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Metrically convex functions in normed spaces

by

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Abstract. Properties of metrically convex functions in normed spaces (of any dimension) are considered. The main result, Theorem 4.2, gives necessary and sufficient conditions for a function to be metrically convex, expressed in terms of the classical convexity theory.

Introduction. The concept of metric convexity follows ideas introduced by Menger for metric spaces ([6], see also [2]). An important and inescapable question is how it corresponds to the standard convexity. The appropriate field for investigating these relations are normed spaces.

Let $(X, \|\cdot\|)$ be a normed linear space. The metric is defined in the usual way: $d(x,y) = \|x-y\|$ for $x, y \in X$. The two notions of convexity can be introduced parallelly via two different concepts of a segment, as follows.

In *algebraic convexity*, the *line segment* connecting points $x, y \in X$ is the set $[x,y] = \{(1-\lambda)x + \lambda y \mid 0 \le \lambda \le 1\}$. Then a subset $A \subset X$ is said to be *convex* if for all $x, y \in A$ the segment $[x,y]$ is contained in A. A function $f : X \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, is said to be *convex* if $f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y)$ for all $x, y \in X$ and $\lambda \in [0,1]$.

In *metric convexity* [8], by the *metric segment* (or briefly: *d-segment*) connecting $x, y \in X$ we mean the set $[x,y]_d = \{z \in X \mid d(x,z) + d(z,y) = d(x,y)\}$. A subset $A \subset X$ is *metrically convex* (or *d-convex*) if $[x,y]_d \subset A$ for all $x, y \in A$. For $\lambda \in [0,1]$ and $x, y \in X$ we define the λ-*layer* of $[x,y]_d$ to be $[x,y]_d^\lambda = \{z \in [x,y]_d \mid d(x,z) = \lambda d(x,y)\}$. A function $f : X \to \overline{\mathbb{R}}$ is said to be *metrically convex* (or *d-convex*) if $f(z) \le (1-\lambda)f(x) + \lambda f(y)$ for all $\lambda \in [0,1]$, $x, y \in X$ and $z \in [x,y]_d^\lambda$.

Remarks. (1) Menger's original definitions [6] concerning metric convexity in general metric spaces, when applied to normed linear spaces, imply weaker notions than ours. For instance, from [6] one can derive the following: a subset $A \subset X$ is metrically convex (in Menger's sense) if $([x,y]_d \setminus \{x,y\}) \cap A \ne \emptyset$ for every pair of distinct points $x, y \in A$.

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(2) Metrically convex functions were discussed for the first time in [1], for some particular subspaces of $\mathbb{R}^n$ with the Manhattan metric.

Several immediate consequences can be derived from the definitions formulated above. For instance, if $x, y \in X$ and $\lambda \in [0,1]$ then $(1-\lambda)x + \lambda y \in [x,y]_d^\lambda$, and $[x,y] \subset [x,y]_d$. Moreover, if $\mathcal{C}(X)$, $\mathcal{C}_d(X)$ denote the families of, respectively, convex and d-convex subsets of X, and $\mathcal{F}(X)$, $\mathcal{F}_d(X)$ the families of, respectively, convex and d-convex functions, then $\mathcal{C}_d(X) \subset \mathcal{C}(X)$ and $\mathcal{F}_d(X) \subset \mathcal{F}(X)$.

Many interesting results concerning properties of $\mathcal{C}_d(X)$ can be found in [3] and [8]. Almost nothing is known, however, about the structure of the family $\mathcal{F}_d(X)$ (cf. [8]).

The aim of the present paper is to characterize d-convex functions in normed spaces, in terms of the standard convexity theory. The central part in our characterization is played by affine subspaces parallel to extremal subsets of the closed unit ball $B = \{x \in X \mid \|x\| \le 1\}$. The main result is Theorem 4.2, where necessary and sufficient conditions for a function $f : X \to \overline{\mathbb{R}}$ to be d-convex are formulated.

The organization of the paper is the following. In the next section we recall some necessary definitions and basic facts from the theory of convexity. In Section 2 two new lemmas concerning convex sets and functions in finite-dimensional spaces are proved. Basic facts from metric convexity theory are covered by Section 3. Important properties of d-convex functions in $\mathbb{R}^2$ are also proved there. They are principal steps in proving, in Section 4, the main theorem, which concerns general normed spaces. Some corollaries from this theorem are also contained in Section 4.

1. Preliminaries. The material exploited here is standard and belongs partly to the theory of normed spaces (cf. [4]) and partly to convex analysis (cf. [7]). The notation used throughout the paper follows, in principal, the two monographs.

We consider a normed space $(X, \|\cdot\|)$ with the zero-vector θ, the induced metric d, and the closed unit ball B. The open segment with endpoints $x, y \in X$ is $(x,y) = \{(1-\lambda)x + \lambda y \mid 0 < \lambda < 1\}$. Other types of segments are $[x,y) = (x,y) \cup \{x\}$, etc.

