STUDIA MATHEMATICA 105 (1) (1993)

Some estimates concerning the Zeeman effect

by

WIESŁAW CUPAŁA (Wrocław)

Abstract. The Itô integral calculus and analysis on nilpotent Lie grops are used to estimate the number of eigenvalues of the Schrödinger operator for a quantum system with a polynomial magnetic vector potential. An analogue of the Cwikel–Lieb–Rosenblum inequality is proved.

Introduction. In an external magnetic field there occurs a splitting of energy levels. This phenomenon is known as the Zeeman effect. According to the Hamilton theory, the energy of a classical system with an electric potential V and a vector magnetic potential $A = (A_1, \ldots, A_d)$ is, in the traditional notation,

$$E = \frac{1}{2}\left(p - \frac{eA}{c}\right)^2 - V(q)\,. \tag{1}$$

If, when studying the Zeeman effect, we neglect the spin and concentrate only on the splitting which results from the existence of the orbital momentum (this is possible in a strong magnetic field and called the Panchen–Back effect), then the behaviour of a quantum system in the external magnetic field can be described in terms of the spectral characteristics of the quantum-mechanical hamiltonian, the symbol of which is the right-hand side of the equation (1). Assuming that the electron charge e and the speed of light c are both equal to one, we can reduce the study of the Zeeman effect to the spectral analysis of the Schrödinger operator

$$H = -\frac{1}{2}\sum_{j=1}^{d}(\partial_j - iA_j)^2 + V\,.$$

Let P be the spectral measure of the operator H, and let $N(H, \lambda)$ mean the dimension of the spectral projection $P(-\infty, \lambda)$. Let us consider the same quantum system without the external magnetic field. Let $S = -\Delta + V$ be

1991 *Mathematics Subject Classification*: Primary 35P15.
Key words and phrases: estimation of eigenvalues, Schrödinger operator.

the corresponding hamiltonian. The physical reality of the Zeeman effect allows us to ask whether or not

$$N(H,\lambda) \le N(S,\lambda)\,. \tag{2}$$

The aim of this paper is to demonstrate the following estimate, which is weaker than inequality (2): for $V \ge 0$,

$$\begin{aligned} N(H,\lambda) \\ \le C|\{(q,p) : \max_{k,j} |\partial_k A_j(q) - \partial_j A_k(q)| \le c(\lambda+1),\ p^2 + V(q) < \lambda + 1\}|\,, \end{aligned} \tag{3}$$

where $|\cdot|$ denotes the Lebesgue measure, the constant C depends only on the dimension d, and the constant c depends on d and on $\max\{\deg(A_1),\ldots \ldots,\deg(A_d)\}$. In the case $A = 0$, this estimate is the well-known Cwikel–Lieb–Rosenblum inequality [10].

If we denote the right side of (3) by $M(H,\lambda)$ then we get a "substitute" of inequality (2):

$$M(H,\lambda) \le M(S,\lambda)\,. \tag{4}$$

All the theorems of this paper are concerned with electric potentials V and magnetic vector potentials for which C_c^∞ is the essential domain of H (this condition has been thoroughly explored in [9]).

The Feynman–Kac formula. Let us consider a nilpotent Lie group G with Lie algebra $\mathfrak{g}$. Let $X,\ldots,X_d$ be left-invariant vector fields on G such that

$$\mathrm{Lie}\{X_1,\ldots,X_d\} = \mathfrak{g}\,.$$

If the dimension of G is n then, in fixed coordinates $(x_1,\ldots,x_n)$, we can write

$$X_j = \sum_{i=1}^{n} q_{ji}\partial_i$$

with q_{ji} ($j = 1,\ldots,d$, $i = 1,\ldots,n$) being polynomials on G.

Let us set (for $i = 1,\ldots,n$)

$$a_i = \frac{1}{2}\sum_{j=1}^{d} X_j q_{ji}\,.$$

DEFINITION 1. The *weak Wiener process* (generated by the fields $X_1,\ldots,X_d$) starting from the point $x_0 \in G$ is the diffusion process $\xi(t) = (\xi_1(t),\ldots,\xi_n(t))$, $t \ge 0$, defined as the strong solution of the stochastic differential equation

$$d\xi_i = a_i(\xi(t))\,dt + \sum_j q_{ji}(\xi(t))\,dW_j(t)\,, \tag{1}$$

where $W_1(t),\ldots,W_d(t)$ are independent copies of the standard Wiener process on the real line, $i = 1,\ldots,n$, and the initial condition is $\xi(0) = x_0$.

