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The decomposability of operators relative to two subspaces

by

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Abstract. Let M and N be nonzero subspaces of a Hilbert space H satisfying $M \cap N = \{0\}$ and $M \vee N = H$ and let $T \in \mathcal{B}(H)$. Consider the question: If T leaves each of M and N invariant, respectively, intertwines M and N, does T decompose as a sum of two operators with the same property and each of which, in addition, annihilates one of the subspaces? If the angle between M and N is positive the answer is affirmative. If the angle is zero, the answer is still affirmative for finite rank operators but there are even trace class operators for which it is negative. An application gives an alternative proof that no distance estimate holds for the algebra of operators leaving M and N invariant if the angle is zero, and an analogous result is obtained for the set of operators intertwining M and N.

1. Introduction. Given a linear space $\mathcal{A}$ of operators on a Hilbert space, an important question is whether it is possible to approximate an operator T in $\mathcal{A}$ by simpler elements of $\mathcal{A}$, or even better, to decompose T as a (perhaps finite) sum of such elements. For example, if $\mathcal{A}$ is a nest algebra, any finite rank operator in $\mathcal{A}$ can be decomposed as a finite sum of rank one operators in $\mathcal{A}$ and these sums are ultraweakly dense in $\mathcal{A}$ [4]. If $\mathcal{A} = \operatorname{Alg}\mathcal{L}$ for a complete atomic Boolean lattice $\mathcal{L}$, the first property holds [16], but the second does not [13] (see also [1, Addendum]). Related results are in [1, 5, 6, 8, 14, 17].

Another important possible property of an operator algebra $\mathcal{A}$ is the estimation of, in the sense of Arveson [2], the (metric) distance of an operator from $\mathcal{A}$ in terms of the invariant subspaces of the algebra. Arveson [3] showed that this is equivalent to a decomposability property, this time of the pre-annihilator ${}^{\perp}\mathcal{A}$ of $\mathcal{A}$. This holds, for instance, in nest algebras [2, 12], but exactly which algebras have this property has proved an elusive question.

Questions such as the above are discussed in this paper. For instance, given two subspaces M and N of a Hilbert space H such that $M \cap N$

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$= \{0\}$ and $M \vee N = H$, we investigate the question of whether an operator, which leaves them invariant or intertwines them, decomposes as a sum of two operators with the same property and each of which, in addition, annihilates one of the subspaces. If the angle between M and N is positive this can be done for all such operators. On the other hand, if the angle is zero this remains true for all finite rank operators but there exist even trace class operators for which it fails.

As a corollary we give a transparent proof of the result of Papadakis [17] that no distance estimate holds for the algebra of operators leaving M and N invariant if M and N are at a zero angle. We also prove the analogous result for the set of operators that intertwine M and N.

Throughout this paper we shall use the following terminology and notation. The letter H will denote a complex nonzero Hilbert space with inner product denoted by $\langle\cdot,\cdot\rangle$. The orthogonal complement of a subset L of H is denoted by $L^{\perp}$. By a *subspace* of H we mean a closed linear manifold. If K and L are subspaces of H, $K \vee L$ denotes their closed linear span. The closed linear span of a family $\{L_\gamma\}_\Gamma$ of subspaces is denoted by $\bigvee_\Gamma L_\gamma$ or simply $\bigvee L_\gamma$. The linear span of two vectors x and y is $[x,y]$. Two subspaces K and L of H satisfying $K \cap L = \{0\}$ are *at a zero angle* if $\sup|\langle x,y\rangle| = 1$ (equivalently, $\inf\|x-y\| = 0$) where the supremum (correspondingly, the infimum) is taken over all unit vectors x of K and y of L. It is well known that K and L are at zero angle if and only if the vector sum $K+L$ is not closed. Two subspaces M and N of H are *in generic position* (in H) if $M \cap N = M^{\perp} \cap N^{\perp} = M^{\perp} \cap N = M \cap N^{\perp} = \{0\}$.