For a subset C in X, $\operatorname{cor} C$ is its algebraic interior (core of C), $\operatorname{icr} C$ is its relative algebraic interior (intrinsic core). The family of all convex C-extremal sets is denoted by $\operatorname{Ext} C$. If $x \in C$ then $\mathrm{T}_C(x)$ and $\mathrm{N}_C(x)$ denote, respectively, the tangent and normal cones at $x \in C$. By $\mathcal{A}(C)$ we denote the family of all affine subspaces parallel to $\operatorname{aff} C$, i.e. $\mathcal{A}(C) = \{L \subset X \mid \exists a \in X\ L = a + \operatorname{aff} C\}$.

We shall distinguish several special objects connected with extremal subsets of the unit ball B, essential for our results. By $\mathcal{S}$ we denote the family of extremal proper subsets of B with nonempty intrinsic core, $\mathcal{S} = \{E \in \operatorname{Ext} B \mid E \ne B,\ \operatorname{icr} E \ne \emptyset\}$. For any nonzero $x \in X$, $S(x)$ denotes the smallest B-extremal set containing $x/\|x\|$. One can show that $S(X) \in \mathcal{S}$. The convex cone spanned by $S(x)$ will be denoted by $K(x)$. Similarly, for any B-extremal set $E \in \mathcal{S}$, $K(E)$ is the convex cone spanned by E. For any $E \in \mathcal{S}$ the linear subspace parallel to E will be denoted by $P(E)$, i.e. $P(E) = \operatorname{span}(E - E)$. Finally, the linear subspace spanned by all $P(E)$, $E \in \mathcal{S}$, will be denoted by $\widetilde{P}$, i.e. $\widetilde{P} = \operatorname{span} \bigcup \{P(E) \mid E \in \mathcal{S}\}$.

Note that the subspace $\widetilde{P}$ is uniquely determined by the family of B-extremal sets which are not extreme points of B. Moreover, one can easily prove that B is strictly convex if and only if $\widetilde{P} = \{\theta\}$.

For a function $f : X \to \overline{\mathbb{R}}$ and a point $x \in X$, $\operatorname{lev}_x f$ denotes the level set determined by the value $f(x)$, i.e. $\operatorname{lev}_x f = \{z \in X \mid f(z) \le f(x)\}$.

The set of points minimizing a function f on a given $C \subset X$ will be denoted by $\operatorname{argmin}_C f$, i.e.

$$\operatorname{argmin}_C f = \{x \in C \mid \forall y \in C\ f(y) \ge f(x)\}.$$

The set of global minima is $\operatorname{argmin} f = \operatorname{argmin}_X f$.

2. Two technical lemmas

LEMMA 2.1. *Let X be a finite-dimensional linear space, C and D convex subsets of X, and M a convex cone. Suppose that*

(i) $\operatorname{int} C \ne \emptyset$, $\operatorname{cl} C = C$;
(ii) $(\operatorname{int} C) \cap D \ne \emptyset$;
(iii) $\forall x \in C \cap D\ \mathrm{N}_C(x) \subset M^*$.

Then $(C + M) \cap D \subset C$.

Proof. Suppose $y \notin C$ and $y \in (C + M) \cap D$. Fix $z \in \operatorname{int} C \cap D$. Then there is a unique $x \in [z,y] \cap \operatorname{bd} C$. Since y and z are in D, also $x \in D$. Moreover, $\operatorname{bd} C \subset C$, therefore $x \in C \cap D$, and (iii) gives $\mathrm{N}_C(x) \subset M^*$.

We have $x + \mathrm{T}_C(x) \supset C$ and $\operatorname{cl} \mathrm{T}_C(x) = \mathrm{N}_C(x)^* \supset M^{**} \supset M$. It follows that $x + \operatorname{cl} \mathrm{T}_C(x) = x + \operatorname{cl} \mathrm{T}_C(x) + M \supset C + M$. But we also have $y \in C + M$. Hence $y - x \in \operatorname{cl} \mathrm{T}_C(x)$.

Since $z \in \operatorname{int} C$, $z - x$ is in $\operatorname{int} \mathrm{T}_C(x)$, and therefore $z - x \in \operatorname{int}(\operatorname{cl} \mathrm{T}_C(x))$. Finally,

$$[z - x, y - x) \subset \operatorname{int}(\operatorname{cl} \mathrm{T}_C(x)) = \operatorname{int} \mathrm{T}_C(x).$$

But $x \in [z,y]$, so $\theta \in [z - x, y - x)$. It follows that $\theta \in \operatorname{int} \mathrm{T}_C(x)$. This is only possible in case $\mathrm{T}_C(x) = X$, or, equivalently, when $x \in \operatorname{int} C$, contradicting $x \in \operatorname{bd} C$. ■

LEMMA 2.2. *Let X be a finite-dimensional linear space, $f : X \to \overline{\mathbb{R}}$ a convex function, $x \in \operatorname{cor}(\operatorname{dom} f)$, and $w \in X \setminus \{\theta\}$. Assume that there is a* *neighbourhood U of x and $\varepsilon \in (0, \frac{1}{2})$ such that for every $z \in [x, x+w] \cap U$,*

$$(*) \qquad f((\tfrac{1}{2}+\varepsilon)x + (\tfrac{1}{2}-\varepsilon)z) \le \tfrac{1}{2}f(x) + \tfrac{1}{2}f(z).$$

Then $f'_w(x) \ge 0$.