In order to prove the existence of the strong solution we choose triangular coordinates on G (which exist by the Engel theorem [2]). In such coordinates, q_{ji} and a_i depend only on the variables $x_1,\ldots,x_{i-1}$, for all $j = 1,\ldots,d$, $i = 1,\ldots,n$, and equation (1) can be solved "step by step". The resulting solution is independent of the choice of coordinates. To prove this we use the Itô formula [1]. A simple calculation shows that for any twice differentiable function ϕ on G, we have the stochastic differential equation

$$d\phi(\xi(t)) = \mathcal{L}\phi(\xi(t))dt + \sum_j X_j\phi(\xi(t))\,dW_j(t)\,, \tag{2}$$

where $\mathcal{L} = \frac{1}{2}\sum_j X_j^2$. If ϕ is a coordinate on G, then (2) reduces to (1).

The diffusion $\xi(t)$ is a time-homogeneous Markov process. If we denote by $\xi_x(t)$ the weak Wiener process starting from $x \in G$, then the operators defined by the equation

$$T_t f(x) = Ef(\xi_x(t)) \tag{3}$$

form a semigroup (because of the homogeneity in time). From (2) and (3) we obtain

$$T_t f(x) = \int_0^t E\mathcal{L}f(\xi_x(s))\,ds\,.$$

Thus $\mathcal{L}$ is the infinitesimal generator of the semigroup T_t.

The term "Wiener process" is justified here by the properties of the semigroup T_t which are parallel to those of the heat semigroup. For example, $T_t f = f * p_t$ with a certain smooth function p_t, the differential and growth properties of which are similar to those of the Gaussian kernel [4].

If π is a unitary representation of the group G and $\xi(t)$ is the weak Wiener process starting from the unity of G, then

$$\pi_{p_t} = E\pi_{\xi(t)}\,. \tag{4}$$

If we set $\pi_{\mathcal{L}} = H$, then (4) may be symbolically rewritten as

$$\exp(-tH) = E\pi_{\xi(t)}\,.$$

Let us assume that π is monomial, i.e., it is the representation induced by a one-dimensional representation of a subgroup G_1 of G. If we denote by S the space of right cosets of G_1 in G and by $L^2(S)$ the space of square integrable functions on S (relative to the G-invariant measure), then we can express $\pi_x f$, where $x \in G$ and $f \in L^2(S)$, using the formula

$$\pi_x f(s) = \exp(i\phi(s,x))f(sx)\,, \tag{5}$$

where sx denotes the action of $x \in G$ on $s \in S$ and ϕ is a function defined on $S \times G$ with the property

$$\phi(s,x) + \phi(sx,y) = \phi(s,xy)\,. \tag{6}$$

LEMMA 1. *If $\xi(t)$ is the weak Wiener process starting from the unity of the group G, and V is a function on S such that $\int_0^t V(s\xi(u))\,du \geq a$ for $t \geq 0$, $s \in S$ and some $a \in \mathbb{R}$ with probability* 1, *then the equation*

$$T_t f(s) = E\pi_{\xi(t)} f(s) \exp\Big(-\int_0^t V(s\xi(u))\,du\Big)$$

defines a semigroup with generator $\pi_{\mathcal{L}} + V$.

