[13] D. R. Larson and W. R. Wogen, *Reflexivity properties of $T \oplus 0$*, J. Funct. Anal. 92 (1990), 448–467.
[14] C. Laurie and W. E. Longstaff, *A note on rank one operators in reflexive algebras*, Proc. Amer. Math. Soc. (2) 89 (1983), 293–297.
[15] W. E. Longstaff, *Strongly reflexive lattices*, J. London Math. Soc. (2) 11 (1975), 491–498.
[16] —, *Operators of rank one in reflexive algebras*, Canad. J. Math. 28 (1976), 19–23.
[17] M. Papadakis, *On hyperreflexivity and rank one density for non-CSL algebras*, Studia Math. 98 (1991), 11–17.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ATHENS
PANEPISTIMIOPOLIS
15784 ATHENS, GREECE

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CRETE
71409 IRAKLION
CRETE, GREECE

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WESTERN AUSTRALIA
NEDLANDS, WESTERN AUSTRALIA 6009
AUSTRALIA

Received June 27, 1991 (2814)

STUDIA MATHEMATICA 105 (1) (1993)

Total subspaces in dual Banach spaces which are not norming over any infinite-dimensional subspace

by

M. I. OSTROVSKIĬ (Kharkov)

Abstract. The main result: the dual of separable Banach space X contains a total subspace which is not norming over any infinite-dimensional subspace of X if and only if X has a nonquasireflexive quotient space with a strictly singular quotient mapping.

1. Introduction. Let X be a Banach space and X^* be its dual space. Let us recall some basic definitions.

A subspace M of X^* is said to be *total* if for every $0 \neq x \in X$ there is an $f \in M$ such that $f(x) \neq 0$.

A subspace M of X^* is said to be *norming over a subspace* $L \subset X$ if for some $c > 0$ we have

$$(\forall x \in L)\Big(\sup_{f \in S(M)} |f(x)| \geq c\|x\|\Big),$$

where $S(M)$ is the unit sphere of M. If $L = X$ then M is called *norming*.

The following natural questions arise:

1) How far could total subspaces be from norming ones? (Of course, there are many different concretizations of this question.)

2) What is the structure of Banach spaces whose duals contain total "very" nonnorming subspaces?

3) What is the structure of total subspaces?

These questions were studied by many authors: [Al], [B, pp. 208–216], [BDH], [DJ], [DL], [D], [F], [G], [Ma], [Mc], [M1], [M2], [O1], [O2], [P], [PP], [S1], [S2]. The results obtained find applications in the theory of Fréchet spaces [BDH], [DM], [MM1], [MM2], [M2]; in the theory of improperly posed problems [O3], [PP, pp. 185–196]; and in the theory of universal bases [Pl, p. 31].

The present paper is devoted to the following natural class of subspaces which are far from being norming. A subspace M of X^* is said to be *nowhere*

1991 *Mathematics Subject Classification*: Primary 46B20.

norming if it is not norming over every infinite-dimensional subspace of X. If X is such that X^* contains a total nowhere norming subspace then we write $X \in TNNS$. This class was introduced by W. J. Davis and W. B. Johnson in [DJ], where the first example of a total nowhere norming subspace was constructed. In the same paper it was noted that J. C. Daneman proved that every infinite-dimensional subspace of l_1 is norming over some infinite-dimensional subspace of c_0. In [O2] a class of spaces with the $TNNS$ property was exhibited. A. A. Albanese [Al] proved that the $C(K)$ spaces are not in $TNNS$. The problem of description of Banach spaces with the $TNNS$ property arises in a natural way.

Our main result (Theorem 2.1) states that for a separable Banach space X we have $X \in TNNS$ if and only if for some nonquasireflexive Banach space Y there exists a surjective strictly singular operator $T : X \to Y$.

Section 3 is devoted to the proof of the auxiliary Theorem 2.4. Using the same method we are able to prove the following result (Theorem 3.1):

A Banach space M is isomorphic to a total nonnorming subspace of the dual of some Banach space if and only if M^* contains a closed norming subspace of infinite codimension.

Thus the class of total nonnorming subspaces coincides with the class of Banach spaces which give a negative solution to J. J. Schäffer's problem [Sc, p. 358] (see [DJ, p. 366]).

Section 4 provides several remarks concerning general (not necessarily separable) spaces, in particular, we show that Banach spaces with the Pełczyński property are not $TNNS$.

Section 5 presents an example of a nonquasireflexive separable Banach space without the Pełczyński property and also without $TNNS$.

We hope that our notation is standard and self-explanatory. For a subset A of a Banach space X, $\operatorname{lin} A$, $A^\perp$ and $\operatorname{cl} A$ are, respectively, the linear span of A, the set $\{x^* \in X^* : (\forall x \in A)(x^*(x) = 0)\}$ and the closure of A in the strong topology. For a subset A of a dual Banach space X^*, w^*-$\operatorname{cl} A$ and $A^\top$ are, respectively, the closure of A in the weak* topology and the set $\{x \in X : (\forall x^* \in A)(x^*(x) = 0)\}$. For an operator $T : X \to Y$ the notation $T|_Z$ means the restriction of T to the subspace Z of X.