By an *operator* on H we mean an everywhere defined bounded linear transformation from H to H. The set of operators on H is denoted by $\mathcal{B}(H)$. A *projection* on H is a self-adjoint idempotent of $\mathcal{B}(H)$ and the projection onto L, for a subspace L of H, is denoted by P_L. For an operator $A \in \mathcal{B}(H)$, its graph $G(A)$ is the subset $\{(x,Ax) : x \in H\}$ of $H \oplus H$, its adjoint is A^* and its trace, if it is of trace class, is denoted by $\operatorname{tr}(A)$. If $\mathcal{A}$ is a subset of $\mathcal{B}(H)$, $\mathcal{A}^*$ is the set $\{A^* : A \in \mathcal{A}\}$ and the *pre-annihilator* ${}^{\perp}\mathcal{A}$ of $\mathcal{A}$ is $\{B \in \mathcal{B}(H) : B$ is of trace class and $\operatorname{tr}(AB) = 0$, for all $A \in \mathcal{A}\}$. The *rank* of an operator on H is the algebraic dimension of its range, and for vectors $e,f \in H$ the operator $x \mapsto \langle x,e\rangle f$ is denoted by $e \otimes f$. If $A \in \mathcal{B}(H)$ and K, L are subspaces of H we say that A *intertwines* them if $A(K) \subseteq L$ and $A(L) \subseteq K$. The algebra of operators leaving invariant all the members of a family $\mathcal{F}$ of subspaces of H is denoted by $\operatorname{Alg}\mathcal{F}$. If a lattice $\mathcal{F}$ of subspaces of H is complete with lattice operations (arbitrary) closed linear spans and intersections and it contains the trivial subspaces $\{0\}$ and H we say that $\mathcal{F}$ is a *subspace lattice* on H. A totally ordered subspace lattice $\mathcal{N}$ is called a *nest* and the corresponding algebra $\operatorname{Alg}\mathcal{N}$ is a *nest algebra*. Nests are special cases of completely distributive subspace lattices. We shall not give the precise definition here but refer the reader to [15]. The following characterization of complete distributivity is given in [15]. If $K, L \in \mathcal{F}$ we denote by L_- the subspace $L_- = \bigvee\{M \in \mathcal{F} : L \not\subseteq M\}$, by K_* the subspace $K_* = \bigcap\{L_- : L \in \mathcal{F}$ and $L \not\subseteq K\}$ and by $K_\#$ the subspace $K_\# = \bigvee\{L \in \mathcal{F} : K \not\subseteq L_-\}$. Then the complete distributivity of $\mathcal{F}$ is equivalent to the condition $K = K_*$ for all $K \in \mathcal{F}$ and is also equivalent to $K = K_\#$ for all $K \in \mathcal{F}$.

2. Density.

2. Density. If $\mathcal{F}$ is a subspace lattice on the Hilbert space H we denote by $\mathcal{S}(\mathcal{F})$ the set of all operators on H that annihilate all the operators of rank at most one in $\operatorname{Alg}\mathcal{F}$, that is,

$$\mathcal{S}(\mathcal{F}) = \{S \in \mathcal{B}(H) : \operatorname{tr}(SR) = 0 \text{ for every } R \in \operatorname{Alg}\mathcal{F} \text{ of rank at most one}\}.$$

LEMMA 2.1. *For any subspace lattice $\mathcal{F}$ on H,*

$$\mathcal{S}(\mathcal{F}) = \{S \in \mathcal{B}(H) : S(K) \subseteq K_- \text{ for every } K \in \mathcal{F}\}.$$

Proof. It is shown in [15] that, for any vectors $e,f \in H$, the operator $R = e \otimes f$ is in $\operatorname{Alg}\mathcal{F}$ if and only if there is a $K \in \mathcal{F}$ such that $f \in K$ and $e \in (K_-)^{\perp}$. Thus $\operatorname{tr}(SR) = \langle Sf, e\rangle$ vanishes for all such R in $\operatorname{Alg}\mathcal{F}$ if and only if $S(K) \subseteq K_-$ for every $K \in \mathcal{F}$.

The above lemma is not essentially new. The set $\mathcal{S}(\mathcal{F})$ has been investigated in [5] and [6]. The following proposition and corollary can be deduced from the former but we include a more direct proof here for the reader's convenience and for future reference.

PROPOSITION 2.2. *Let $\mathcal{F}$ be a subspace lattice on H and let $e,f \in H$ be vectors. The following are equivalent.*

(i) $e \otimes f \in \mathcal{S}(\mathcal{F})$,
(ii) $f \in L$ *and* $e \in (L_\#)^{\perp}$ *for some* $L \in \mathcal{F}$,
(iii) $f \in K_*$ *and* $e \in K^{\perp}$ *for some* $K \in \mathcal{F}$.

Proof. (i)⇒(ii). Let $e \otimes f \in \mathcal{S}$. Put $L = \bigcap\{N \in \mathcal{F} : f \in N\}$. Then $f \in L$. We claim that $e \in (L_\#)^{\perp}$. Suppose that $M \in \mathcal{F}$ and $L \not\subseteq M_-$. Then $f \notin M_-$ so, since by Lemma 2.1, $e \otimes f(M) \subseteq M_-$, it follows that $e \in M^{\perp}$. Hence $e \in \bigcap\{M^{\perp} : L \not\subseteq M_-\} = (L_\#)^{\perp}$.

(ii)⇒(iii). Suppose that $f \in L$ and $e \in (L_\#)^{\perp}$ for some $L \in \mathcal{F}$. Put $K = L_\#$. Then $e \in K^{\perp}$. By [15, Lemma 5.2], $L \subseteq L_{\#*}$ so $L \subseteq K_*$. Thus $f \in K_*$.

(iii)⇒(i). Suppose that $f \in K_*$ and $e \in K^{\perp}$ for some $K \in \mathcal{F}$. Let $M \in \mathcal{F}$. If $M \not\subseteq K$ then $K_* \subseteq M_-$ so $e \otimes f(M) \subseteq [f] \subseteq K_* \subseteq M_-$ and $e \otimes f(M) \subseteq M_-$. If instead $M \subseteq K$, then $e \in M^{\perp}$ and so $e \otimes f(M) = \{0\} \subseteq M_-$. By Lemma 2.1 we have $e \otimes f \in \mathcal{S}$ as required.