Proof. We may assume that U is convex. The finite derivative $f'_w(x)$ exists (cf. [7], Theorem 23.4). Suppose that $f'_w(x) < 0$. Then there exists $z_0 \in U \cap [x, x+w]$ such that $f(z_0) < f(x)$.

Consider the sequence $\{z_i\}_{i=1}^{\infty}$, where $z_i = (\frac{1}{2}+\varepsilon)x + (\frac{1}{2}-\varepsilon)z_{i-1}$ for $i = 1, 2, \ldots$ From $(*)$ one obtains $f(z_i) \le \frac{1}{2}f(x) + \frac{1}{2}f(z_{i-1})$ for all $i = 1, 2, \ldots$, which gives $f(z_i) \le (1-2^{-i})f(x) + 2^{-i}f(z_0) < (1-2^{-i})f(x) + 2^{-i}f(x) = f(x)$.

The above relations imply for all $i = 1, 2, \ldots$,

$$\text{(a)} \qquad |f(z_i) - f(x)| = f(x) - f(z_i) \ge 2^{-i}(f(x) - f(z_0)).$$

Note that $\|x - z_i\| = (\frac{1}{2}-\varepsilon)\|x - z_{i-1}\|$, hence for $i = 1, 2, \ldots$,

$$\text{(b)} \qquad \|x - z_i\| = (\tfrac{1}{2}-\varepsilon)^i\|x - z_0\|.$$

Now, by (a) and (b),

$$|f'_w(x)| = \|w\| \lim_{i\to\infty} \frac{|f(z_i) - f(x)|}{\|z_i - x\|} \ge \|w\| \lim_{i\to\infty} \left[2^{-i}\left(\frac{1}{2}-\varepsilon\right)^{-i} \frac{f(x) - f(z_0)}{\|x - z_0\|}\right]$$
$$= \|w\| \frac{f(x) - f(z_0)}{\|x - z_0\|} \lim_{i\to\infty} (1-2\varepsilon)^{-i} = +\infty,$$

a contradiction. ■

3. Metric convexity in a normed space

3.1. *Basic properties of metric segments.* We discuss the metric convexity in a normed linear space $(X, \|\cdot\|)$ with the distance d induced by the norm. In addition to notions introduced previously we shall consider here metric segments without one or both "endpoints", defined as follows: $(x, y)_d = \bigcup_{0<\lambda<1}[x, y]_d^{\lambda}$, $[x, y)_d = \bigcup_{0\le\lambda<1}[x, y]_d^{\lambda}$ and $(x, y]_d = \bigcup_{0<\lambda\le1}[x, y]_d^{\lambda}$. Note that $(x, x)_d = \{x\}$ and if $x \ne y$ then $(x, y)_d = [x, y]_d \setminus \{x, y\}$.

PROPOSITION 3.1 ([8], Theorem 11.22). *If $x, y \in X$, $x \ne y$, then $[x, y]_d = (x + K(y-x)) \cap (y + K(x-y))$.* ■

COROLLARY 3.1.1 ([8], Corollary 11.11). *Every metric segment is a convex set.* ■

COROLLARY 3.1.2. *For every $x, y \in X$, $\operatorname{icr}[x, y]_d \ne \emptyset$.*

Proof. By Proposition 3.1, $-(\frac{1}{2}x + \frac{1}{2}y) + [x, y]_d$ is absolutely convex, which yields the assertion. ■

From Proposition 3.1 one obtains the following characterization of a λ-layer.

PROPOSITION 3.2. *For every $x, y \in X$ and $0 \le \lambda \le 1$,*

$$[x, y]_d^{\lambda} = (x + \lambda d(x, y)S(y-x)) \cap (y + (1-\lambda)d(x, y)S(x-y)). \ ■$$

COROLLARY 3.2.1. *The λ-layer $[x, y]_d^{\lambda}$ is contained in an affine subspace parallel to $P(S(x-y))$.* ■

PROPOSITION 3.3. *If $x, y \in X$ and $\delta \in \mathbb{R}$ are such that $d(x, y) \le \delta$ then $[x, y]_d \subset x + \delta B$.*

Proof. From $z \in [x, y]_d$ it follows that $d(x, z) = d(x, y) - d(z, y) \le \delta$. ■

3.2. *Metrically convex functions in the 2-dimensional space.* We prove two lemmas describing important properties of d-convex functions in $X = \mathbb{R}^2$. In the next section they will be used for proving much stronger results concerning general normed spaces of any dimension.

