM 3.8. *Let Ω be an open subset of $\mathbb{C}^n$, $D \subset \Omega$ be a open subset, with compact closure and a piecewise C^1 boundary, $F \subset D$ a compact set, and α and β be as in the last lemma. Suppose $f \in A(\Omega, X)$. Then* (6)

$$f(a) = \frac{1}{(2\pi i)^n} \int_{\partial D} (\alpha(z) + \beta(z)^T)^{-1} \times (\overline{\partial}_z(\alpha(z) + \beta(z)^T)^{-1})^{n-1} f(z) s_1 \wedge \ldots \wedge s_n \, dz. \tag{3.9}$$

Proof. Since $s_* f = f s_1 \wedge \ldots \wedge s_n$, and $f \in A(\Omega, X)$, we have

$$(\alpha + \overline{\partial}_z) r_*(f s_1 \wedge \ldots \wedge s_n) = 0.$$

Now let F_1 and F_2 be compact subsets of Ω such that

$$F \subset \mathrm{int}(F_1) \subset F_1 \subset \mathrm{int}(F_2) \subset F_2 \subset D.$$

Let $\phi \in C^\infty(\Omega)$ such that $\phi = 0$ on F_1 and $\phi = 1$ on $\Omega \setminus F_2$; and let

$$g(z) = (-1)^{n-1}\phi(z)(\alpha(z) + \beta(z)^T)^{-1} \times (\overline{\partial}_z(\alpha(z) + \beta(z)^T)^{-1})^{n-1} f(z) s_1 \wedge \ldots \wedge s_n \wedge dz$$

for $z \in \Omega \setminus F$, and $g(z) = 0$ otherwise. Then, by Proposition 3.4, Lemma 3.6, and Theorem 2.5, we have $g \in B(\Omega, X)$, and

$$(\alpha + \overline{\partial}_z) r_* g = r_*(f s_1 \wedge \ldots \wedge s_n).$$

So, by the discussion in Section 1, we have

$$i_*^{-1}(f s_1 \wedge \ldots \wedge s_n) = f s_1 \wedge \ldots \wedge s_n - (\alpha + \overline{\partial}_z) g.$$

Therefore

$$\begin{aligned} f(a) &= \frac{1}{(2\pi i)^n} \int_\Omega R_\alpha f(z) \wedge dz \\ &= \frac{1}{(2\pi i)^n} \int_\Omega (-1)^n \pi_* i_*^{-1} s_* f(z) \wedge dz \\ &= \frac{1}{(2\pi i)^n} \int_\Omega \overline{\partial}_z \phi(z)(\alpha(z) + \beta(z)^T)^{-1} \\ &\qquad \times (\overline{\partial}_z(\alpha(z) + \beta(z)^T)^{-1})^{n-1} f(z) s_1 \wedge \ldots \wedge s_n \wedge dz \\ &= \frac{1}{(2\pi i)^n} \int_D d(\phi(z)(\alpha(z) + \beta(z)^T)^{-1} \\ &\qquad \times (\overline{\partial}_z(\alpha(z) + \beta(z)^T)^{-1})^{n-1} f(z) s_1 \wedge \ldots \wedge s_n \wedge dz) \\ &= \frac{1}{(2\pi i)^n} \int_{\partial D} (\alpha(z) + \beta(z)^T)^{-1} \\ &\qquad \times (\overline{\partial}_z(\alpha(z) + \beta(z)^T)^{-1})^{n-1} f(z) s_1 \wedge \ldots \wedge s_n \wedge dz. \end{aligned}$$

(6) Note, for $n = 1$, this equation reduces to the Dunford–Taylor integral.