COROLLARY 2.3. *Let $\mathcal{F}$ be a subspace lattice on H different from $\{\{0\},H\}$. If $K\in\mathcal{F}$ and $K\neq\{0\},H$ then for every pair e,f of nonzero vectors satisfying $f\in K$, $e\in K^{\perp}$, the rank one operator $e\otimes f$ belongs to $\mathcal{S}(\mathcal{F})$. Every rank one operator of $\mathcal{S}(\mathcal{F})$ arises in this way if and only if $\mathcal{F}$ is completely distributive.*

Proof. First, let e,f be nonzero vectors satisfying $f\in K$, $e\in K^{\perp}$ where $K\in\mathcal{F}$ and $K\neq\{0\},H$. Then, as $K\subseteq K_*$ (see [15]) we have $f\in K_*$. Hence $e\otimes f\in\mathcal{S}(\mathcal{F})$ by Proposition 2.2.

Next, assume that every rank one operator of $\mathcal{S}(\mathcal{F})$ arises in this way. To show that $\mathcal{F}$ is completely distributive it is enough to show that $K=K_*$ for every $K\in\mathcal{F}$. As $K\subseteq K_*$ we need only show that $K_*\subseteq K$, or equivalently, that K_* and $K^{\perp}$ are orthogonal. Let $f\in K_*$ and $e\in K^{\perp}$. By the preceding proposition, $e\otimes f\in\mathcal{S}(\mathcal{F})$. If $e=0$ or $f=0$, e and f are certainly orthogonal. Otherwise, by our assumption, $f\in L$ and $e\in L^{\perp}$ for some element $L\in\mathcal{F}$. In particular, e and f are orthogonal. Thus K_* and $K^{\perp}$ are orthogonal, as required.

Finally, let $\mathcal{F}$ be completely distributive. Let $e\otimes f\in\mathcal{S}(\mathcal{F})$ be a rank one operator. By the preceding proposition, $f\in K_*$ and $e\in K^{\perp}$ for some $K\in\mathcal{F}$. By complete distributivity $K=K_*$ and the proof is complete.

Remarks. 1. Clearly $\mathcal{S}(\mathcal{F})=\{0\}$ if $\mathcal{F}=\{\{0\},H\}$. Thus, for any subspace lattice $\mathcal{F}$, $\mathcal{S}(\mathcal{F})$ contains a rank one operator if and only if $\mathcal{F}\neq\{\{0\},H\}$.

2. It follows from [5, Theorem 9.4] that, for a completely distributive subspace lattice $\mathcal{F}$, $\operatorname{Alg}\mathcal{F}$ is the set of operators that annihilate every operator of rank at most one of $\mathcal{S}(\mathcal{F})$. Again, this is not difficult to show directly.

In the remainder of this paper we fix two arbitrary distinct subspaces M and N of the Hilbert space H. To avoid trivialities we shall assume that neither M nor N is $\{0\}$ or H. We denote by $\mathcal{L}$ the subspace lattice generated by M and N so that generally $\mathcal{L}=\{\{0\},M\cap N,M,N,M\vee N,H\}$ although some of these subspaces may coincide. However, whatever the case may be, we always have $(M\cap N)_-=\{0\}$ and $H_-=M\vee N$. By $\mathcal{A}$ we denote the algebra $\operatorname{Alg}\mathcal{L}$ consisting of those operators leaving M and N invariant, and by $\mathcal{S}$ we denote the set $\mathcal{S}(\mathcal{L})$ defined as above.

Our first result concerning the two subspaces M and N is the following.

THEOREM 2.4. *The intersection $\mathcal{A}\cap\mathcal{S}^*$ is $\{0\}$.*

Proof. Let $A\in\mathcal{A}\cap\mathcal{S}$. If M and N are comparable, say $M\subset N$, we have $A^3(H)=A^2(AH)\subseteq A^2(N)=A(AN)\subseteq A(M)=\{0\}$ so $A^3=0$. On the other hand, if M and N are noncomparable, we have $A(M)\subseteq M$ and $A(M)\subseteq M_-=N$ so $A(M)\subseteq M\cap N$. Similarly, $A(N)\subseteq M\cap N$, so

$A(M\vee N)\subseteq M\cap N$. Thus $A^3(H)=A^2(AH)\subseteq A^2(M\vee N)=A(A(M\vee N))\subseteq A(M\cap N)=\{0\}$ and again $A^3=0$.

Now let $T\in\mathcal{A}\cap\mathcal{S}^*$. Then $T^*T\in\mathcal{S}$ so $B=(T^*T)^2\in\mathcal{A}$. But $B=B^*$ so $B\in\mathcal{A}\cap\mathcal{A}^*$. The latter is a C^*-algebra so the unique positive square root of B, namely T^*T, also belongs to $\mathcal{A}\cap\mathcal{A}^*$. Thus $T^*T\in\mathcal{A}\cap\mathcal{S}$ so $(T^*T)^3=0$ and $T=0$.