Proof. Let $(\xi_1(r))_{r\geq 0}$ and $(\xi_2(r))_{r\geq 0}$ be two independent copies of the Wiener process starting from the unity. Because $\mathcal{L}$ commutes with left translations, the Markovian property and time-homogeneity imply that the random element $\xi_1(r_1)\xi_2(r_2)$ has the same distribution as $\xi_1(r_1+r_2)$. Therefore

$$\begin{aligned} T_t T_r f(s) &= E\exp(i\phi(s,\xi_1(t))T_r f(s\xi_1(t))\exp\Big(-\int_0^t V(s\xi_1(u))\,du\Big) \\ &= E\exp(i\phi(s,\xi_1(t)\xi_2(r)))f(s\xi_1(t)\xi_2(r)) \\ &\quad\times\exp\Big(-\int_0^t V(s\xi_1(u))\,du - \int_0^r V(s\xi_1(t)\xi_2(u))\,du\Big) \\ &= E\exp(i\phi(s,\xi_1(t+r))) \\ &\quad\times\exp\Big(-\int_0^{t+r} V(s\xi_1(u))\,du\Big) f(s\xi_1(t+r)) = T_{t+r}f(s)\,. \end{aligned}$$

It follows that the operators $(T_t)_{t\geq 0}$ form a semigroup. Applying the Itô formula to the process defined on $G \times \mathbb{R}$ by the stochastic differential equations

$$d\eta_i = d\xi_i \quad \text{for } i = 1,\ldots,n, \qquad d\eta_{n+1}(t) = V(s\xi_1(t))\,dt$$

and to the function $F(g,r) = \pi_g f(s)\exp(-r)$ we can prove that the generator of this semigroup is $\pi_{\mathcal{L}} + V$.

According to the Campbell–Hausdorff formula [2], for any polynomial w on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$\exp(\partial_x - iw)f(y) = \exp\Big(-i\sum_{k=0}^N c_k \partial_x^k w(y)\Big) f(y+x)\,, \tag{7}$$

where ∂_x is the directional derivative, $c_0,\ldots,c_N$ are the Campbell–Hausdorff constants, and N depends on the degree of w.

Let $A = (A_1,\ldots,A_d)$ be a polynomial magnetic vector potential and let M be the smallest $\mathbb{R}^d$-invariant linear space of polynomials that contains $\partial_k A_j - \partial_j A_k$ for $1 \leq j,k \leq d$.

Let us introduce the notation:

$$A_x = \sum_k c_k \sum_j x_j \partial_x^k A_j\,, \quad \sigma_x F(y) = F(x+y)\,, \quad P_{x,y} = A_x + \sigma_x A_y - A_{x+y}\,.$$

Defining multiplication on $\mathbb{R}^d \times M$ by

$$(x,w)(y,v) = (x+y, w + \sigma_x v + P_{x,y})\,,$$

we obtain a nilpotent Lie group G whose Lie algebra is isomorphic to $\mathrm{Lie}\{\partial_j - iA_j : 1 \leq j \leq d\}$. M may be treated as an abelian normal subgroup of G. The representation π of G induced by the one-dimensional representation

$$w \mapsto \exp(-iw(0))$$

of the subgroup M acts on $L^2(\mathbb{R}^d)$ and, for $f \in L^2(\mathbb{R}^d)$,

$$\pi_{(x,w)} f(y) = \exp(-iA_x(y) - iw(y)) f(x+y)\,.$$

Let $X_1,\ldots,X_d$ be the left-invariant vector fields on G corresponding to the operators $\partial_1 - iA_1,\ldots,\partial_d - iA_d$ and let $\mathcal{L} = -2^{-1}\sum_j X_j^2$. Then

$$\pi_{\mathcal{L}} = -2^{-1}\sum_j (\partial_j - iA_j)^2\,.$$

Solving equation (1) with the initial condition $\xi(0) = 0$ we see that $\xi(t) = (W(t), w_t)$, where $W(t)$ is the standard Wiener process on $\mathbb{R}^d$ and w_t a certain stochastic process on M. Using the notation from the above construction we may rewrite Lemma 1 as follows:

PROPOSITION 1. *If $A = (A_1,\ldots,A_d)$ is a polynomial magnetic vector potential, $H_0 = -2^{-1}\sum_j(\partial_j - iA_j)^2$, V is a function on $\mathbb{R}^d$ such that $\int_0^t V(x+W(s))\,ds > -\infty$ with probability* 1, *for any $t \geq 0$ and $x \in \mathbb{R}^d$, $T_t = \exp(-t(H_0+V))$, and $\varrho_t = A_{W(t)} + w_t$, then*

$$T_t f(x) = E\exp(-i\varrho_t(x))\exp\Big(-\int_0^t V(x+W(s))\,ds\Big) f(x+W(t))$$

for $f \in L^2(\mathbb{R}^d)$.