I wish to express my gratitude to V. M. Kadets for his valuable advice.

2. Main result. Our references for basic concepts and results in Banach space theory are [DS], [LT], [W]. The unit ball and the unit sphere of a Banach space X are denoted by $B(X)$ and $S(X)$ respectively. The term "operator" means a bounded linear operator, and "subspace" means a closed linear subspace.

Let us recall some definitions.

A Banach space X is called *quasireflexive* if its canonical image has finite codimension in X^{**}. The number $\dim(X^{**}/X)$ is called the *order of quasireflexivity* of X and is denoted by $\operatorname{Ord} X$.

An operator $T : X \to Y$ is called *strictly singular* if the restriction of T to any infinite-dimensional subspace of X is not an isomorphism.

The main result of this paper is the following.

2.1. THEOREM. *Let X be a separable Banach space. Then $X \in TNNS$ if and only if for some nonquasireflexive Banach space Y there exists a surjective strictly singular operator $T : X \to Y$.*

Proof. Suppose that such an operator T exists. We may assume without loss of generality that T is a quotient map. Then T^* is an isometric embedding of Y^* into X^*. The subspace $M_1 := T^*(Y^*)$ is nowhere norming because T is strictly singular. Moreover, M_1 is not total. Our aim is to find an isomorphism $Q : X^* \to X^*$ which is a small perturbation of the identity operator and is such that under its action M_1 becomes total but remains nowhere norming.

Since Y is nonquasireflexive, by [DJ, p. 360] there exist a weak* null basic sequence $\{y_n\} \subset Y^*$, a bounded sequence $\{g_n\} \subset Y^{**}$ and a partition $\{I_n\}_{n=1}^\infty$ of the integers into pairwise disjoint infinite subsets such that

$$g_k(y_n) = \begin{cases} 1 & \text{if } n \in I_k, \\ 0 & \text{if } n \notin I_k. \end{cases}$$

Set $u_n^* = T^* y_n$. Since T^* is weak* continuous and isometric, $\{u_n^*\}_{n=1}^\infty$ is a weak* null basic sequence in X^* and there exists a bounded sequence $\{v_n^{**}\}_{n=1}^\infty$ in X^{**} such that

$$v_k^{**}(u_n^*) = \begin{cases} 1 & \text{if } n \in I_k, \\ 0 & \text{if } n \notin I_k. \end{cases}$$

Let $\{s_k^*\}_{k=1}^\infty$ be a normalized sequence spanning a total subspace in X^*. Let the operator $Q : X^* \to X^*$ be given by

$$Q(x^*) = x^* + \sum_{k=1}^\infty 4^{-k} v_k^{**}(x^*) s_k^* / \|v_k^{**}\| .$$

It is clear that Q is an isomorphism. Let $M = Q(M_1)$. We show that M is a total nowhere norming subspace. Let $0 \neq x \in X$ and let $k \in \mathbb{N}$ be such that $s_k^*(x) \neq 0$. Since $\{u_n^*\}_{n=1}^\infty$ is weak* null, we can choose $n \in I_k$ such that $|u_n^*(x)| < 4^{-k} s_k^*(x)/\|v_k^{**}\|$. We have

$$(Q(u_n^*))(x) = u_n^*(x) + 4^{-k} s_k^*(x)/\|v_k^{**}\| \neq 0 .$$

Hence M is total.

Recall that if U, V are subspaces of a Banach space, then the number

$$\delta(U,V) = \inf\{\|u - v\| : u \in S(U), v \in V\}$$

is called the *inclination* of U to V.

We now prove that $M \subset X^*$ is nowhere norming. Suppose that this is not the case and let an infinite-dimensional subspace $L \subset X$ be such that M is norming over L. By strict singularity of T there is no infinite-dimensional subspace of X with nonzero inclination to $\ker T$. Using a standard reasoning with basic sequences (see [Gu]) we can find in L a normalized basic sequence $\{z_i\}$ such that for some sequence $\{t_i\} \subset \ker T$ we have $\|z_i - t_i\| \le 2^{-i}$; we may, moreover, require that

$$(\forall n \in \mathbb{N})(\lim_{i\to\infty} s_n^*(t_i) = 0)\,. \tag{1}$$

Let $c > 0$ be such that

$$(\forall x \in L)(\exists f \in S(M))(|f(x)| \ge c\|x\|)\,.$$

In particular,

$$(\forall i \in \mathbb{N})(\exists f_i \in S(M))(|f_i(z_i)| \ge c)\,.$$

By the definition of M we can find $y_i^* \in Y^*$ such that

$$f_i = T^* y_i^* + \sum_{k=1}^{\infty} 4^{-k} v_k^{**}(T^* y_i^*) s_k^* / \|v_k^{**}\|\,.$$