4. Special cases. Let, unless otherwise stated, X be a Banach space, and $a = (a_1, \ldots, a_n)$ be a commuting tuple of bounded linear operators on X. For $z \in \mathbb{C}^n$, let

$$\alpha(z) = (z_1 - a_1)s_1 + \ldots + (z_n - a_n)s_n,$$

where $s = (s_1, \ldots, s_n)$ is a tuple of indeterminates; and let

$$\overline{\partial}_z = \frac{\partial}{\partial \overline{z}_1} d\overline{z}_1 + \ldots + \frac{\partial}{\partial \overline{z}_n} d\overline{z}_n.$$

DEFINITION 4.1. The *joint spectrum* of a on X is

$$\mathrm{Sp}(a, X) = \{z \in \mathbb{C}^n : F(\alpha(z), X) \text{ is not exact}\}.$$

EXAMPLE 4.2. Let X be a Hilbert space, and

$$\beta(z) = (a_1 - z_1)^* s_1 + \ldots + (a_n - z_n)^* s_n,$$

where $z \in \mathbb{C}^n$. Then $\alpha(z)^* = \beta(z)^T$; and hence by Corollary 2.2 in Vasilescu [11], $\beta(z)$ splits $\alpha(z)$ for $z \notin \mathrm{Sp}(a, X)$. Moreover, equation (3.9) reduces to Vasilescu's Martinelli type formula.

LEMMA 4.3. *Let $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ be a commuting tuple of endomorphisms on a K-module, X. Let*

$$\alpha = a_1 s_1 + \ldots + a_n s_n \quad \textit{and} \quad \beta = b_1 s_1 + \ldots + b_n s_n.$$

Then

$$(\alpha + \beta^T)^2 = \alpha\beta^T + \beta^T\alpha = \sum_{p=1}^n a_p b_p.$$

Proof. Since $(a_1, \ldots, b_n)$ is a commuting tuple, $\alpha^2 = 0$ and $(\beta^T)^2 = 0$. Hence, if $\sigma \in \Lambda[s, X]$ we have

$$\begin{aligned} (\alpha + \beta^T)^2 \sigma &= \alpha\beta^T\sigma + \beta^T\alpha\sigma \\ &= \sum_{p=1}^n \sum_{q=1}^n a_p b_q s_p \wedge (s_q \rfloor \sigma) + \sum_{q=1}^n \sum_{p=1}^n a_p b_q s_q \rfloor (s_p \wedge \sigma) \\ &= \sum_{p=1}^n a_p b_p \sigma, \end{aligned}$$

by properties (1) and (3) of $\rfloor$.

Let $L(X)$ be the set of bounded linear operators on X.

LEMMA 4.4. *Let Ω be an open set in $\mathbb{C}^n$, $\Psi \in C^2(\Omega, L(X))$, and*

$$\beta(z) = \Psi(z) b_1(z) s_1 + \ldots + \Psi(z) b_n(z) s_n,$$

where $b(z) = (b_1(z), \ldots, b_n(z))$ is a commuting tuple, for each $z \in \Omega$, and $b \in C^2(\Omega, L(X)^n)$. Also, let

$$\Psi b_i = b_i \Psi, \quad \Psi \overline{\partial}_z b_i = \overline{\partial}_z b_i \Psi,$$

for each $1 \le i \le n$. Then

$$\beta^T(\overline{\partial}_z \beta^T)^{n-1} s_1 \wedge \ldots \wedge s_n = (-1)^{n(n-1)/2}(n-1)!\Psi^n \sum_{p=1}^{n} (-1)^{p-1} b_p \wedge \bigwedge_{q \ne p} \overline{\partial}_z b_q \wedge dz_1 \wedge \ldots \wedge dz_n.$$

Proof. See [1].

EXAMPLE 4.5. Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain given by a defining function ϱ, i.e. $\Omega = \{z \in \mathbb{C}^n : \varrho(z) < 0\}$, where ϱ is C^∞ in an open neighbourhood of $\overline{\Omega}$ and

$$\left(\frac{\partial \varrho}{\partial z_1}, \ldots, \frac{\partial \varrho}{\partial z_n}\right) \ne 0 \quad \text{on } \partial\Omega = \{z \in \mathbb{C}^n : \varrho(z) = 0\}.$$

We will now show that we can use Theorem 3.8 to obtain the integral formula of Janas [5].