LEMMA 3.4. *Let $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be d-convex and $x, y \in \operatorname{dom} f$, $x \ne y$. Then f is continuous on $(x, y)_d$.*

Proof. Define $g : \mathbb{R}^2 \to \overline{\mathbb{R}}$ by $g(z) = f(z)$ for $z \in (x, y)_d$, $g(z) = +\infty$ for $z \in \mathbb{R}^2 \setminus (x, y)_d$. Then, clearly, g is a convex function and $\operatorname{dom} g = (x, y)_d$. By Theorem 10.2 of [7], g is upper semicontinuous on $(x, y)_d$. It suffices to prove that g is lower semicontinuous or, equivalently, that $f(z) = \operatorname{cl} g(z)$ for every $z \in (x, y)_d$.

By Theorem 7.4 of [7], since $\operatorname{dom} g = (x, y)_d$ we obtain $f(z) = g(z) = \operatorname{cl} g(z)$ for $z \in \operatorname{icr}(x, y)_d$.

If the B-extremal set $S(x-y)$ is a singleton then $[x, y]_d = [x, y]$ and $(x, y)_d = (x, y) = \operatorname{icr}(x, y)_d$. Hence, in this case, $f(z) = g(z) = \operatorname{cl} g(z)$ as required. It follows that only the case of $S(x-y)$ being a nondegenerate line segment deserves attention.

The metric segment $[x, y]_d$ is a parallelepiped, and x, y are its opposite vertices. Let v_1 and v_2 denote the other two vertices.

Consider the set $D = (x, y)_d \setminus \operatorname{icr}(x, y)_d$. It may be expressed as the disjoint union $D = D_1 \cup D_2 \cup D_3$, where $D_1 = (x, v_1) \cup (x, v_2)$, $D_2 = (y, v_1) \cup (y, v_2)$, and $D_3 = \{v_1, v_2\}$.

Fix $z \in D_1$. There exist $b_0 \in \operatorname{icr}(x, y)_d$ and sequences $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ converging to z such that $a_k \in (x, z)$, $b_k \in [b_0, z)$ and $z \in [a_k, b_k]_d^{1/2}$. Then, by metric convexity of f,

$$g(z) = f(z) \le \tfrac{1}{2}f(a_k) + \tfrac{1}{2}f(b_k) = \tfrac{1}{2}g(a_k) + \tfrac{1}{2}g(b_k).$$

Since a convex function is continuous on every open subset of its domain (relative $\operatorname{aff}(\operatorname{dom} f)$), we have $\lim_{k\to\infty} g(a_k) = g(z)$. Moreover, by Theorem 7.5 of [7], $\lim_{k\to\infty} g(b_k) = \operatorname{cl} g(z)$. Hence $g(z) \le \frac{1}{2}g(z) + \frac{1}{2}\operatorname{cl} g(z)$. It follows that $g(z) \leq \mathrm{cl}\, g(z)$. The reverse inequality is true by the definition of the closure of a convex function. Therefore $f(z) = g(z) = \mathrm{cl}\, g(z)$.

The same can be shown analogously for $z \in D_2$.

If $z \in D_3$, i.e. $z = v_1$ or $z = v_2$, then similarly to the former case two sequences will be considered: $\{c_k\}_{k=1}^{\infty}$ and $\{e_k\}_{k=1}^{\infty}$, both converging to z, but now $c_k \in (x, z)$, $e_k \in (y, z)$ and $z \in (c_k, e_k)_d^{1/2}$.

The first part of the proof yields $g(c_k) = \mathrm{cl}\, g(c_k)$ and $g(e_k) = \mathrm{cl}\, g(e_k)$. By Corollary 7.5.1 of [7] we get $\lim_{k\to\infty} g(c_k) = \mathrm{cl}\, g(z)$ and $\lim_{k\to\infty} g(e_k) = \mathrm{cl}\, g(z)$. Hence

$$g(z) = f(z) \leq \tfrac{1}{2}f(c_k) + \tfrac{1}{2}f(e_k) = \tfrac{1}{2}g(c_k) + \tfrac{1}{2}g(e_k) = \tfrac{1}{2}\mathrm{cl}\, g(c_k) + \tfrac{1}{2}\mathrm{cl}\, g(e_k).$$

It follows that $g(z) \leq \mathrm{cl}\, g(z)$, and, finally, $g(z) = \mathrm{cl}\, g(z)$. This finishes the proof. ∎

If X is a finite-dimensional space then its dual space X^* can be identified with X. Therefore all subsets of X^* appearing in the next lemma (where $X = \mathbb{R}^2$) will be treated as subsets of X. For $x \in X$ and $x^* \in X^*$, $\langle x, x^*\rangle$ is the usual scalar product in $\mathbb{R}^2$.