From this equality we obtain

$$\|f_i\| \ge (2/3)\|T^* y_i^*\|\,.$$

Hence, for every positive integer i,

$$\begin{aligned} c &\le |f_i(z_i)| \le |f_i(z_i - t_i)| + |f_i(t_i)| \\ &\le 2^{-i} + \Big|\sum_{k=1}^{\infty} 4^{-k} v_k^{**}(T^* y_i^*) s_k^*(t_i)/\|v_k^{**}\|\Big| \\ &\le 2^{-i} + (3/2)\sum_{k=1}^{\infty} 4^{-k}|s_k^*(t_i)|\,. \end{aligned}$$

Using (1) and the boundedness of $\{s_n^*\}$ and $\{t_i\}$ we arrive at a contradiction. Hence M is nowhere norming.

Now we begin to prove the converse statement. We need the following result, which follows easily from the arguments of [DJ, p. 358].

2.2. LEMMA. *Let X be a separable Banach space and let N be a subspace of X^* such that the strong closure of the canonical image of X in N^* is of infinite codimension. Then N contains a weak* null basic sequence $\{u_n^*\}_{n=1}^{\infty}$ such that for some bounded sequence $\{v_k^{**}\}_{k=1}^{\infty}$ in X^{**} and some partition $\{I_k\}_{k=1}^{\infty}$ of the positive integers into pairwise disjoint infinite subsets we have*

$$v_k^{**}(u_n^*) = \begin{cases} 1 & \text{if } n \in I_k, \\ 0 & \text{if } n \notin I_k. \end{cases}$$

It turns out that a total nowhere norming subspace need not satisfy the condition of Lemma 2.2.

2.3. PROPOSITION. *There exists a total nowhere norming subspace L of $(l_1)^*$ such that the canonical image of l_1 is dense in L^*.*

Proof. Let a Banach space X be such that X^* is separable, contains closed norming subspaces of infinite codimension and does not contain subspaces isomorphic to l_1. We may take e.g. $X = (\sum \oplus J)_2$, where J is James' space [LT, p. 25].

We need the following definition. Let $a \ge 0$, $b \ge 0$. We say that a subset $A \subset X^*$ is (a,b)-*norming* if the following conditions are satisfied:

$$(\forall x \in X)(\sup\{|x^*(x)| : x^* \in A\} \ge a\|x\|)\,,$$
$$\sup\{\|x^*\| : x^* \in A\} \le b\,.$$

Let $K \subset X^*$ be a closed norming subspace of infinite codimension. Let $\alpha : l_1 \to K$ be some quotient mapping. Hence $\alpha(B(l_1))$ is $(c,1)$-norming for some $c > 0$. Let $\{z_i\}_{i=1}^{\infty} \subset X^*$ be a sequence whose image under the quotient map $X^* \to X^*/K$ is minimal. Define the operator $\beta : l_1 \to X^*$ by

$$\beta(\{a_i\}_{i=1}^{\infty}) = (c/2)\sum_{i=1}^{\infty} a_i z_i/\|z_i\| + \alpha(\{a_i\}_{i=1}^{\infty})\,.$$

It is clear that β is injective and that $\beta(B(l_1))$ is $(c/2, 1 + c/2)$-norming. Let $\{y_i\}_{i=1}^{k} \subset X^*$ be a (finite or infinite) sequence whose image under the quotient mapping $X^* \to X^*/\operatorname{cl}\beta(l_1)$ is minimal and $X^* = \operatorname{cl}(\operatorname{lin}(\{y_i\}_{i=1}^{k} \cup \beta(l_1)))$. Represent l_1 as $l_1 \oplus l_1^k$ (or $l_1 \oplus l_1$ if k is infinite) and define the operator $\gamma : l_1 \oplus l_1^k \to X^*$ by

$$\gamma(\{a_i\}_{i=1}^{\infty}, \{b_i\}_{i=1}^{k}) = \beta(\{a_i\}_{i=1}^{\infty}) + \sum_{i=1}^{k} b_i y_i/\|y_i\|\,.$$

It is clear that γ is injective, its image is dense in X^* and $\gamma(B(l_1))$ is $(c/2, 1 + c/2)$-norming. Moreover, γ is strictly singular since X^* contains no subspaces isomorphic to l_1.

Let $L = \gamma^*(X) \subset (l_1)^*$. This subspace is total since γ is injective. Since $\gamma(B(l_1))$ is $(c/2, 1+c/2)$-norming, $\gamma^*|_X$ is an isomorphic embedding. Therefore the strict singularity of γ implies that L is nowhere norming. On the other hand, it is easy to check that the canonical image of l_1 in L^* may be identified with $\gamma(l_1)$ and therefore is dense. The proposition is proved.