Let f be continuous on $\overline{\Omega}$ and holomorphic in Ω, and assume that $\mathrm{Sp}(a, X) \subset \Omega$. Then we wish to show the following:

$$f(a) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n} \int_{\partial\Omega} f(z) M(z,a)^{-n} W(z),$$

where

$$M(z,a) = \sum_{p=1}^{n} \frac{\partial \varrho}{\partial z_p}(z)(z_p - a_p),$$

$$W(z) = (n-1)! \sum_{p=1}^{n} (-1)^{(p-1)} \frac{\partial \varrho}{\partial z_p}(z) \wedge \bigwedge_{q \ne p} \frac{\partial}{\partial \overline{z}_q} \frac{\partial \varrho}{\partial z_q}(z) d\overline{z}_q \wedge dz.$$

Remark 1. By Taylor's spectral mapping theorem [10], we have

$$\mathrm{Sp}(M(z,a), X) = \left\{\xi \in \mathbb{C}^n : \xi = \sum_{p=1}^{n} \frac{\partial \varrho}{\partial z_p}(z)(z_p - w_p),\ w \in \mathrm{Sp}(a, X)\right\}.$$

Thus, $M(z,a)$ is invertible in a neighbourhood, N, of $\partial\Omega$.

Remark 2. The term $(-1)^{n(n-1)/2}$ is needed in Henkin's proof of Henkin's integral representation theorem (see [4]) so that in the special case the representation reduces to the Martinelli–Bochner integral representation (see [12] or [8]).

Now, let $\{D_k\}_{k=1}^{\infty}$ be a suitable exhaustion of $\overline{\Omega}$ by open subsets of Ω which contain $\mathrm{Sp}(a, X)$, have compact closure and piecewise C^1 boundaries

contained in the neighbourhood N. Let

$$\beta(z) = M(z,a)^{-1} \frac{\partial \varrho}{\partial z_1}(z) s_1 + \ldots + M(z,a)^{-1} \frac{\partial \varrho}{\partial z_n}(z) s_n,$$

for $z \in N$. Then, by Lemma 4.3, we have

$$(\alpha(z) + \beta(z)^T)^2 = \alpha(z)\beta(z)^T + \beta(z)^T \alpha(z) = I,$$

the identity operator. Hence, as α is holomorphic, we have

$$\alpha \overline{\partial}_z \beta^T = -\overline{\partial}_z \beta^T \alpha.$$

So, by Theorem 3.8, we have, for each k,

$$f(a) = \frac{1}{(2\pi i)^n} \int_{\partial D_k} (\alpha(z) + \beta(z)^T)^{-1} \times (\overline{\partial}_z(\alpha(z) + \beta(z)^T)^{-1})^{n-1} f(z) s_1 \wedge \ldots \wedge s_n\, dz$$
$$= \frac{1}{(2\pi i)^n} \int_{\partial D_k} \beta(z)^T (\overline{\partial}_z \beta(z)^T)^{n-1} f(z) s_1 \wedge \ldots \wedge s_n\, dz.$$

Using Lemma 4.4, this reduces to

$$f(a) = \frac{(-1)^{n(n-1)/2}}{(2\pi i)^n} \int_{\partial D_k} f(z) M(z,a)^{-n} W(z).$$

Hence, passing to the limit $k \to \infty$, we obtain the result (7).

EXAMPLE 4.6. Let $a = (a_1, \ldots, a_n)$ be a commuting tuple of bounded linear operators on X, with real spectra, i.e. $\mathrm{Sp}(a, X) \subseteq \mathbb{R}^n$. Let Ω be an open neighbourhood of $\mathrm{Sp}(a, X)$, and

$$\gamma(z) = (\overline{z}_1 - a_1)s_1 + \ldots + (\overline{z}_n - a_n)s_n.$$

Then, by Lemma 4.3, we have

$$(\alpha(z) + \gamma(z)^T)^2 = \sum_{p=1}^{n} (\Re(z_p) - a_p)^2 + \Im(z_p)^2,$$

where $\Re(z_p)$ and $\Im(z_p)$ are respectively the real and imaginary parts of z_p. So, by Taylor's spectral mapping theorem [10], $\gamma(z)$ splits $\alpha(z)$ when $z \in \Omega \setminus \mathrm{Sp}(a, X)$.