LEMMA 3.5. *Let $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be a d-convex function such that* $\mathrm{cor}(\mathrm{dom}\, f) \neq \emptyset$, *let E be a B-extremal set, $E \in \mathcal{S}$, and $L \in \mathcal{A}(E)$. Then $f|_{L\cap \mathrm{cor}(\mathrm{dom}\, f)} \equiv$* const.

Proof. Let $L_0 = L \cap \mathrm{cor}(\mathrm{dom}\, f)$. If L_0 is empty or contains exactly one element, then the assertion is trivially satisfied. Therefore assume that $\dim L = \dim L_0 = 1$. It follows that E is a line segment, $\dim E = 1$, and $\mathrm{cor}\, K(E) \neq \emptyset$. Recall that $P(E)$ denotes the linear subspace parallel to E (in this case it is a line). Note that for every $L \in \mathcal{A}(E)$ and every $a \in L$, the normal cone $\mathrm{N}_L(a)$ is the subspace orthogonal to $P(E)$, i.e. $\mathrm{N}_L(a) = P(E)^{\perp}$.

Suppose that $f|_{L_0} \not\equiv$ const. It follows that there exists $x \in L_0$ such that $x \notin \mathrm{argmin}_{L_0} f$. We shall show that this leads to a contradiction. The proof will consist of three steps. In the first step it will be proved that there exist a convex cone M and a neighbourhood U_x of x such that $\mathrm{cor}\, M \cap P(E) \neq \emptyset$ and $(\mathrm{lev}_y f + M) \cap U_x \subset \mathrm{lev}_y f$ for every $y \in U_x$. In the second step we shall show that $f'_a(x) \geq 0$ for every $a \in \mathrm{cor}\, K(E)$, or, equivalently, $f(y) \geq f(x)$ for $y \in x + \mathrm{cor}\, K(E)$. The third step will be the proof that $f|_{U_x \cap (x + \mathrm{cor}\, K(E))} \equiv$ const $= f(x)$. This immediately yields $x \in \mathrm{argmin}_{L_0} f$, contradicting our assumption.

Step 1. We have assumed that $x \in L_0$ and $x \notin \mathrm{argmin}_{L_0} f$. By Theorem 27.4 of [7], $\partial f(x) \cap \mathrm{N}_{L_0}(x) = \emptyset$, and so $\partial f(x) \cap P(E)^{\perp} = \emptyset$ since $\mathrm{N}_{L_0}(x) = P(E)^{\perp}$.

Let $\Delta = \inf\{d(z_1, z_2) \mid z_1 \in \partial f(x),\ z_2 \in P(E)^{\perp}\}$. By compactness of the subdifferential $\partial f(x)$ (cf. Theorem 23.4 of [7]), $\Delta > 0$. Let $\varepsilon_1 = \frac{1}{2}\Delta$. From Corollary 24.5.1 of [7] there exists a neighbourhood U_x of x such that

(A) $$\forall y \in U_x \quad \partial f(y) \subset \partial f(x) + \varepsilon_1 B.$$

Again by [7, Theorem 27.4] we get

(B) $$\forall y \in U_x \quad y \notin \mathrm{argmin}_{y+P(E)} f.$$

(This fact will be used in step 3 of the proof.)

Without loss of generality we may assume that $\mathrm{cl}\, U_x \subset \mathrm{cor}(\mathrm{dom}\, f)$ and that $[x, z]_d \subset U_x$ for every $z \in U_x$ (cf. Proposition 3.3).

Let $G = \mathrm{cl}(\partial f(x) + \varepsilon_1 B)$. The set G is convex and satisfies

(C) $$G \cap P(E)^{\perp} = \emptyset.$$

It follows from (A) that $\mathrm{cone}\, \partial f(y) \subset \mathrm{cone}\, G$ for every $y \in U_x$. Since $y \notin \mathrm{argmin}_{y+P(E)} f$, and therefore $y \notin \mathrm{argmin} f$, from [7, Corollary 23.7.1] we get

(D) $$\forall y \in U_x \quad \mathrm{N}_{\mathrm{lev}_y f}(y) \subset \mathrm{cone}\, G.$$

Let M denote the dual cone, $M = (\mathrm{cone}\, G)^*$. It is clear from (D) that $\mathrm{N}_{\mathrm{lev}_y f}(y) \subset M^*$ for every $y \in U_x$. Now, if we put $C = \mathrm{lev}_y f \cap \mathrm{cl}\, U_x$ and $D = U_x$, then the assumptions of Lemma 2.1 are satisfied, thus

(E) $$\forall y \in U_x \quad (\mathrm{lev}_y f + M) \cap U_x \subset \mathrm{lev}_y f.$$

The cone M has one more property which is important for further steps of the proof, namely from (C) and the closedness of $M^* = \mathrm{cone}\, G$ it follows that $P(E) \cap \mathrm{cor}\, M \neq \emptyset$. Let $b \in P(E) \cap \mathrm{cor}\, M$. Obviously $b \neq \theta$.