In order to make Lemma 2.2 applicable for our purposes we need the following result.

2.4. THEOREM. *Let X be a Banach space and M be a total nowhere norming subspace of X^*. Then there exists an isomorphic embedding $E : M \to X^*$ such that $E(M)$ is also a nowhere norming subspace and the closure of $E^*(X)$ in the strong topology has infinite codimension in M^*.*

We postpone the proof until Section 3.

Let $X \in TNNS$ and let $M \subset X^*$ be a total nowhere norming subspace. Applying Theorem 2.4 we find an embedding $E : M \to X^*$ such that $N = E(M)$ is a nowhere norming subspace satisfying the condition of Lemma 2.2. Let $\{u_n^*\}_{n=1}^\infty \subset E(M)$, $\{v_k^{**}\}_{k=1}^\infty \subset X^{**}$ and $\{I_k\}_{k=1}^\infty$ be sequences obtained by application of Lemma 2.2 to $N = E(M)$.

We need the following definition [JR].

A sequence $\{x_n^*\}_{n=1}^\infty \subset X^*$ is called *weak* basic* provided that there is a sequence $\{x_n\}_{n=1}^\infty \subset X$ so that $\{x_n, x_n^*\}$ is biorthogonal and for each $x^* \in w^*\text{-cl}(\text{lin}\{x_n^*\}_{n=1}^\infty)$,

$$x^* = w^*\text{-}\lim_{n\to\infty} \sum_{i=1}^n x^*(x_i)x_i^* .$$

By [JR, p. 82] (see also [LT, p. 11]) every weak* null sequence bounded away from 0 in the dual of a separable Banach space has a weak* basic subsequence. Therefore, we may select a weak* basic subsequence $\{u_{n(j)}^*\}_{j=1}^\infty \subset \{u_n^*\}_{n=1}^\infty$. Moreover, by arguments of [JR] we may suppose that $I_k \cap \{n(j)\}_{j=1}^\infty$ is infinite for every $k \in \mathbb{N}$.

Let $Y = X/((\{u_{n(j)}\}_{j=1}^\infty)^\top)$ and let $T : X \to Y$ be the quotient map. Then Y^* may be naturally identified with $w^*\text{-cl}(\text{lin}\{u_{n(j)}^*\}_{j=1}^\infty)$. The space Y is nonquasireflexive because the intersections $I_k \cap \{n(j)\}_{j=1}^\infty$ $(k \in \mathbb{N})$ are infinite. By the well-known properties of weak* basic sequences [LT, p. 11] it follows that for some $\lambda < \infty$ we have

$$B(Y^*) \subset \lambda w^*\text{-cl}\, B(\text{lin}\{u_{n(j)}^*\}_{j=1}^\infty) \subset \lambda w^*\text{-cl}\, B(E(M)) .$$

Therefore Y^* is a nowhere norming subspace of X^*. Hence T is strictly singular. The proof of Theorem 2.1 is complete.

2.5. COROLLARY. *If X is a separable Banach space which contains a complemented subspace Y with $Y \in TNNS$ then $X \in TNNS$.*

Proof. The composition of a strictly singular surjection $T : Y \to Z$ and any projection $P : X \to Y$ is a required surjection.

2.6. COROLLARY. *For every separable Banach space X we have $X \oplus l_1 \in TNNS$.*

To prove this we need only recall that there is a surjective strictly singular operator $T : l_1 \to c_0$ [LT, pp. 75, 108].

2.7. Remark. There exists a space X for which $X \in TNNS$ but X contains no subspaces isomorphic to l_1. In fact, by the James–Lindenstrauss theorem [LT, p. 26] there exists a separable space Z for which Z^{**}/Z is isomorphic to c_0 and Z^{***} is isomorphic to $Z^* \oplus l_1$. Let $X = Z^{**}$. Then there exists a quotient map $T : X \to c_0$. It must be strictly singular because otherwise X contains a subspace isomorphic to c_0 [LT, p. 53], which contradicts the fact that X is a separable dual [LT, p. 103]. At the same time, X contains no subspaces isomorphic to l_1 because X^* is separable.

3. Total nonnorming subspaces in dual Banach spaces.

In this section we prove Theorem 2.4 and the following characterization of total nonnorming subspaces.

3.1. THEOREM. *A Banach space M is isomorphic to a total nonnorming subspace of the dual of some Banach space if and only if M^* contains a closed norming subspace of infinite codimension.*

We need the following lemmas.