COROLLARY 2.5. *Every Hilbert–Schmidt operator in $\mathcal{A}$ (respectively $\mathcal{S}$) can be arbitrarily well approximated in the Hilbert–Schmidt norm by finite sums of rank one operators in $\mathcal{A}$ (respectively $\mathcal{S}$).*

Proof. If a Hilbert–Schmidt operator T in $\mathcal{A}$ (respectively $\mathcal{S}$) is orthogonal to the linear manifold of all finite sums of rank one operators in $\mathcal{A}$ (respectively $\mathcal{S}$) then T^* annihilates all rank one operators in $\mathcal{A}$ (respectively $\mathcal{S}$). By the definition of $\mathcal{S}$ (respectively, the second Remark following Corollary 2.3), $T^*\in\mathcal{S}$ (respectively $T^*\in\mathcal{A}$) so that $T\in\mathcal{A}\cap\mathcal{S}^*$ (respectively $T^*\in\mathcal{A}\cap\mathcal{S}^*$). By Theorem 2.4, $T=0$ (in either case). Since the linear manifold of Hilbert–Schmidt operators in $\mathcal{A}$ (respectively $\mathcal{S}$) is closed in the Hilbert–Schmidt norm the required result follows.

The above density property for $\mathcal{A}$ can also be obtained, using an appropriate decomposition of H and a duality argument, from a result of Papadakis [17], who shows that in the case $M\cap N=\{0\}$ and $M\vee N=H$ we can approximate in the ultraweak topology any operator in $\mathcal{A}$ by finite sums of rank one operators in $\mathcal{A}$. A result of Harrison quoted in [1] (see also [11]) shows that these sums can be taken in the unit ball.

3. Decomposability and distance estimates. To motivate the results that follow for our two subspaces M, N and for $\mathcal{L}=\{\{0\},M\cap N,M,N,M\vee N,H\}$, consider the special cases

(a) $M\subset N$, in which case $\mathcal{L}$ is a nest,
(b) $M\cap N=\{0\}$, $M\vee N=H$, in which case $\mathcal{L}$ is a Boolean lattice.

In either case it follows from more general theorems of Ringrose [4] and Longstaff [16] respectively (but also can be seen directly) that every finite rank operator in $\mathcal{A}$ is a finite sum of rank one operators of $\mathcal{A}$, a property not shared by all completely distributive lattices, as shown in [8]. It is not difficult to extend this to the general $\mathcal{A}$ considered here, and we include a short proof. It turns out that $\mathcal{S}$ also has this property. Indeed, Lemma 1.2 of [6] shows that, for any nest $\mathcal{N}$, every finite rank operator of $\mathcal{S}(\mathcal{N})$ is a finite sum of rank one operators of $\mathcal{S}(\mathcal{N})$, so only the case where M and N are noncomparable needs further consideration.

Let us return to the general situation. By Corollary 2.5 every Hilbert–Schmidt operator in $\mathcal{A}$ can be approximated by finite sums of rank one operators in $\mathcal{A}$. In the special case (b) each rank one operator in $\mathcal{A}$ annihilates precisely one of M and N. It is natural to ask whether every Hilbert–Schmidt operator in $\mathcal{A}$ can be written as a sum of two others with the same property. Perhaps surprisingly, in view of the above, this is not true in general. It holds precisely when the angle between M and N is positive (equivalently, when the vector sum $M+N$ is closed).

The following "replacement lemma" is the basis of the proof of Theorem 5.2 of [16].

LEMMA 3.1. *Let K and L be subspaces of a Hilbert space H and let $F=\sum_{i=1}^{n} e_i\otimes f_i$ be a finite rank operator on H. If $F(L)\subseteq K$ and $e_1\notin L^{\perp}$, then F can be written as $F=e_1\otimes f_1'+\sum_{i=2}^{n} e_i'\otimes f_i$ with $f_1'\in K$.*

Proof. Since $e_1\notin L^{\perp}$, $P_Le_1\neq 0$. Now $0=(I-P_K)FP_L=P_Le_1\otimes(I-P_K)f_1+\sum_{i=2}^{n}P_Le_i\otimes(I-P_K)f_i$, so $P_Le_1\otimes(I-P_K)f_1=-\sum_{i=2}^{n}P_Le_i\otimes(I-P_K)f_i$. Hence $(I-P_K)f_1=\sum_{i=2}^{n}\lambda_i(I-P_K)f_i$ where

$$\lambda_i=-\langle P_Le_1,P_Le_i\rangle/\|P_Le_1\|^2 \quad (2\le i\le n).$$

Thus

$$\begin{aligned}F&=e_1\otimes P_Kf_1+e_1\otimes\Big(\sum_{i=2}^{n}\lambda_i(I-P_K)f_i\Big)+\sum_{i=2}^{n}e_i\otimes f_i\\&=e_1\otimes\Big[P_K\Big(f_1-\sum_{i=2}^{n}\lambda_if_i\Big)\Big]+\sum_{i=2}^{n}(e_i+\overline{\lambda}_ie_1)\otimes f_i\\&=e_1\otimes f_1'+\sum_{i=2}^{n}e_i'\otimes f_i\end{aligned}$$

where $f_1'=P_K(f_1-\sum_{i=2}^{n}\lambda_if_i)\in K$.