Let

$$\beta(z) = (\alpha(z) + \gamma(z)^T)^{-2}(\overline{z}_1 - a_1)s_1 + \ldots + (\alpha(z) + \gamma(z)^T)^{-2}(\overline{z}_n - a_n)s_n,$$

(7) Note that for the special case where $a \in \mathbb{C}^n$, we obtain Henkin's integral representation.

where $z \in \Omega \setminus \mathrm{Sp}(a, X)$. Then, by Lemma 4.4, equation (3.9) reduces to ([8])

$$f(a) = \frac{1}{(2\pi i)^n} \int_{\partial D} f(z) M(z,a)^{-n} W(z,a),$$

where

$$M(z,a) = \sum_{p=1}^{n} (\Re(z_p) - a_p)^2 + \Im(z_p)^2,$$

$$W(z,a) = (-1)^{n(n-1)/2}(n-1)! \sum_{p=1}^{n} (-1)^{p-1}(\overline{z}_p - a_p) \wedge \bigwedge_{q \neq p} d\overline{z}_q \wedge dz.$$

EXAMPLE 4.7. Let $a = (a_1, \ldots, a_n)$ be a tuple of strongly commuting bounded linear operators ([9]) on X, i.e. there exists a tuple $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ of commuting bounded linear operators on X with real spectra and such that for each p, $a_p = u_p + iv_p$. Let

$$\beta(z) = M(z,a)^{-1}(\overline{z}_1 - u_1 + iv_1)s_1 + \ldots + M(z,a)^{-1}(\overline{z}_n - u_n + iv_n)s_n,$$

where

$$M(z,a) = \sum_{p=1}^{n} (\Re(z_p) - u_p)^2 + (\Im(z_p) - v_p)^2.$$

Then by an argument similar to the one in the last example, we can show that $\beta(z)$ splits $\alpha(z)$, for $z \notin \mathrm{Sp}(a, X)$, and that equation (3.9) reduces to ([10])

$$f(a) = \frac{1}{(2\pi i)^n} \int_{\partial D} f(z) M(z,a)^{-n} W(z,a),$$

where

$$W(z,a) = (-1)^{n(n-1)/2}(n-1)! \sum_{p=1}^{n} (-1)^{p-1}(\overline{z}_p - u_p + iv_p) \wedge \bigwedge_{q \neq p} d\overline{z}_q \wedge dz.$$

Let $b = (b_1, \ldots, b_n)$; then we will denote by $(b)'$ and $(b)''$, respectively, the following sets:

$$(b)' = \{c \in L(X) : cb_i = b_i c \ \forall 1 \le i \le n\},$$
$$(b)'' = \{c \in L(X) : cd = dc \ \forall d \in (b)'\}.$$

([8]) Note that for the special case $a \in \mathbb{C}^n$, this equation reduces to an integral representation for a function which is holomorphic on an open subset of $\mathbb{R}^n$.

([9]) Note that commuting tuples of normal operators on a Hilbert space, and commuting tuples of regular generalised scalar operators (in the sense of Colojoară and Foiaş [3]), are examples of strongly commuting tuples (see McIntosh *et al.* [7]).

([10]) Note that for the special case $a \in \mathbb{C}^n$, this equation reduces to the Martinelli–Bochner integral representation.

EXAMPLE 4.8. In this example we consider the special case that Arens [2] did, namely when f is a holomorphic function on an open subset of $\mathbb{C}^n$, Ω, which contains $\mathrm{Sp}(a, (a)'')$.