3.2. LEMMA [B, p. 39]. *If U and V are Banach spaces and $P : U \to V$ is an operator with nonclosed image then the closure of $P(B(U))$ in the strong topology does not contain interior points.*

3.3. LEMMA [LT, p. 79]. *If $P : U \to V$ is an operator with nonclosed image and $F : U \to V$ is a finite rank operator then $P + F$ has nonclosed image.*

3.4. LEMMA. *Let $P : U \to V$ be an operator with nonclosed image and let $\varepsilon > 0$. Then there exist a functional $f \in V^*$ and an operator $P_1 : U \to V$ such that f does not vanish on $\text{im}\, P$, the image of $P - P_1$ is one-dimensional, $\|P - P_1\| \le \varepsilon$ and $\text{im}\, P_1 \subset \ker f \cap \text{cl}(\text{im}\, P)$.*

Proof. By Lemma 3.2 the closed convex set $\text{cl}\, P(B(U))$ does not have interior points in the subspace $V_0 = \text{cl}(\text{im}\, P) \subset V$. Therefore there exists a functional $f_0 \in S(V_0^*)$ such that

$$(\forall v \in P(B(U)))(|f_0(v)| \le \varepsilon/2) .$$

It is clear that f_0 does not vanish on $\text{im}\, P$. Let $v_0 \in V_0$ be such that $f_0(v_0) = 1$ and $\|v_0\| \le 2$. Define an operator $P_1 : U \to V$ by $P_1(u) = P(u) - f_0(P(u))v_0$. Let f be any continuous extension of f_0 onto the whole V. It can be directly verified that P_1 and f satisfy all the requirements of Lemma 3.4.

Let X be a Banach space and let M be a subspace of its dual. Every element of X may be considered as a functional on M, so there is a natural map of X into M^*. We denote it by H.

3.5. PROPOSITION. *Let X be a Banach space and let M be a total nonnorming subspace in X^*. Then there exists an isomorphic embedding $E: M \to X^*$ such that the closure of $E^*(X)$ in the strong topology is of infinite codimension in M^* and $E^*|_X - H$ is a nuclear operator.*

Proof. In our case the map H is injective because M is total, and is not an isomorphic embedding because M is nonnorming. By the open mapping theorem the image of H is nonclosed.

Let us apply Lemma 3.4 to $P = H$ and $\varepsilon = 1/4$, and denote the obtained functional by f_1 and the obtained operator by H_1. By Lemma 3.3, H_1 also has nonclosed image. Applying Lemma 3.4 to $P = H_1$ and $\varepsilon = 1/8$ we find a functional f_2 and an operator H_2. We continue in an obvious way.

We have $\|H_{i-1} - H_i\| < 2^{-i-1}$. Therefore the sequence $\{H_i\}_{i=1}^{\infty}$ is uniformly convergent. Denote by R its limit.

The operator $R - H$ is nuclear and satisfies the inequality

$$\|R - H\| < 2^{-1}. \tag{2}$$

It is clear that $H(B(X))$ is $(1,1)$-norming. Hence (2) shows that $R(B(X))$ is $(1/2, 3/2)$-norming. Moreover, $\mathrm{cl}(\mathrm{im}\, R) \subset \bigcap_{i=1}^{\infty} \ker f_i$. The sequence $\{f_i\}_{i=1}^{\infty}$ is linearly independent because by construction f_{i+1} does not vanish on $\bigcap_{k=1}^{i} \ker f_k$. Therefore $\mathrm{cl}(\mathrm{im}\, R)$ is of infinite codimension in M^*. Define an operator $E: M \to X^*$ by $(E(m))(x) = (R(x))(m)$. It is an isomorphic embedding because $R(B(X))$ is $(1/2, 3/2)$-norming.

It is easy to see that the restriction of E^* to X coincides with R. Therefore $E^*|_X - H$ is nuclear and $\mathrm{cl}\, E^*(X)$ is of infinite codimension in M^*. The proof is complete.

Proof of Theorem 2.4. A nowhere norming subspace is of course nonnorming. So we can apply Proposition 3.5. It should be noted that for M nowhere norming the operator H is strictly singular.

Let $E: M \to X^*$ be the operator constructed in Proposition 3.5. We need only check that $E(M)$ is nowhere norming. But this follows immediately from the fact that $E^*|_X = R = H + (R - H)$ is strictly singular as the sum of two strictly singular operators [LT, p. 76].

Proof of Theorem 3.1. The necessity follows immediately from Proposition 3.5 and the fact that $\mathrm{cl}\, E^*(X)$ is a norming subspace in M^*.

Suppose that M is a Banach space for which there exists a closed norming subspace $V \subset M^*$ of infinite codimension. Let $\{z_i\}_{i=1}^{\infty}$ be a normalized basic sequence in M^*/V and let $m_i^* \in M^*$ ($i \in \mathbb{N}$) be such that $\|m_i^*\| \le 2$ and $Q(m_i^*) = z_i$, where $Q: M^* \to M^*/V$ is the quotient map.