THEOREM 3.2. *Any nonzero finite rank operator in $\mathcal{A}$ (respectively $\mathcal{S}$) can be written as a finite sum of rank one operators in $\mathcal{A}$ (respectively $\mathcal{S}$).*

Proof. If M and N are comparable, $\mathcal{L}$ is a nest and result follows from Ringrose's theorem [4] and [6, Lemma 1.2].

Suppose that M and N are noncomparable. Put $M_1=M\ominus(M\cap N)$ and $N_1=N\ominus(M\cap N)$. Then M_1 and N_1 are nonzero and $M_1\cap N_1=\{0\}$, $M_1\vee N_1=(M\vee N)\ominus(M\cap N)$. Relative to the decomposition $H=(M\cap N)\oplus(M_1\vee N_1)\oplus(M\vee N)^{\perp}$ operators on H are represented by 3×3 operator matrices. An operator is of finite rank if and only if each of the entries in its matrix is. Let F be a nonzero finite rank operator with matrix $[F_{ij}]$.

Suppose that $F\in\mathcal{A}$. Since F leaves $M\cap N$ and $M\vee N$ invariant, $[F_{ij}]$ is upper triangular. Since F leaves M and N invariant, F_{22} leaves M_1 and N_1 invariant. By [16, Theorem 5.2], F_{22} can be written as a finite sum of rank one operators in $\mathcal{B}(M_1\vee N_1)$ each leaving both M_1 and N_1 invariant. The desired result easily follows.

Suppose that $F\in\mathcal{S}$. Then $F_{11}=F_{21}=F_{31}=F_{32}=F_{33}=0$, and $F_{22}(M_1)\subseteq N_1$, $F_{22}(N_1)\subseteq M_1$. Suppose that F_{22} has rank $n>1$. Then $F_{22}=\sum_{i=1}^{n}e_i\otimes f_i$ for some nonzero vectors e_i,f_i $(1\le i\le n)$ in $M_1\vee N_1$. The vector e_1 cannot belong to both $M_1\vee N_1\ominus M_1$ and $M_1\vee N_1\ominus N_1$. Suppose that $e_1\notin M_1\vee N_1\ominus M_1$. By Lemma 3.1, $F_{22}=e_1\otimes f_1'+\sum_{i=2}^{n}e_i'\otimes f_i$ for some vectors f_1',e_i' $(2\le i\le n)$ in $M_1\vee N_1$ with $f_1'\in N_1$. Now $f_1'\neq 0$ so $f_1'\notin M_1$. By Lemma 3.1, since F_{22}^* maps $M_1\vee N_1\ominus M_1$ into $M_1\vee N_1\ominus N_1$ and $F_{22}^*=f_1'\otimes e_1+\sum_{i=2}^{n}f_i\otimes e_i'$, we have $F_{22}^*=f_1'\otimes e_1'+\sum_{i=2}^{n}f_i'\otimes e_i'$ for some vectors e_1',f_i' $(2\le i\le n)$ in $M_1\vee N_1$ with $e_1'\in M_1\vee N_1\ominus N_1$. Then $F_{22}=e_1'\otimes f_1'+\sum_{i=2}^{n}e_i'\otimes f_i'$ with $f_1'\in N_1$ and $e_1'\in M_1\vee N_1\ominus N_1$. Similarly, if $e_1\notin M_1\vee N_1\ominus N_1$ then $F_{22}=e_1''\otimes f_1''+\sum_{i=2}^{n}e_i''\otimes f_i''$ with $f_1''\in M_1$ and $e_1''\in M_1\vee N_1\ominus M_1$. By induction, F_{22} is a finite sum of rank one operators R in $\mathcal{B}(M_1\vee N_1)$ satisfying $R(M_1)\subseteq N_1$ and $R(N_1)\subseteq M_1$. The desired result easily follows.

Remark. It is not known if this decomposability property of finite rank operators of $\mathcal{S}(\mathcal{B})$ holds if $\mathcal{B}$ is an atomic Boolean subspace lattice with more than two atoms.

Putting $\mathcal{L}_1=\{\{0\},M\cap N,M,M\vee N,H\}$ and $\mathcal{L}_2=\{\{0\},M\cap N,N,M\vee N,H\}$, so that $\mathcal{L}_1$ and $\mathcal{L}_2$ are nests, we write $\mathcal{A}_i=\operatorname{Alg}\mathcal{L}_i$ and write $\mathcal{S}_i$ for the set of operators that annihilate all rank one operators in $\mathcal{A}_i$ $(i=1,2)$. By Lemma 2.1, $S\in\mathcal{S}_i$ if and only if $S(K)\subseteq K_-$ for every $K\in\mathcal{L}_i$, where K_- is now calculated in $\mathcal{L}_i$ $(i=1,2)$. Note that $\mathcal{S}_i\subseteq\mathcal{S}$.