Let F_1 and F_2 be compact subsets of Ω such that

$$\mathrm{Sp}(a, (a)'') \subset \mathrm{int}(F_1) \subset F_1 \subset \mathrm{int}(F_2) \subset F_2 \subset \Omega.$$

Now take any $z \in F_2 \setminus \mathrm{int}(F_1)$. Then, since ([11])

$$\mathrm{Sp}(a, (a)'') = \mathbb{C}^n \setminus \Big\{z \in \mathbb{C}^n : \exists d_1, \ldots, d_n \in (a)'' \text{ s.t. } \sum_{p=1}^{n} d_p(z_p - a_p) = I\Big\},$$

there exist $d_1, \ldots, d_n \in (a)''$ such that

$$\sum_{p=1}^{n} d_p(z_p - a_p) = I.$$

Then, by Lemma 4.3, there exists a neighbourhood of z, $N(z)$, such that

$$\beta_{N(z)}(z) = \sum_{p=1}^{n} d_p \Big\{ \sum_{q=1}^{n} d_q(z_q - a_q) \Big\}^{-1} s_p$$

splits $\alpha(z)$, for $z \in N(z)$.

Now, since $F_2 \setminus \mathrm{int}(F_1)$ is compact, there exists a finite number of such neighbourhoods, say $N_1, \ldots, N_m$, which cover $F_2 \setminus \mathrm{int}(F_1)$.

Let $\{\phi_p\}_{p=1}^m$ be a partition of unity subordinate to the cover $\{N_p\}_{p=1}^m$, and let

$$\beta(z) = \sum_{p=1}^{m} \beta_{N_p}(z)\phi_p(z),$$

for $z \in F_2 \setminus \mathrm{int}(F_1)$. Then, by Taylor's spectral mapping theorem [10], $\beta(z)$ splits α when $z \in F_2 \setminus \mathrm{int}(F_1)$; and so by Theorem 3.8, we can obtain an integral representation of Taylor's functional calculus [10] for functions holomorphic in a neighbourhood of $\mathrm{Sp}(a, (a)'')$.

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The Słodkowski spectra and higher Shilov boundaries

by

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Abstract. We investigate relations between the spectra defined by Słodkowski [14] and higher Shilov boundaries of the Taylor spectrum. The results generalize the well-known relation between the approximate point spectrum and the usual Shilov boundary.

Let $A = (A_1, \ldots, A_n)$ be an n-tuple of commuting operators in a Banach space X. We recall the definitions of the Taylor and Słodkowski spectra ([16] and [14]). Let Λ be the exterior algebra with n generators $e_1, \ldots, e_n$. Denote by Λ^p $(0 \le p \le n)$ the subset of Λ consisting of all elements of degree p and set $K^p = X \otimes \Lambda^p$. The *Koszul complex* $K(A)$ of the n-tuple $A = (A_1, \ldots, A_n)$ is the cochain complex

$$0 \longrightarrow K^0 \xrightarrow{d_A^0} K^1 \xrightarrow{d_A^1} \ldots \xrightarrow{d_A^{n-1}} K^n \longrightarrow 0$$

where the operators $d_A^p : K^p \to K^{p+1}$ $(0 \le p \le n-1)$ are operators of "multiplication" by $A_1 e_1 + \ldots + A_n e_n$. More precisely,

$$d_A^p(x e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{n} (A_j x) e_j \wedge e_{i_1} \wedge \ldots \wedge e_{i_p}$$
$$= \sum_{s=0}^{p} (-1)^s \sum_{i_s < j < i_{s+1}} (A_j x) e_{i_1} \wedge \ldots \wedge e_{i_{s-1}} \wedge e_j \wedge e_{i_s} \wedge \ldots \wedge e_{i_p}$$

for all $x \in X$ and $1 \le i_1 < \ldots < i_p \le n$.

The Słodkowski spectra $\sigma_{\pi,k}$ and $\sigma_{\delta,k}$ $(k = 0, \ldots, n)$ are defined as follows:

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. Then λ does not belong to $\sigma_{\pi,k}$ if and only if the Koszul complex $K(A-\lambda)$ is exact at $K^0, \ldots, K^k$ and $d_{A-\lambda}^k$ has closed range. Similarly, $\lambda \notin \sigma_{\delta,k}(A)$ if and only if $K(A-\lambda)$ is exact at $K^n, \ldots, K^{n-k}$. Clearly

$$\sigma_{\pi,0}(A) \subset \sigma_{\pi,1}(A) \subset \ldots \subset \sigma_{\pi,n}(A) = \sigma_{\mathrm{T}}(A)$$

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