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Commutators based on the Calderón reproducing formula

by

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Abstract. We prove the Schatten–Lorentz ideal criteria for commutators of multiplications and projections based on the Calderón reproducing formula and the decomposition theorem for the space of symbols corresponding to commutators in the Schatten ideal.

1. Introduction and summary. This paper is devoted to the study of commutators of multiplications and projections based on the Calderón reproducing formula. Commutators of multiplications and projections are usually defined in the context of a Hilbert space with reproducing kernel. Let $H \subset L^2(X, d\mu)$ be such a space, $P : L^2 \to H$ the orthogonal projection from L^2 onto H, and let b be a function defined on X. The *commutator* of M_b and P is

$$C_b = [M_b, P] = M_bP - PM_b\,,$$

where M_b denotes the operator of multiplication by b. The function b is called the *symbol* of C_b. The commutator C_b is closely related to the Hankel operator

$$H_b = (I - P)M_bP\,,$$

namely $H_b = C_bP$, $C_b = H_b - H_b^*$.

The Calderón reproducing formula defines a class of Hilbert spaces with reproducing kernels. For a particular choice of the wavelet function which appears in the Calderón reproducing formula (and after a minor modification) the commutators based on the Calderón reproducing formula are unitarily equivalent to the commutators on weighted Bergman spaces on the upper half-plane. There has been an extensive study of Hankel operators on Bergman spaces ([Ax], [AFP], [BBCZ], [J], [Str], [Z2]). We refer to Zhu's book [Z1] for the background and more references. Our method of studying commutators does not rely on complex analytic tools nor on the formulas for reproducing kernels as is often done in the case of Bergman

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spaces. However, the choices of questions we consider and results we prove are strongly influenced by the model of Hankel operators on Bergman spaces. Also, our results apply to Bergman spaces and give various known and some new results.

We next describe the Calderón reproducing formula and we discuss our results.

We denote by G the "$ax+b$"-group, i.e. $G=\{\zeta=(v,t): v\in\mathbb{R}^d,\ t>0\}$ with the group law $(u,s)(v,t)=(sv+u,st)$ and the left-invariant measure $d\zeta=t^{-d-1}dv\,dt$. The group G acts on $L^2(\mathbb{R}^d)$ via translations and dilations, i.e. for $\zeta=(v,t)$

$$U_\zeta f(x)=f_\zeta(x)=t^{-d/2}f\left(\frac{x-v}{t}\right) \tag{1.1}$$

is a unitary representation of G. The left-invariant metric d on G is given by the length element $ds^2=t^{-2}(dv^2+dt^2)$.

The *Calderón reproducing formula* is the following resolution of the identity:

$$\langle f,g\rangle=\int_G \langle f,\psi_\zeta\rangle\langle\psi_\zeta,g\rangle d\zeta,\qquad \text{where } f,g\in L^2(\mathbb{R}^d)\,,$$

and $\psi\in L^2(\mathbb{R}^d)$ is an admissible wavelet, i.e. for almost every $\xi\in\mathbb{R}^d$,

$$\int_0^\infty |\widehat{\psi}(s\xi)|^2\,\frac{ds}{s}=1, \tag{1.2}$$

where $\widehat{\psi}$ is the Fourier transform of ψ, i.e.

$$\widehat{\psi}(\xi)=\int_{\mathbb{R}^d}\psi(x)e^{-2\pi ix\xi}\,dx\,.$$

It is not hard to check that the above admissibility condition is not only sufficient but also necessary for the Calderón reproducing formula to hold.

For a fixed ψ, the functions on G of the form $\langle f,\psi_\zeta\rangle$, where $f\in L^2(\mathbb{R}^d)$, form a Hilbert space with reproducing kernel and a subspace of $L^2(G)$. The orthogonal projection P onto this subspace, called the space of *wavelet transforms* or *Calderón transforms*, is given by integration against the reproducing kernel

$$k(\zeta,\eta)=\langle\psi_\zeta,\psi_\eta\rangle\,.$$

A commutator C_b is an integral operator given by the kernel

$$(b(\zeta)-b(\eta))\langle\psi_\eta,\psi_\zeta\rangle. \tag{1.3}$$

The general question is to find the relations between the singular numbers of commutators and their symbols. We use Schatten–Lorentz ideals to

describe the behavior of the singular numbers. We recall that for a compact operator T acting on a Hilbert space its nth *singular number* is defined as

$$s_n(T)=\inf\{\|T-A_n\|: A_n \text{ is } n\text{-dimensional}\}\,.$$

The *Schatten–Lorentz ideal* consists of those compact operators T which satisfy

$$\|T\|_{S^{pq}}=\|\{s_n(T)\}\|_{l^{pq}}<\infty\,,$$

where l^{pq} denotes the Lorentz space, and $0<p<\infty$, $0<q\le\infty$. The symbol $S^{\infty q}$ denotes the set of bounded operators and $S^p=S^{pp}$ the Schatten ideal. For the background for Schatten–Lorentz ideals we refer to [GK], [Si].