COROLLARY 3.3. *Let F be a nonzero finite rank operator on H. Then*

(i) *If $F\in\mathcal{S}$, there exist finite rank operators $F_1\in\mathcal{S}_1$, $F_2\in\mathcal{S}_2$ such that $F=F_1+F_2$.*

(ii) *If $F\in\mathcal{A}$, there exist finite rank operators $F_1,F_2\in\mathcal{A}$ such that $F=F_1+F_2$ and $F_1(M)\subseteq M\cap N$, $F_2(N)\subseteq M\cap N$.*

Proof. If $M\subseteq N$ take $F_2=0$ in (i) and (ii). If $N\subseteq M$ take $F_1=0$. Suppose that M and N are noncomparable.

(i) Let $F\in\mathcal{S}$. By Theorem 3.2, $F=\sum_{i=1}^{n}e_i\otimes f_i$ with each rank one operator $e_i\otimes f_i$ in $\mathcal{S}$. By Corollary 2.3 for every i, $f_i\in K_i$ and $e_i\in K_i^{\perp}$ for some $K_i\in\mathcal{L}$. Now either $K_i\subseteq M$ or $N\subseteq K_i$. If $K_i\subseteq M$ it is easily verified that $e_i\otimes f_i\in\mathcal{S}_1$; if $N\subseteq K_i$ then $e_i\otimes f_i\in\mathcal{S}_2$. By grouping terms appropriately $F=F_1+F_2$ with $F_1\in\mathcal{S}_1$ and $F_2\in\mathcal{S}_2$ finite rank operators.

(ii) Let $F\in\mathcal{A}$. By Theorem 3.2, $F=\sum_{i=1}^{n}e_i\otimes f_i$ with each rank one operator $e_i\otimes f_i$ in $\mathcal{A}$. For each i, $f_i\in K_i$ and $e_i\in(K_{i-})^{\perp}$ for some $K_i\in\mathcal{L}$ [15, Lemma 3.1]. Either $K_i\subseteq M$ or $N\subseteq K_i$. If $K_i\subseteq M$, $e_i\otimes f_i(N)\subseteq M\cap N$.

If $N \subseteq K_i$, $e_i \otimes f_i(M) \subseteq M \cap N$. By grouping terms appropriately the desired result follows.

In spite of the fact that finite rank operators in $\mathcal{A}$ (respectively $\mathcal{S}$) decompose as in Corollary 3.3, as soon as we allow infinite rank operators this remains valid only when the vector sum $M+N$ is closed (if M and N are noncomparable, this is equivalent to $M \ominus (M \cap N)$ and $N \ominus (M \cap N)$ being at a positive angle).

THEOREM 3.4. *If the vector sum $M+N$ is closed, then*

(i) *every operator $T \in \mathcal{A}$ can be written in the form $T = T_1 + T_2$ where $T_1, T_2 \in \mathcal{A}$ with $T_1(M) \subseteq M \cap N$, $T_2(N) \subseteq M \cap N$, and*

(ii) *every operator $S \in \mathcal{S}$ can be written in the form $S = S_1 + S_2$ with $S_1 \in \mathcal{S}_1$, $S_2 \in \mathcal{S}_2$.*

If $M+N$ is not closed, there exist trace class operators $T \in \mathcal{A}$, $S \in \mathcal{S}$ which do not decompose as in (i), (ii).

Proof. Suppose first that $M+N$ is closed. Let $T \in \mathcal{A}$, $S \in \mathcal{S}$. If $M \subseteq N$ take $T_2 = S_2 = 0$. If $N \subseteq M$ take $T_1 = S_1 = 0$. Suppose that M and N are noncomparable. The (not necessarily orthogonal) projection P_1 in $\mathcal{B}((M \vee N) \ominus (M \cap N))$ onto $M \ominus M \cap N$ along $N \ominus M \cap N$ is bounded, hence extends to a bounded operator P on H defined to be the identity on $(M \cap N) \oplus (M \vee N)^{\perp}$. We may take $T_1 = T(I-P)$, $T_2 = TP$, $S_1 = S(I-P)$ and $S_2 = SP$.

Suppose now that $M+N$ is not closed. Then, letting $M_1 = M \ominus (M \cap N \oplus M \cap N^{\perp})$ and $N_1 = N \ominus (M \cap N \oplus M^{\perp} \cap N)$ and observing that $M \vee N = (M_1 \vee N_1) \oplus (M \cap N) \oplus (M \cap N^{\perp}) \oplus (M^{\perp} \cap N)$ we conclude that $M_1 + N_1$ is not closed either. It may be verified that M_1 and N_1 are in generic position in their span $M_1 \vee N_1$ so by [7,Theorem 3] we may take $M_1 \vee N_1 = H_1 \oplus H_1$, $M_1 = G(B)$ and $N_1 = G(-B)$ where $B \in \mathcal{B}(H_1)$ is a positive injective contraction. The fact that $M_1 + N_1$ is not closed implies that B is not invertible, and since B is injective, zero is an accumulation point of its spectrum. We may therefore choose a decreasing sequence $\{\lambda_n\}$ of positive numbers such that, for each $n \in \mathbb{Z}^+$, $\lambda_n < n^{-3}$ and the spectral projection of B corresponding to the interval $(\lambda_{n+1}, \lambda_n]$ is nonzero, hence contains a rank one projection P_n.