Step 2. Fix any point y_0 in $U_x \cap (x + \mathrm{cor}\, K(E))$. It follows that $[x, y_0]_d \subset U_x$. The $\frac{1}{2}$-layer $[x, y_0]_d^{1/2}$ is not a singleton, thus it is a line segment. Corollary 3.2.1 implies that it is parallel to $P(S(y_0 - x)) = P(E)$. Let $w_0 = \frac{1}{2}x + \frac{1}{2}y_0$. Since $b \in P(E)$, there exists $w_0' \in \mathrm{icr}[x, y_0]_d^{1/2}$ such that $w_0' = w_0 - \tau b$ for some $\tau > 0$.

Let y_0' be a point in $(w_0' + M) \cap [x, y_0]$ closest to x. Since $b \in \mathrm{cor}\, M$, and $[x, y_0]_d^{1/2}$ is a line segment parallel to b, $y_0' \neq w_0$ and hence $d(x, y_0') < d(x, w_0) = \frac{1}{2}d(x, y_0)$. Obviously $w_0' \in \mathrm{lev}_{w_0'} f$. It follows that

$$y_0' \in (\mathrm{lev}_{w_0'} f + M) \cap [x, y_0] \subset (\mathrm{lev}_{w_0'} f + M) \cap U_x.$$

From (E) we get $y_0' \in \mathrm{lev}_{w_0'} f$. The function f is d-convex and $w_0' \in [x, y_0]_d^{1/2}$, therefore

(F) $$f(y_0') \leq f(w_0') \leq \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y_0).$$

If $y_0' = x$ then (F) implies $f(x) \leq f(y_0)$. If $y_0' \neq x$, let

$$\varepsilon = \varepsilon(y_0) = \frac{1}{2} - \frac{d(x, y_0')}{d(x, y_0)}.$$

Then $y'_0 = (\frac{1}{2}+\varepsilon)x + (\frac{1}{2}-\varepsilon)y_0$ and from (F), $f((\frac{1}{2}+\varepsilon)x + (\frac{1}{2}-\varepsilon)y_0) \le \frac{1}{2}f(x) + \frac{1}{2}f(y_0)$.

The same argument can be carried out for every $y \in [x, y_0]$. If we let $w = \frac{1}{2}x + \frac{1}{2}y$ and $w' \in [x, w'_0] \cap \{w - \tau b \mid \tau > 0\}$ be the counterparts of w_0 and w'_0, then we get a point y' and a number $\varepsilon(y)$ with analogous properties to y'_0 and $\varepsilon = \varepsilon(y_0)$.

By the Thales theorem, for every $y \in [x, y_0]$ and the corresponding y' we have

$$\text{(G}'\text{)} \qquad y' = x \Leftrightarrow y'_0 = x\,,$$

$$\text{(G}''\text{)} \qquad \varepsilon(y) = \varepsilon\,.$$

Therefore, for every $y \in [x, y_0]$,

$$\text{(H)} \qquad y' = (\tfrac{1}{2}+\varepsilon)x + (\tfrac{1}{2}-\varepsilon)y\,,$$

$$\text{(I)} \qquad f(y') \le \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y)\,.$$

If $x = y'_0$, then from (G′) and (I), $f(x) \le f(y)$ for every $y \in [x, y_0]$, which implies $f'_{y_0-x}(x) \ge 0$. If $x \ne y'_0$, the same conclusion can be obtained by Lemma 2.2.

We have thus proved that $f'_a(x) \ge 0$ for every $a \in \operatorname{cor} K(E)$. Since f is convex, it follows that $f(y) \ge f(x)$ for every $y \in x + \operatorname{cor} K(E)$.

Step 3. Let $x' \in U_x \cap (x + \operatorname{cor} K(E))$. By (B), $x' \notin \operatorname{argmin}_{x'+P(E)} f$, which immediately yields $x' \notin \operatorname{argmin}_{L'_0} f$, where $L'_0 = (x' + P(E)) \cap \operatorname{dom} f$. The same can be repeated with x replaced by x', $K(E)$ by $-K(E)$, and L_0 by L'_0. One can prove that way that $f(y) \ge f(x')$ for every $y \in x' - \operatorname{cor} K(E)$. Note, however, that $x \in x' - \operatorname{cor} K(E)$, hence $f(x) \ge f(x')$. Consequently, $f(x) = f(x')$ for every $x' \in U_x \cap (x + \operatorname{cor} K(E))$, i.e.

$$f|_{U_x \cap (x + \operatorname{cor} K(E))} \equiv \text{const} = f(x)\,.$$

By convexity of f it follows that $f(z) \ge f(x)$ for every $z \in \operatorname{aff}(U_x \cap (x + \operatorname{cor} K(E))) = \mathbb{R}^2$. An immediate conclusion is $x \in \operatorname{argmin}_{L_0} f$, and this contradicts the initial assumption. ∎