In most of our results we need to assume a certain decay of the reproducing kernel. A convenient condition of this sort is

$$|\langle\psi_\zeta,\psi_\eta\rangle|\le c_M e^{-Md(\zeta,\eta)}\,, \tag{1.4}$$

where M is some positive constant. In Section 2 we discuss the question when this condition is satisfied. In particular, if ψ is a Schwartz class function and $\widehat{\psi}$ is compactly supported in $\mathbb{R}^d\setminus\{0\}$, then (1.4) is satisfied for all positive M. We do not try to sort out the optimal condition for the wavelet, but rather concentrate on the question what happens with commutators when the wavelet is sufficiently smooth, has good decay at infinity, and a sufficient number of vanishing moments.

Singular numbers of commutators are closely related to the oscillation numbers

$$\operatorname{osc}_{D_r}b(\eta_i)=c\Big(\int_{\eta_iD_r}\int_{\eta_iD_r}|b(\zeta)-b(\eta)|^2\,d\zeta\,d\eta\Big)^{1/2}, \tag{1.5}$$

where D_r denotes the hyperbolic ball with radius r centered at $e=(0,\ldots,0,1)$. The above relation is the main theme of this paper.

We prove that for $0<p<\infty$, $0<q\le\infty$, if $\{\eta_i\}$ is a hyperbolic lattice (i.e. η_iD_r are pairwise disjoint if r is small, η_iD_r cover G if r is large) then for any suitably large r

$$\|C_b\|_{S^{pq}}\cong\|\operatorname{osc}_{D_r}b(\eta_i)\|_{l^{pq}}\,.$$

The inspiration for the proof of the direct estimate comes from the paper by Rochberg and Semmes [RS1], and for the converse for $0<p\le1$ from the paper by Semmes [Se]. The converse for $1<p\le\infty$ holds for all admissible wavelets. The proof relies on the properties of the Fourier algebra of Eymard. The background for the oscillation spaces and the S^p results for $p\ge2$ for the Bergman space case are presented in Zhu's book [Z1].

In the case of a harmonic b and $p>d$

$$\|\operatorname{osc}_D b(\eta_i)\|_{l^{pq}}\cong\|b\|_{\dot{B}^{pq}}$$

(see [RS1], [RS2]). Thus our results extend the S^p criteria for Hankel operators with harmonic symbols proved by Arazy–Fisher–Peetre in [AFP]. Our results also imply that there is a cut-off for harmonic symbols at $S^{d\infty}$. That is, if C_b belongs to S^{dq}, $q < \infty$, and b is harmonic, then b is constant. At the end point we have

$$\|C_b\|_{S^{d\infty}} \cong \|\mathrm{osc}_D b(\eta_i)\|_{l^{d\infty}},$$

and this shows that the nature of the cut-off is the same as in the case studied by Rochberg and Semmes in [RS2]. In particular, the space of symbols corresponding to $S^{d\infty}$ commutators contains $\dot{B}^{d1}$, and is contained in $\dot{B}^{d\infty}$.

We also show that for a general symbol b, if $C_b \in S^p$, $1 < p < \infty$, then there is unique decomposition of b,

$$b = b_{\mathrm{h}} + b_{\mathrm{c}}, \tag{1.6}$$

where b_{h} is a harmonic function satisfying

$$\|\mathrm{osc}_D b_{\mathrm{h}}(\gamma_i)\|_{l^p} < \infty,$$

and

$$\|b_{\mathrm{c}}\|_{l^p(L^2)} = \Big\|\Big(\int\limits_{\eta_i D_r} |b_{\mathrm{c}}(\zeta)|^2\,d\zeta\Big)^{1/2}\Big\|_{l^p} < \infty.$$

Such a decomposition result was known in the case of weighted Bergman spaces for $p = 2$ (see [AFP]).