Let $X = V \oplus l_1$. Define an operator $H: X \to M^*$ by

$$H(v, \{a_i\}_{i=1}^{\infty}) = v + \sum_{i=1}^{\infty} (a_i/i) m_i^*.$$

It is clear that it is injective but is not an isomorphic embedding.

The restriction of H^* to M is an isomorphic embedding because V is a norming subspace. The subspace $H^*(M) \subset X^*$ is total because H is injective, and is not norming because H is not an isomorphic embedding. This completes the proof of Theorem 3.1.

3.6. COROLLARY. *If M is a total subspace of X^* and M is quasireflexive then X is quasireflexive and* $\mathrm{Ord}(X) = \mathrm{Ord}(M)$.

Proof. It is known [CY] that $\mathrm{Ord}(X^*) = \mathrm{Ord}(X)$ and that the order of quasireflexivity of a subspace is not greater than that of the whole space.

It is well-known and easy to see that the duals of quasireflexive spaces have no norming subspaces of infinite codimension. Therefore by Theorem 3.1, $M \subset X^*$ is norming. Hence X is isomorphic to a subspace of M^*. Using the above-mentioned result we obtain $\mathrm{Ord}(X) \le \mathrm{Ord}(M^*) = \mathrm{Ord}(M)$.

Using the above result once more yields $\mathrm{Ord}(M) \le \mathrm{Ord}(X^*) = \mathrm{Ord}(X)$. The proof is complete.

3.7. Remark. By [DJ, p. 355] nonquasireflexivity of X does not yield the existence in X^* of an infinite-codimensional norming subspace. Therefore there exist nonquasireflexive spaces which are not isomorphic to total nonnorming subspaces.

4. Remarks on the nonseparable case and on spaces with the Pełczyński property. Theorem 2.1 and Corollaries 2.5 and 2.6 are not valid in the nonseparable case. In order to prove this let us show that the space $X = l_1 \oplus l_2(\Gamma)$ does not have the *TNNS* property if $\mathrm{card}(\Gamma) > 2^{\mathfrak{c}}$.

Let M be a total subspace in X^*. Then $2^{\mathrm{card}(M)} \ge \mathrm{card}(X) \ge \mathrm{card}(\Gamma)$. Consequently, $\mathrm{card}(M) > \mathfrak{c}$. Therefore M contains a set of functionals of cardinality greater than $\mathfrak{c}$ whose restrictions to l_1 coincide. Therefore the intersection of M with the subspace of X^* which vanishes on l_1 is an infinite-dimensional subspace in $\{0\} \oplus l_2(\Gamma)$. If we "transfer" this subspace to X then we obtain a subspace over which M is norming.

PROBLEM. Characterize *TNNS* in the nonseparable setting.

At the moment it is known [Al] that $C(K) \notin$ *TNNS* for every compact K.

4.1. PROPOSITION. *Let X be a Banach space such that every strictly singular operator $T: X \to Y$ is weakly compact. Then $X \notin$ TNNS.*

Proof. Evidently it is sufficient to consider the case when X is nonreflexive. Let M be a total nowhere norming subspace in X^*. Let X_M be the completion of X under the norm

$$\|x\|_M = \sup\{|f(x)| : f \in S(M)\}.$$

Let $T : X \to X_M$ be the natural embedding. Then T is strictly singular because M is nowhere norming. Hence T is weakly compact.

The subspace $M \subset X^*$ may also be considered as a subspace of $(X_M)^*$. Moreover, the restriction of $T^* : (X_M)^* \to X^*$ to M is an isometry.

The operator T^* is weakly compact by V. Gantmacher's theorem [DS, VI.4.8]. Therefore M is reflexive, hence weak* closed in X^* by the Krein–Shmul'yan theorem [DS, V.5.7]. Since M is a total subspace of X^* we obtain $M = X^*$. This contradiction completes the proof.

4.2. Remark. For separable spaces this proposition follows immediately from Theorem 2.1.

The conditions of Proposition 4.1 are satisfied by spaces with the Pełczyński property. Let us recall the definition.

A Banach space X has the *Pełczyński property* if for every subset $K \subset X^*$ that is not relatively weakly compact there exists a weakly unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ in X such that

$$\inf_n \sup_{x^* \in K} x^*(x_n) > 0.$$

This property was introduced by A. Pełczyński in [Pe] (under the name "property (V)"). In the same paper it was proved that for any compact Hausdorff space S the space $C(S)$ has property (V). For other spaces with the Pełczyński property see [W, pp. 166–172].

The fact that spaces with the Pełczyński property satisfy the conditions of Proposition 4.1 follows from the next proposition.

4.3. Proposition [W, p. 172]. *Suppose X has the Pełczyński property. Then for every operator $T : X \to Y$ that is not weakly compact there exists a subspace $X_1 \subset X$ such that X_1 is isomorphic to c_0 and the restriction of T to X_1 is an isomorphic embedding.*

The spaces which fail the *TNNS* property need not have the Pełczyński property and need not satisfy the conditions of Proposition 4.1.