4. A characterization of metrically convex functions in normed spaces. Let X be a normed space of any dimension. Let $E \in \mathcal{S}$. For a d-convex set $A \subset X$ we define

$$\mathrm{I}_E(A) = \bigcup \{(x, y)_d \mid x, y \in A,\ x - y \in \operatorname{icr} K(E)\}\,.$$

The set $\mathrm{I}_E(A)$ can be expressed as the disjoint union

$$\mathrm{I}_E(A) = \mathrm{I}_E^0(A) \cup \mathrm{I}_E^1(A)\,,$$

where $\mathrm{I}_E^0(A) = \{z \in A \mid \mathrm{T}_A(z) \supset \operatorname{span} E\}$ and $\mathrm{I}_E^1(A) = \mathrm{I}_E(A) \setminus \mathrm{I}_E^0(A)$.

Note that if $x, y \in A$, $x \ne y$, are such that $x - y \in \operatorname{icr} K(E)$, then $\operatorname{icr}[x, y]_d \subset \mathrm{I}_E^0(A)$ (since $\operatorname{aff}[x, y]_d \in \mathcal{A}(\operatorname{span} E)$). It follows that $\mathrm{I}_E(A) \ne \emptyset$ if and only if $\mathrm{I}_E^0(A) \ne \emptyset$. However, it may happen that $\mathrm{I}_E^1(A) = \emptyset$ even when $\mathrm{I}_E(A) \ne \emptyset$ (e.g. for $A = X$).

THEOREM 4.1. *Let $f : X \to \overline{\mathbb{R}}$ be d-convex and $E \in \mathcal{S}$ be such that $\mathrm{I}_E(\operatorname{dom} f) \ne \emptyset$. Then $f|_{L \cap \mathrm{I}_E(\operatorname{dom} f)} \equiv \text{const}$ for every $L \in \mathcal{A}(E)$.*

Proof. Let I_E, I_E^0, I_E^1 denote respectively $\mathrm{I}_E(\operatorname{dom} f)$, $\mathrm{I}_E^0(\operatorname{dom} f)$ and $\mathrm{I}_E^1(\operatorname{dom} f)$. We first show that $f|_{L \cap \mathrm{I}_E^0} \equiv \text{const}$. Let $x, y \in L \cap \mathrm{I}_E^0$, $x \ne y$. Put $L_0 = (\operatorname{aff}\{x, y\}) \cap \mathrm{I}_E^0$. We prove that $f|_{L_0} \equiv \text{const}$. Without loss of generality we may assume that $L_0 \cap \operatorname{icr} E \ne \emptyset$ (if necessary, the linear transformation $z \mapsto z + (e - a)$ for $z \in X$ can be applied, where $e \in \operatorname{icr} E$ and $a \in L_0$ are chosen arbitrarily).

It is easy to see that x and y are linearly independent (since $x, y \in \operatorname{aff} E$, and $E \in \mathcal{S}$). Therefore the subspace $X_1 = \operatorname{span}\{x, y\}$ is 2-dimensional. It is a normed space with norm $\|\cdot\|_1 = \|\cdot\||_{X_1}$. The corresponding metric in X_1 will be denoted by d_1, and the closed unit ball is $B_1 = B \cap X$. The set $E_1 = E \cap X$ is B_1-extremal. Note that $E_1 \in \mathcal{S}(B_1)$ and $\dim E_1 = 1$.

The function $f_1 = f|_{X_1}$ is d_1-convex. It is clear that $L_0 \subset L \cap X_1 \in \mathcal{A}_1(E_1)$, where $\mathcal{A}_1(E_1)$ is the family of affine subspaces in X_1 parallel to E_1. Moreover, $L_0 \subset \operatorname{cor}(\operatorname{dom} f_1)$. From Lemma 3.5 we get $f_1|_{L_0} \equiv \text{const}$. It follows that $f(x) = f_1(x) = f_1(y) = f(y)$.

Thus we have shown that $f|_{L \cap \mathrm{I}_E^0} \equiv \text{const} = c$.