The Cwikel–Lieb–Rosenblum inequality. We now estimate the number $N(H_0+V,\lambda)$. In the case $A \equiv 0$, this result is due to Cwikel [3], Lieb [5], and Rosenblum [8]. Our proof is an adaptation of Lieb's method published in [7].

PROPOSITION 2. *For $d \geq 3$ there is a constant $c = c(d)$ such that for any polynomials $A_1, \ldots, A_d$ on $\mathbb{R}^d$ and any nonpositive potential $V \in L^\infty$,*

$$N(H_0 + V, 0) \leq c \int |V(x)|^{d/2}\, dx\,.$$

Proof. First we prove the inequality in the case $d = 3$. Then we describe the changes which are sufficient for the proof in the general case. We assume that $V \in C_c^\infty$. The result can then be easily extended to L^∞.

We set $F = -V$ and, for $\lambda < 0$, $\lambda = -\kappa^2$. As the first step we show that

$$N(H_0 - F, \lambda) \leq 2\,\mathrm{Tr}(F((H_0 + \kappa^2)^{-1} - (H_0 + F + \kappa^2)^{-1}))\,. \tag{1}$$

For a selfadjoint operator H we define the nth *characteristic number* as

$$\mu_n(H) = \sup_{\dim K = n-1} \inf_{\substack{f \in D(H)\\ F \perp K, \|f\|=1}} (Hf, f)\,.$$

If $\|f_1\| = \|f_2\| = 1$ then the functions defined by $t \mapsto ((H_0 - tF)f_i, f_i)$, $t > 0$, $i = 1, 2$, are equicontinuous. Hence $t \mapsto \mu_n(H_0 - tF)$ defines a continuous function $\mu_n(t)$. Since $V \leq 0$ we have $\mu_n(t + h) < \mu_n(t)$. Using the minimax principle [7], we see that $N(H_0 - F, \lambda) = |\{n : \mu_n(1) < \lambda\}| = |\{n : \mu_n(t) = \lambda \text{ for some } 0 < t < 1\}|$ for $\lambda < 0$.

Let η be a function which satisfies the equation

$$(H_0 - tF)\eta = \lambda\eta\,.$$

Then $\psi = F^{1/2}\eta$ satisfies

$$F^{1/2}(H_0 + \kappa^2)^{-1}F^{1/2}\psi = t^{-1}\psi\,,$$
$$F^{1/2}(H_0 + F + \kappa^2)F^{1/2}\psi = (1 + t)^{-1}\psi\,.$$

Therefore, if we set

$$K = F^{1/2}[(H_0 + \kappa^2)^{-1} - (H_0 + F + \kappa^2)^{-1}]F^{1/2}$$

then

$$K\psi = [t^{-1} - (1 + t)^{-1}]\psi\,.$$

Hence, $N(H_0 + V, \lambda)$ does not exceed the number of eigenvalues of K greater than $1/2$. Since K is positive,

$$N(H_0 + V, \lambda) \leq 2\,\mathrm{Tr}(K)\,,$$

which is equivalent to (1).

The next step is to show the inequality

$$N(H_0 + V, 0) \leq 2 \int_0^\infty \mathrm{Tr}(F(\exp(-tH_0) - \exp(-t(H_0 + F))))\, dt\,.$$

Formally, (2) is a consequence of (1) and the Laplace transform

$$(\kappa^2 + H)^{-1} = -\int_0^\infty \exp(-t(\kappa^2 + H))\, dt\,.$$

To complete such a formal proof we must show that we can change the order of trace and integral. For this purpose, we notice that $F\exp(-rH_0)$ has a square integrable kernel. This is evident because $H_0 = \pi_{\mathcal{L}}$, where π and $\mathcal{L}$ denote the representation and the differential operator explored in the proof of Proposition 1. The assumption $F \geq 0$ implies that $F^{1/2}\exp(-r(H_0 + F))$ is also a Hilbert–Schmidt operator. Therefore, the operator

$$A = \exp(-s(H_0 + F))F\exp(-(t - s)(H_0 + F))$$

is of trace class for any $t > s > 0$.