A corresponding example is given by James' space J. Let us recall its definition [LT, p. 25]. The space J consists of all sequences of scalars $x = (a_1, a_2, \ldots)$ for which $\lim_{n\to\infty} a_n = 0$ and

$$\|x\| = \sup 2^{-1/2}((a_{p_1} - a_{p_2})^2 + (a_{p_2} - a_{p_3})^2 + \ldots + (a_{p_{m-1}} - a_{p_m})^2)^{1/2} < \infty$$

where the supremum is taken over all choices of m and $p_1 < p_2 < \ldots < p_m$.

It is easy to see that the identity operator $T : J \to c_0$ is a non-weakly compact strictly singular operator. On the other hand, any total subspace in J^* is norming over J. This follows from the well-known properties of quasireflexive spaces.

In fact, there are nonquasireflexive spaces of this type as is shown in the next section.

5. A nonquasireflexive separable Banach space without the Pełczyński property whose dual has no nowhere norming subspaces

5.1. Theorem. *There exists a nonquasireflexive Banach space $X \notin TNNS$ such that there exists a strictly singular non-weakly compact operator $T : X \to c_0$.*

Proof. Let $X = (\sum_{n=1}^{\infty} \oplus J)_2$. The unit vectors in J are denoted by $\{e_i\}_{i=1}^{\infty}$. It is known [LT, p. 25] that $\{e_i\}$ is a shrinking basis of J, therefore its biorthogonal functionals $\{e_i^*\}_{i=1}^{\infty}$ form a basis of J^*.

It is clear that the vectors

$$e_{n,j} = (0, \ldots, 0, e_j, 0, \ldots)$$

(where e_j is in the nth place), after any numeration preserving order in each sequence $\{e_{n,j}\}_{j=1}^{\infty}$, form a basis of X. We need the following two lemmas about X and its dual.

5.2. Lemma. *Every weakly null sequence $\{x_m\}_{m=1}^{\infty}$ in X for which $\inf \|x_m\| > 0$ contains a subsequence equivalent to the unit vector basis of l_2.*

5.3. Lemma. *Every infinite-dimensional subspace of X^* contains a subspace isomorphic to l_2.*

These lemmas easily follow by well-known arguments (see [An], [HW]).

Consider an operator $T : (\sum \oplus J)_2 \to c_0 = (\sum \oplus c_0)_0$ defined by

$$T(x_1, \ldots, x_n, \ldots) = (x'_1, \ldots, x'_n, \ldots),$$

where (x_i) is a sequence of elements of J and (x'_i) is the sequence of elements of c_0 with the same coordinates. It is clear that T is continuous. It is not weakly compact because for any $n \in \mathbb{N}$ the sequence $(T(\sum_{j=1}^{k} e_{n,j}))_{k=1}^{\infty}$ has no limit points in the weak topology. At the same time, T is strictly singular because by Lemma 5.2 the space X contains no subspaces isomorphic to c_0.

Suppose that $X \in TNNS$. Then by Theorem 2.1 there exists a surjective strictly singular operator $T : X \to Z$ where Z is a certain Banach space. Consequently, Z^* is isomorphic to a subspace of X^*. By Lemma 5.3, Z^* contains a subspace isomorphic to l_2. Denote this subspace by U. Let $R : Z \to Z/U^\top$ be the quotient map. The space $(Z/U^\top)^*$ may be naturally identified with w^*-cl U. Since U is reflexive, by Krein–Shmul'yan theorem [DS, V.5.7] we have w^*-cl $U = U$. Therefore $Z/U^\top$ is isomorphic to l_2. Let $\{u_i\}_{i=1}^\infty$ be a sequence in $Z/U^\top$ equivalent to the unit vector basis of l_2. By Lemma 2 of [GR] we can find in X a weakly null sequence $\{x_i\}_{i=1}^\infty$ for which $\{RTx_i\}_{i=1}^\infty$ is a subsequence of $\{u_n\}$. By Lemma 5.2, $\{x_i\}$ contains a subsequence $\{x_{n_i}\}_{i=1}^\infty$ which is equivalent to the unit vector basis of l_2. The restriction of RT to the closed linear span of $\{x_{n_i}\}_{i=1}^\infty$ is an isomorphism. Because T is strictly singular this gives us a contradiction.

References

[Al] A. A. Albanese, *On total subspaces in duals of spaces of type $C(K)$ or L^1*, preprint.

[An] A. Andrew, *James' quasi-reflexive space is not isomorphic to any subspace of its dual*, Israel J. Math. 38 (1981), 276–282.

[B] S. Banach, *Théorie des opérations linéaires*, Monografje Mat. 1, Warszawa 1932.

[BDH] E. Behrends, S. Dierolf and P. Harmand, *On a problem of Bellenot and Dubinsky*, Math. Ann. 275 (1986), 337–339.