Now let $x \in L \cap \mathrm{I}_E^1$. There exist $a, b \in \operatorname{dom} f$ such that $b - a \in \operatorname{icr} K(E)$ and $x \in (a, b)_d$. Then $x \notin \operatorname{icr}(a, b)_d$, since $x \notin \mathrm{I}_E^0$. Without loss of generality it may be assumed that $a = \theta$. Consider the 2-dimensional subspace $X_1 = \operatorname{span}\{x, b\}$, with $\|\cdot\|_1$, d_1 B_1, E_1 defined as above. Note that $\operatorname{icr}[a, b]_{d_1} \subset \mathrm{I}_E^0$, hence $f_1|_{\operatorname{icr}(a,b)_{d_1}} \equiv \text{const} = c$. By Lemma 3.4 we have $f(x) = f_1(x) = \operatorname{cl}(f_1|_{(a,b)_{d_1}})(x) = c$. This finishes the proof: we have shown that $f|_{L \cap \mathrm{I}_E} \equiv \text{const}$. ∎

THEOREM 4.2. *A function $f : X \to \overline{\mathbb{R}}$ is metrically convex if and only if*:

(i) *f is convex*;
(ii) *$\operatorname{dom} f$ is a metrically convex set*;
(iii) $\forall E \in \mathcal{S}\ \forall L \in \mathcal{A}(E)\ f|_{L \cap \mathrm{I}_E(\operatorname{dom} f)} \equiv \text{const}$.

Proof. The necessity of (i) and (ii) is obvious. Condition (iii) is necessary by Theorem 4.1.

Now assume that f satisfies (i)–(iii). Take $x, y \in X$, $x \ne y$, and consider the λ-layer $[x, y]_d^\lambda$ for any $\lambda \in (0, 1)$. If $x \notin \operatorname{dom} f$ or $y \notin \operatorname{dom} f$ then clearly $f(z) \le (1-\lambda)f(x) + \lambda f(y)$ for every $z \in [x, y]_d^\lambda$.

Therefore, let $x, y \in \operatorname{dom} f$. Note that if $E \in \mathcal{S}$ is such that $E = S(y-x)$, then $[x,y]_d^\lambda \subset L$ for some $L \in \mathcal{A}(E)$ (cf. Corollary 3.2.1). Moreover, if $0 < \lambda < 1$ then $[x,y]_d^\lambda \subset \mathrm{I}_E(\operatorname{dom} f)$. From (iii) and $(1-\lambda)x + \lambda y \in [x,y]_d^\lambda$ it follows that $f(z) = f((1-\lambda)x + \lambda y)$ for every $z \in [x,y]_d^\lambda$. But, from (i), $f((1-\lambda)x+\lambda y) \le (1-\lambda)f(x) + \lambda f(y)$. Finally, $f(z) \le (1-\lambda)f(x) + \lambda f(y)$ for every $z \in [x,y]_d^\lambda$. Thus f is metrically convex. ∎

COROLLARY 4.2.1. *A function $f : X \to \mathbb{R}$ is metrically convex if and only if*

(i) *f is convex;*

(ii) *$f|_{\widetilde{P}} \equiv \text{const}$.*

Proof. The domain of f is now the whole space, $\operatorname{dom} f = X$. We shall prove that in this case the present condition (ii) is equivalent to (iii) of Theorem 4.2.

Recall that $\widetilde{P} = \operatorname{span} \bigcup_{E \in \mathcal{S}} P(E)$. It follows that if $f|_{\widetilde{P}} \equiv \text{const}$ then $f|_{P(E)} \equiv \text{const}$ for every $E \in \mathcal{S}$ and

$$\text{(a)} \qquad \forall E \in \mathcal{S}\, \forall L \in \mathcal{A}(E) \quad f|_L \equiv \text{const}\,.$$

This is condition (iii) of Theorem 4.2, since now $\mathrm{I}_E(\operatorname{dom} f) = X$.

Conversely, assume that (a) is satisfied and consider any $x \in \widetilde{P}$. There exist $n \in \mathbb{N}$, $E_i \in \mathcal{S}$ and $z_i \in P(E_i)$ for $i = 1, \ldots, n$ such that $x = z_1 + \ldots + z_n$. Define $y_0 = \theta$, $y_i = y_{i-1} + z_i$ for $i = 1, \ldots, n$. Then $y_i, y_{i-1} \in L_i$ for some $L_i \in \mathcal{A}(E_i)$, $i = 1, \ldots, n$, and, by (a), $f(y_i) = f(y_{i-1})$ for all i. More precisely, $f(x) = f(y_n) = f(y_{n-1}) = \ldots = f(y_1) = f(\theta)$. Thus $f|_{\widetilde{P}} \equiv \text{const} = f(\theta)$. ∎

COROLLARY 4.2.2. *The family of finite and metrically convex functions in a normed space separates points of the space if and only if the unit ball is strictly convex.*

Proof. If the unit ball B is strictly convex then every proper B-extremal set is a singleton and metric segments coincide with line segments. By the definition of a d-convex function it follows that both families, of convex and of d-convex functions, are equal. Finite convex functions separate points of the space.

Conversely, if B is not strictly convex then there exists $E \in \mathcal{S}$ such that $\dim E > 0$. Hence $\dim P(E) \ge 1$ and, by Corollary 4.2.1, $f|_{P(E)} \equiv \text{const}$ for every d-convex function f. It follows that neither points of the subspace $P(E)$ nor points of any $L \in \mathcal{A}(E)$ can be separated by d-convex functions. ∎

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