Proposition 1 and the Markov property lead to

$$Af(x) = EF(x + W(s))\exp\Big(-\varrho_t(x) - \int_0^t F(x + W(r))\, dr\Big) f(x + W(t))\,. \tag{3}$$

Let Ω_t be the set of trajectories of a Wiener process $(W(r))_{0 \leq r \leq t}$ on $\mathbb{R}^d$ starting from x. We can decompose the Wiener measure on Ω_t into a family of conditional Wiener measures $\{\mu_{x,y,t} : y \in \mathbb{R}^d\}$ in such a way that for any y, $\mu_{x,y,t}$ is supported by $\{\omega \in \Omega_t : \omega(t) = y\}$. According to Definition 1, a weak Wiener process on a nilpotent Lie group is the strong solution of a stochastic differential equation. We can therefore define the process $\varrho_t(x)$ on Ω_t. Using (3), we can express the kernel of A as

$$A(x, y) = \int F(\omega(s))\exp\Big(-i\varrho_t(x, \omega) - \int_0^t F(\omega(r))\, dr\Big)\, d\mu_{x,y,t}(\omega)\,. \tag{4}$$

An elementary reasoning shows that A is a continuous function. As we have shown above, $\mathrm{Tr}(A) < \infty$. Hence

$$\mathrm{Tr}(A) = \int A(x, x)\, dx\,,$$

and the proof of (2) is complete.

Now, we notice that

$$\mathrm{Tr}(F\exp(-t(H_0 + F)))$$
$$= t^{-1}\int_0^t \mathrm{Tr}(\exp(-s(H_0 + F))F\exp(-(s - t)(H_0 + F)))\, ds\,.$$

Let $G(u) = u(1 - \exp(-u))$. Now, (2) and (4) imply that

$$N(H_0+V,0)$$
$$\le 2\Big|\int_0^\infty dt \int dx \int d\mu_{x,x,t}(\omega)\, t^{-1}\exp(-i\varrho_t(x,\omega))G\Big(\int_0^t F(\omega(s))\,ds\Big)\Big|$$
$$\le 2\int_0^\infty dt \int dx \int d\mu_{x,x,t}(\omega)\, t^{-1}G\Big(\int_0^t F(\omega(s))\,ds\Big).$$

Thus the factor $\exp(-i\varrho_t(x))$ (which represents the external magnetic field) does not affect our estimate.

We can rewrite the above estimate as

$$(5)\quad N(H_0+V,0)\le 2\int_0^\infty dt \int dx \int d\mu_{0,0,t}(\omega)\, t^{-1}G\Big(\int_0^t F(x+\omega(s))\,ds\Big).$$

Let $g(u)=u(1-\exp(-u))$ for $0<u\le 2$, $g(u)=G(2)+(u-2)G'(2)$ for $2\le u$. Then, noticing that $G''>0$ on $(0,2)$, and $G''<0$ on $(2,\infty)$, we see that

$$(6)\quad \begin{cases} G(u)\le g(u),\\ g \text{ is a convex function},\\ g(u)\sim u^2 \text{ as } u\to 0,\\ g(u)\sim u \text{ as } u\to\infty. \end{cases}$$

So, by the Jensen inequality,

$$G\Big(\int_0^t F(x+\omega(s))\,ds\Big)\le t^{-1}\int_0^t g(tF(x+\omega(s)))\,ds,$$

and, noticing that $\int d\mu_{0,0,t}=(4\pi t)^{-3/2}$, we conclude that

$$N(H_0+V,0)\le c\int |V(x)|^{3/2}\,dx,$$

where $c=2(4\pi)^{-3/2}\int_0^\infty u^{-5/2}g(u)\,du$ is finite by (6).