After this work was done the author received a copy of Luecking's paper [Lu] which contains, in the case of Hankel operators on Bergman spaces on the disc, some similar results.

We recall the relation between weighted Bergman spaces on the upper half-plane and the Calderón transforms of Hardy space functions with respect to Bergman wavelets . Let ψ^α be the *Bergman wavelet*, i.e. for $\alpha > 0$,

$$\widehat{\psi}^\alpha(\xi) = \begin{cases} c_\alpha \xi^\alpha e^{-2\pi\xi} & \text{for } \xi > 0,\\ 0 & \text{for } \xi \le 0,\end{cases} \tag{1.7}$$

$$c_\alpha = \Big(\int\limits_0^\infty |\xi^\alpha e^{-2\pi\xi}|^2 \frac{d\xi}{\xi}\Big)^{-1/2}.$$

Let $\mathcal{C}_{\psi^\alpha}$ denote the space of the Calderón transforms of functions in the Hardy space $H^2(\mathbb{R})$ with respect to ψ^α. Let A^β stand for the *weighted Bergman space*, i.e. the holomorphic functions on the upper half-plane satisfying

$$\|F\|_{A^\beta} = \Big(\int\limits_0^\infty\int\limits_{-\infty}^\infty |F(u+is)|^2 s^\beta\,du\,ds\Big)^{1/2} < \infty, \tag{1.8}$$

where $\beta > -1$, and let V denote the map given as

$$VF(u,s) = s^{-\alpha-1/2}F(u,s). \tag{1.9}$$

The map V is a unitary map from $\mathcal{C}_{\psi^\alpha}$ onto $A^{2\alpha-1}$ and it provides a unitary equivalence between commutators defined by the wavelet ψ^α and their Bergman space analogues (see [GMP] for details about ψ^α and V).

2. Preliminaries and technical lemmas. This section contains some technical results which for the sake of clarity are separated from the main part of the paper. We begin by stating basic facts related to lattices in the hyperbolic metric. Next we discuss conditions on the wavelet that provide the decay of the reproducing kernel described in (1.4). The part that follows describes different norms on the oscillation spaces and the mixed norm spaces and some of their properties. We end this section by recalling the description of the spectrum of the Fourier algebra.

The left-invariant metric d is given by the length element $ds^2 = t^{-2}(dv^2 + dt^2)$. This metric is also Möbius invariant, and satisfies

$$\cosh(d(\zeta,\eta)) = 1 + \frac{|\zeta-\eta|^2}{2st},$$

where $\zeta = (u,s)$, $\eta = (v,t)$. This shows that for $e = (0,\ldots,0,1)$,

$$e^{-Md(\zeta,e)} \cong \begin{cases} s^M(1+|u|^2)^{-M} & \text{for } s \le 1,\\ s^{-M}(1+|u/s|^2)^{-M} & \text{for } s > 1,\end{cases} \tag{2.1}$$

and for $M > d$, $e^{-Md(\zeta,e)}$ is integrable on G. We refer to [B], [H], [T] for the background in hyperbolic geometry.

A sequence $\{\gamma_i\} \subset G$ is a δ,κ *lattice* if $\{\gamma_i D_\delta\}$ cover G and $\{\gamma_i D_\kappa\}$ are pairwise disjoint. A sequence is a *lattice* if it is a δ,κ lattice for some δ, κ.

If $\{\gamma_i\} \subset G$ is a lattice then

$$\sum_i e^{-Md(\gamma_i,\zeta)} \le c, \tag{2.2}$$

$$\sum_i \chi_{\gamma_i D_R}(\zeta) \le c_1, \tag{2.3}$$

$$\sum_i \chi_{\gamma_i D_r}(\zeta) \ge c_2, \tag{2.4}$$

provided that $M > d$ and r is large enough. The constants c, c_1, c_2 do not depend on ζ. The letter χ denotes the characteristic function of a set.

A sequence $\{(2^k m, 2^k)\}$, where $k \in \mathbb{Z}$, $m \in \mathbb{Z}^d$, is called a *standard lattice* and is denoted as $\{\eta_{mk}\}$ or $\{\eta_i\}$. It induces a standard partition $\{U_{mk}\}$ of G, where $m = (m_1,\ldots,m_k)$ and

$$U_{mk} = [2^k m_1, 2^k(m_1+1)) \times \ldots \times [2^k m_d, 2^k(m_d+1)) \times [2^{k-1}, 2^k).$$

All the elements of this partition have the same measure $d\zeta$. For many purposes this partition plays the same role as the covering $\{\eta_{mk}D_r\}$.