Since $(\lambda_n n^2)^{-1}\|BP_n\| \le n^{-2}$, the sums

$$\sum(\lambda_n n^2)^{-1}\begin{pmatrix} P_nB & 0 \\ 0 & BP_n \end{pmatrix}, \quad \sum(\lambda_n n^2)^{-1}\begin{pmatrix} P_nB & 0 \\ 0 & -BP_n \end{pmatrix}$$

converge in trace norm to trace class operators T and S respectively. Extend T and S to the whole space by defining them to be zero on $(M_1 \vee N_1)^{\perp}$. It is easily checked that $T \in \mathcal{A}$ and $S \in \mathcal{S}$.

Suppose that $T = T_1 + T_2$ as in (i). We will arrive at a contradiction. Indeed, the compression T_3 of T_1 to $M_1 \vee N_1$ must leave N_1 invariant and annihilate M_1. A short calculation, using the fact that B is injective (hence it has dense range), shows that T_3 must have the form

$$\begin{pmatrix} YB & -Y \\ -BYB & BY \end{pmatrix}$$

for some $Y \in \mathcal{B}(H_1)$. Since the compression T_4 of T_2 to $M_1 \vee N_1$ must leave M_1 invariant and annihilate N_1, it must have the form

$$\begin{pmatrix} ZB & Z \\ BZB & BZ \end{pmatrix}.$$

Since

$$T_3 + T_4 = \sum(\lambda_n n^2)^{-1}\begin{pmatrix} P_nB & 0 \\ 0 & BP_n \end{pmatrix}$$

we obtain $Y = Z$ and thus $2BY = \sum(\lambda_n n^2)^{-1}BP_n$, hence $2BYP_n = (\lambda_n n^2)^{-1}BP_n$ or $2YP_n = (\lambda_n n^2)^{-1}P_n$, since B is injective. But this is impossible, since Y is bounded while $(\lambda_n n^2)^{-1} > n$.

Finally, note that the compression of an operator in $\mathcal{S}_1$ to $M_1 \vee N_1$ is easily seen to have the form

$$\begin{pmatrix} YB & -Y \\ BYB & -BY \end{pmatrix}$$

while the compression of an operator of $\mathcal{S}_2$ has the form

$$\begin{pmatrix} ZB & Z \\ -BZB & -BZ \end{pmatrix}$$

where Y and Z are bounded operators on H_1. As above, we conclude that the operator S cannot be decomposed as in (ii). This completes the proof.

An important and much studied property of a reflexive algebra $\operatorname{Alg}\mathcal{F}$ is the validity of a "distance estimate" [2, 12], namely the existence of a constant c such that, for any operator T, we have

$$d(T, \operatorname{Alg}\mathcal{F}) \le c\sup\{\|P^{\perp}TP\| : P \in \mathcal{F}\}$$

(where projections and subspaces are identified in the usual way). Arveson [3] has shown that this is equivalent to the following decomposability property for the pre-annihilator ${}^{\perp}\operatorname{Alg}\mathcal{F}$: Every trace class operator which annihilates $\operatorname{Alg}\mathcal{F}$ can be written as an *absolutely convergent* sum (in the trace norm) of rank one operators which annihilate $\operatorname{Alg}\mathcal{F}$.

As an application of our work, we show that Theorem 3.4 implies that there is no distance estimate for our algebra $\mathcal{A}$ if $M+N$ is not closed. If M and N are comparable a distance estimate (with $c=1$) holds by Arveson's distance formula for nests [2, 3, 12]. If M is orthogonal to N it is not hard to see that a distance estimate will hold with $c = 1$. More generally, if M and N are noncomparable and $M+N$ is closed, then $M \ominus M \cap N$ and $N \ominus M \cap N$ can be orthogonalized by a similarity. Thus a distance estimate will hold, although the constant c will depend on the norm of the similarity.

COROLLARY 3.5. *A distance estimate holds for $\mathcal{A}$ if and only if $M + N$ is closed.*

Proof. Suppose that every trace class operator $X \in {}^{\perp}\mathcal{A}$ can be written as $X = \sum R_n$ where each R_n is a rank one operator in ${}^{\perp}\mathcal{A}$ and $\sum \|R_n\|_1 < \infty$. Each R_n belongs to $\mathcal{S}$, hence either belongs to $\mathcal{S}_1$ or $\mathcal{S}_2$ (or both). Letting $X_1 = \sum\{R_n : R_n \in \mathcal{S}_1\}$, absolute convergence ensures that X_1 is a well-defined trace class operator in $\mathcal{S}_1$. Then $X_2 = X - X_1 = \sum\{R_n : R_n \in \mathcal{S} \setminus \mathcal{S}_1\}$ must be in $\mathcal{S}_2$. Thus, if a distance estimate holds, every trace class operator X in ${}^{\perp}\mathcal{A}$ decomposes as a sum $X = X_1 + X_2$, with $X_i \in \mathcal{S}_i$ of trace class.