For $d=3$, the proof is complete. For $d>3$ the proof is incorrect—we have to replace $\int_0^\infty u^{-5/2}g(u)\,du$ by $\int_0^\infty u^{-d/2-1}g(u)\,du$, which is infinite. But, if we use, instead of K, the operator K' defined by

$$K'=F^{1/2}\sum_{j=0}^m(-1)^j\binom{m}{j}(H_0+jF+\kappa^2)^{-1}$$

for a fixed natural m, then

$$K'\psi=t^{-1}R_m(t)\psi,$$

with R_m defined by

$$R_m(y)=\sum_{j=0}^m(-1)^j\binom{m}{j}(1+jy)^{-1}=\int_0^\infty e^{-s}(1-e^{-sy})\,ds.$$

R_m is a monotone function. So, if t runs over the interval $(0,1)$, then $t^{-1}R_m(t)$ runs from ∞ to $(m+1)^{-1}$. Hence, we can change inequality (1) to

$$(1')\quad N(H_0+V,0)\le(m+1)\,\mathrm{Tr}\Big(\sum_{j=0}^m(-1)^j\binom{m}{j}(H_0+jF+\kappa^2)^{-1}F\Big).$$

In the same way as in the case $d=3$ we prove that $(1')$ implies

$$(5')\quad N(H_0+V,0)$$
$$\le(m+1)\int_0^\infty dt\int dx\int d\mu_{0,0,t}(\omega)\,t^{-1}G_m\Big(\int_0^t F(x+\omega(s))\,ds\Big),$$

where $G_m(y)=y(1-e^{-y})^m$. We notice that there exists y_m such that $G''>0$ for $y\in(0,y_m)$ and $G''<0$ for $y\in(y_m,\infty)$. We define

$$g_m(y)=\begin{cases}G_m(y) & \text{for } 0<y\le y_m,\\ G_m(y_m)+(y-y_m)G'(y_m) & \text{for } y_m<y.\end{cases}$$

Then $g_m\sim y^{m+1}$ as $y\to 0$, and $g_m\sim y$ as $y\to\infty$. We set

$$c_{dm}=2(4\pi)^{-d/2}\int_0^\infty y^{-d/2-1}g_m(y)\,dy.$$

We can see that this constant is finite when $m>d/2-1$ and conclude that

$$N(H_0+V,0)\le c_{dm}\int |V(x)|^{m/2}\,dx.$$

Corollary. *For $d\ge 3$ there is a constant $c=c(d)$ such that, for any polynomials $A_1,\dots,A_d$ on $\mathbb{R}^d$, a potential $V\ge 0$ and any $\lambda\ge 0$,*

$$N(H_0+V,\lambda)\le c|\{(x,\xi):\xi^2+V(x)<\lambda\}|.$$

Proof. Set $V_\lambda=\min(V-\lambda,0)$. We have

$$((H_0+V)\phi,\phi)\ge\lambda\|\phi\|^2+((H_0+V_\lambda)\phi,\phi),$$

so the minimax principle [7] and Proposition 2 prove the Corollary.

The uncertainty principle and the final estimate. For selfadjoint operators F and H with commutator $[F,H]=iM$, if M is selfadjoint, we have the uncertainty principle:

$$\|H\phi\|^2\|F\phi\|^2\ge 4^{-1}(M\phi,\phi)^2.$$

Let us set $D_k=i\partial_k+A_k$ $(k=1,\dots,d)$. Then $[D_k,D_j]=i(\partial_kA_j-\partial_jA_k)$. The uncertainty principle implies that

$$\sum_{k,j}\|D_k\phi\|^2\|D_j\phi\|^2\ge 4^{-1}\sum_{k,j}\Big(\int(\partial_kA_j-\partial_jA_k)|\phi|^2\Big)^2.$$

Hence, there exists a constant $c = c(d)$ such that

$$\Big(\sum_{k=1}^{d} D_k^2\phi,\phi\Big) \ge c\sum_{k>j}\Big|\int(\partial_k A_j-\partial_j A_k)|\phi|^2\Big|.$$

In the case of a polynomial magnetic vector potential, there is a sharper version of this inequality.

PROPOSITION 3. *Let $A_1,\dots,A_d$ be polynomials and $H_0 = -2^{-1}\sum_k(\partial_k - iA_k)^2$. There is a constant c such that*

$$(H_0\phi,\phi) \ge c\int\sum_{k>j}|\partial_k A_j-\partial_j A_k||\phi|^2 - \|\phi\|^2$$

for $\phi\in C_c^\infty$. The constant c depends only on d and on the largest of the degrees of $A_1,\dots,A_d$.