[CY] P. Civin and B. Yood, *Quasi-reflexive spaces*, Proc. Amer. Math. Soc. 8 (1957), 906–911.

[DJ] W. J. Davis and W. B. Johnson, *Basic sequences and norming subspaces in non-quasi-reflexive Banach spaces*, Israel J. Math. 14 (1973), 353–367.

[DL] W. J. Davis and J. Lindenstrauss, *On total nonnorming subspaces*, Proc. Amer. Math. Soc. 31 (1972), 109–111.

[DM] S. Dierolf and V. B. Moscatelli, *A note on quojections*, Funct. Approx. Comment. Math. 17 (1987), 131–138.

[D] J. Dixmier, *Sur un théorème de Banach*, Duke Math. J. 15 (1948), 1057–1071.

[DS] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience, New York 1958.

[F] R. J. Fleming, *Weak*-sequential closures and the characteristic of subspaces of conjugate Banach spaces*, Studia Math. 26 (1966), 307–313.

[G] B. V. Godun, *On weak* derived sets of sets of linear functionals*, Mat. Zametki 23 (1978), 607–616 (in Russian).

[GR] B. V. Godun and S. A. Rakov, *Banach–Saks property and the three space problem*, ibid. 31 (1982), 61–74 (in Russian).

[Gu] V. I. Gurariĭ, *On openings and inclinations of subspaces of a Banach space*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 1 (1965), 194–204 (in Russian).

[HW] R. Herman and R. Whitley, *An example concerning reflexivity*, Studia Math. 28 (1967), 289–294.

[JR] W. B. Johnson and H. P. Rosenthal, *On w^*-basic sequences and their applications to the study of Banach spaces*, ibid. 43 (1972), 77–92.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer, Berlin 1977.

[Ma] S. Mazurkiewicz, *Sur la dérivée faible d'un ensemble de fonctionnelles linéaires*, Studia Math. 2 (1930), 68–71.

[Mc] O. C. McGehee, *A proof of a statement of Banach about the weak* topology*, Michigan Math. J. 15 (1968), 135–140.

[MM1] G. Metafune and V. B. Moscatelli, *Generalized prequojections and bounded maps*, Results in Math. 15 (1989), 172–178.

[MM2] —, —, *Quojections and prequojections*, in: Advances in the Theory of Fréchet Spaces, T. Terzioğlu (ed.), Kluwer, Dordrecht 1989, 235–254.

[M1] V. B. Moscatelli, *On strongly non-norming subspaces*, Note Mat. 7 (1987), 311–314.

[M2] —, *Strongly nonnorming subspaces and prequojections*, Studia Math. 95 (1990), 249–254.

[O1] M. I. Ostrovskiĭ, *w^*-derived sets of transfinite order of subspaces of dual Banach spaces*, Dokl. Akad. Nauk Ukrain. SSR Ser. A 1987 (10), 9–12 (in Russian).

[O2] —, *On total nonnorming subspaces of a conjugate Banach space*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 53 (1990), 119–123 (in Russian); English transl.: J. Soviet Math. 58 (6) (1992), 577–579.

[O3] —, *Regularizability of superpositions of inverse linear operators*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 55 (1991), 96–100 (in Russian); English transl.: J. Soviet Math. 59 (1) (1992), 652–655.

[Pe] A. Pełczyński, *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 10 (1962), 641–648.

[P] Yu. I. Petunin, *Conjugate Banach spaces containing subspaces of zero characteristic*, Dokl. Akad. Nauk SSSR 154 (1964), 527–529 (in Russian); English transl.: Soviet Math. Dokl. 5 (1964), 131–133.

[PP] Yu. I. Petunin and A. N. Plichko, *The Theory of Characteristic of Subspaces and its Applications*, Vishcha Shkola, Kiev 1980 (in Russian).

[Pl] A. N. Plichko, *On bounded biorthogonal systems in some function spaces*, Studia Math. 84 (1986), 25–37.

[S1] D. Sarason, *On the order of a simply connected domain*, Michigan Math. J. 15 (1968), 129–133.

[S2] —, *A remark on the weak-star topology of l^∞*, Studia Math. 30 (1968), 355–359.

[Sc] J. J. Schäffer, *Linear differential equations and functional analysis. VI*, Math. Ann. 145 (1962), 354–400.

[W] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Stud. Adv. Math. 25, Cambridge University Press, 1991.

INSTITUTE FOR LOW TEMPERATURE PHYSICS AND ENGINEERING
UKRAINIAN ACADEMY OF SCIENCES
47 LENIN AVENUE
KHARKOV 310164, UKRAINE
E-mail: OSTROVSKII%ILT.KHARKOV.UA@RELAY.USSR.EU.NET

Received December 9, 1991
Revised version September 30, 1992 (2873)