However, the trace class operator S constructed in the proof of Theorem 3.4 does not decompose as above if $M + N$ is not closed. But S is in ${}^{\perp}\mathcal{A}$. Indeed, S is the trace-norm sum of operators of $\mathcal{S}$ each of which is of rank at most two, hence each belongs to ${}^{\perp}\mathcal{A}$, by Lemma 2.1. This establishes the failure of a distance estimate if $M + N$ is not closed and the converse has been observed above.

Remark. Corollary 3.5 was also shown by Papadakis [17] in the special case $M \cap N = \{0\}$, $M \vee N = H$. His proof makes use of his ultraweak density result, quoted after Corollary 2.5, and an elaborate application of the spectral theorem. Our proof is simpler and more transparent.

Kraus and Larson [9, 10], extending Arveson's work [2] described above to (ultraweakly closed) subspaces $\mathcal{T}$ of $\mathcal{B}(H)$, show that the decomposability of elements of ${}^{\perp}\mathcal{T}$ as absolutely convergent sums of rank one operators in ${}^{\perp}\mathcal{T}$ is equivalent to the validity of a distance estimate for $\mathcal{T}$, that is, to the existence of a constant c such that

$$d(T, \mathcal{T}) \leq c \sup\{\|Q^{\perp}TP\| : P, Q \text{ projections with } Q^{\perp}\mathcal{T}P = 0\},$$

for every operator T.

Using these concepts, our work implies the following.

COROLLARY 3.6. *A distance estimate holds for $\mathcal{S}$ if and only if $M + N$ is closed.*

Proof. Observe that a rank one operator R annihilates $\mathcal{S}$ if and only if $R \in \mathcal{A}$. It is not difficult to see that for such R, either $R(M) \subseteq M \cap N$ or $R(N) \subseteq M \cap N$. Now, as in Corollary 3.5, we can see that if a trace class operator $T \in {}^{\perp}\mathcal{S}$ can be written as an absolutely convergent sum of rank one operators in ${}^{\perp}\mathcal{S}$, then T can be written as a sum $T = T_1 + T_2$, where $T_i \in \mathcal{A}$ and $T_1(M) \subseteq M \cap N$ and $T_2(N) \subseteq M \cap N$.

But if $M + N$ is not closed, the trace class operator $T \in \mathcal{A}$ constructed in the proof of Theorem 3.4 does not decompose as above. We claim that $T \in {}^{\perp}\mathcal{S}$. Indeed, T is the sum of rank two operators each of which belongs to ${}^{\perp}\mathcal{S}$ by Lemma 2.1. Hence T, being the trace-norm sum of these operators, also lies in ${}^{\perp}\mathcal{S}$.

Thus, if $M + N$ is not closed, the result of Kraus and Larson quoted above implies that no distance estimate can hold for $\mathcal{S}$. The converse is easily seen, as for $\mathcal{A}$.

Remark. It is perhaps interesting to observe that we have constructed operators T and S which are absolutely convergent sums of rank two operators in ${}^{\perp}\mathcal{S}$ (respectively ${}^{\perp}\mathcal{A}$) but which cannot be written as absolutely convergent sums of rank one operators in these spaces. However, T and S can be approximated (in the trace-norm) by finite sums of rank one operators in ${}^{\perp}\mathcal{S}$ (respectively ${}^{\perp}\mathcal{A}$). Indeed, each of the summands in the definition of T is a finite sum of rank one operators in $\mathcal{A}$, by Theorem 3.2, hence is in ${}^{\perp}\mathcal{S}$, by Lemma 2.1 (and similarly for S).

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Total subspaces in dual Banach spaces which are not norming over any infinite-dimensional subspace

by

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Abstract. The main result: the dual of separable Banach space X contains a total subspace which is not norming over any infinite-dimensional subspace of X if and only if X has a nonquasireflexive quotient space with a strictly singular quotient mapping.

1. Introduction. Let X be a Banach space and X^* be its dual space. Let us recall some basic definitions.

A subspace M of X^* is said to be *total* if for every $0 \neq x \in X$ there is an $f \in M$ such that $f(x) \neq 0$.

A subspace M of X^* is said to be *norming over a subspace* $L \subset X$ if for some $c > 0$ we have

$$(\forall x \in L)\big(\sup_{f \in S(M)} |f(x)| \geq c\|x\|\big),$$

where $S(M)$ is the unit sphere of M. If $L = X$ then M is called *norming*.

The following natural questions arise:

1) How far could total subspaces be from norming ones? (Of course, there are many different concretizations of this question.)

2) What is the structure of Banach spaces whose duals contain total "very" nonnorming subspaces?

3) What is the structure of total subspaces?

These questions were studied by many authors: [Al], [B, pp. 208–216], [BDH], [DJ], [DL], [D], [F], [G], [Ma], [Mc], [M1], [M2], [O1], [O2], [P], [PP], [S1], [S2]. The results obtained find applications in the theory of Fréchet spaces [BDH], [DM], [MM1], [MM2], [M2]; in the theory of improperly posed problems [O3], [PP, pp. 185–196]; and in the theory of universal bases [Pl, p. 31].

The present paper is devoted to the following natural class of subspaces which are far from being norming. A subspace M of X^* is said to be *nowhere*

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