Proof. Let $\mathfrak{g}$ be a free nilpotent Lie algebra with free generators $X_1,\dots,X_d$ and with nilpotence class N. Let $\mathcal{L} = -2^{-1}\sum_k X_k^2$ be the sublaplacian on $G=\exp(\mathfrak{g})$. By Folland [4], there exists a constant $c_1>0$ such that

$$\|[X_j,X_k]\phi\| \le c_1\|(\mathcal{L}+1)\phi\|, \tag{1}$$

for $j,k=1,\dots,d$, $\phi\in C_c^\infty(G)$.

Let N (the nilpotence class of G) be so large that

$$X_j\mapsto \partial_j - iA_j,\qquad j=1,\dots,d,$$

defines a representation π of G. (1) implies that

$$\|\pi_{[X_j,X_k]}f\| \le c_1\|\pi_{\mathcal{L}+1}f\|$$

for $\phi\in C_c^\infty(\mathbb{R}^d)$. Thus

$$((H_0+1)^2 f,f) \ge c_2(|\partial_k A_j-\partial_j A_k|^2 f,f).$$

This implies (see [6]) that

$$((H_0+1)f,f) \ge c_3(|\partial_k A_j-\partial_j A_k|f,f),$$

which completes the proof.

THEOREM. *For $d\ge 3$ and any natural number N there exist constants $C=C(d)$ and $c=c(d,N)$ such that for any polynomials $A_1,\dots,A_d$ on $\mathbb{R}^d$, any potential $V\ge 0$ and any $\lambda\ge 0$, if $\max\{\deg(A_1),\dots,\deg(A_d)\}\le N$ then*

$$N(H_0+V,\lambda) \le C\Big|\Big\{(x,\xi):\sum_{k>j}|\partial_k A_j(x)-\partial_j A_k(x)|\le c(\lambda+1),\ \xi^2+V(x)<\lambda+1\Big\}\Big|.$$

Proof. Let c be the constant defined by Proposition 3. Fix $\lambda>0$ and put

$$V_\lambda(x)=0\quad\text{if either }\sum_{k>j}|\partial_k A_j(x)-\partial_j A_k(x)|>(2/c)(\lambda+1)\text{ or }V(x)>\lambda,$$
$$V_\lambda(x)=V(x)-\lambda-1\qquad\text{for the remaining }x.$$

We have

$$((H_0+V)f,f) = ((2^{-1}H_0+V-V_\lambda)f,f)+((2^{-1}H_0+V_\lambda)f,f).$$

By Proposition 3,

$$((2^{-1}H_0+V-V_\lambda)f,f)\ge\lambda\|f\|^2.$$

Thus, using the minimax principle, we see that

$$N(H_0+V,\lambda)\le N(2^{-1}H_0+V_\lambda,0)$$

and Proposition 2 finishes the proof.

References

[1] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, Wiley, New York 1974.

[2] N. Bourbaki, *Groupes et Algèbres de Lie*, Hermann, Paris 1971.

[3] M. Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Ann. of Math. 106 (1977), 93–100.

[4] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. 13 (1975), 161–207.

[5] E. Lieb, *The number of bound states of one-body Schrödinger operators and the Weyl problem*, unpublished.

[6] K. Löwner, *Über monotone Matrixfunktionen*, Math. Z. 38 (1934), 177–216.

[7] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 4, Academic Press, 1978.

[8] G. W. Rosenblum, *The distribution of the discrete spectrum of singular differential operators*, Dokl. Akad. Nauk SSSR 202 (1972), 1012–1015 (in Russian).

[9] B. Simon, *Schrödinger operators with singular magnetic vector potentials*, Math. Z. 131 (1973), 361–370.

[10] —, *Functional Integration and Quantum Physics*, Academic Press, 1979.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WROCŁAW BRANCH
KOPERNIKA 18
51-617 WROCŁAW, POLAND

Received April 23, 1991
Revised version June 8, 1992 and February 8, 1993 (2799)