The next lemma relates the exponential decay of the kernel $\langle\psi_\zeta,\Psi\rangle$ to the decay, smoothness and vanishing moment properties of the functions ψ, Ψ. It is taken from the paper [FJ2] by Frazier and Jawerth.

LEMMA 2.1 (Lemma B1 in [FJ2]). *Suppose that* $0<s\le 1$, $R>d$, $0<\theta\le 1$, $S>L+d+\theta$, L *is a nonnegative integer. If* $\psi,\Psi\in L^1(\mathbb{R}^d)$ *satisfy*

$$|D^\gamma g(x)|\le c(1+|x|)^{-R}\quad \textit{if } |\gamma|\le L, \tag{2.5}$$

$$|D^\gamma g(x)-D^\gamma g(y)|\le |x-y|^\theta \sup_{|z|\le|y-x|}(1+|z-x|)^{-R}\quad \textit{if } |\gamma|=L, \tag{2.6}$$

$$|g(x)|\le(1+|x|)^{-\max\{R,S\}}, \tag{2.7}$$

$$\int_{\mathbb{R}^d} x^\gamma g(x)\,dx=0\quad \textit{if } |\gamma|\le L, \tag{2.8}$$

and $M\le R/2$, $M\le L+\theta+d/2$, *then*

$$|\langle\psi_\zeta,\Psi\rangle|\le ce^{-Md(\zeta,e)}.$$

The following lemma shows that it is also possible to get the decay of the kernel $\langle\psi_\zeta,\Psi\rangle$ by requiring that $\widehat{\Psi}$ is supported in a ring centered at 0 and that the behavior of $\widehat{\psi}$ is properly controled on rings centered at the origin. The proof follows by integration by parts on the Fourier transform side and it is omitted.

LEMMA 2.2. *Let* $\psi,\phi\in L^2(\mathbb{R}^d)$, $\widehat{\psi}(\xi)=|\xi|^\beta\widehat{\phi}(\xi)$, *and let* Ψ *be a Schwartz class function satisfying* $\operatorname{supp}\widehat{\Psi}\subset\{\xi: a\le|\xi|\le b\}$. *If* $d/2+\beta\ge M$, $n\ge M$ *and*

$$\sup_{0<s\le 1}\sum_{\substack{k,i\\ k\le n}} s^k A_i^k(s)<\infty, \tag{2.9}$$

$$\sup_{s\ge 1} s^{M+d/2+\beta}A^0(s)<\infty, \tag{2.10}$$

$$\sup_{s\ge 1} s^{-M+d/2+\beta}\sum_{\substack{k,i\\ k\le n}} s^k A_i^k(s)<\infty, \tag{2.11}$$

where

$$A_i^k(s)=s^{-d}\int_{sa\le|\xi|\le sb}|D_i^k\widehat{\phi}(\xi)|\,d\xi,$$

then

$$|\langle\psi_\zeta,\Psi\rangle|\le ce^{-Md(\zeta,e)}.$$

Remark 2.3. In the case of the Bergman wavelet ψ^α, given in (1.7), the condition

$$|\langle\psi_\eta^\alpha,\psi_\zeta^\alpha\rangle|\le c_\alpha e^{-Md(\zeta,\eta)}$$

is satisfied for $M=\alpha+1/2$. For a Schwartz class function Ψ with $\widehat{\Psi}$ compactly supported in $\mathbb{R}\setminus\{0\}$, also

$$|\langle\psi_\eta^\alpha,\Psi_\eta\rangle|\le c_\alpha e^{-Md(\zeta,\eta)},$$

where $M=\alpha+1/2$ (see [N]). In the scale of weighted Bergman spaces A^β, $M=\alpha+1/2$ corresponds to $M=(\beta+2)/2$. We recall that $\beta=2\alpha-1$, and that the space A^β is defined in (1.8).

The above statements show that our results apply to the Bergman space case provided β is large enough.

For b a locally square integrable function defined on G and a hyperbolic ball B centered at e the *oscillation* of b with respect to B at γ is defined as

$$\begin{aligned}\operatorname{osc}_B b(\gamma)&=\Big(|B|^{-1}\int_{\gamma B}\Big|b(\zeta)-|B|^{-1}\int_{\gamma B}b(\eta)\,d\eta\Big|^2 d\zeta\Big)^{1/2}\\ &=\Big(\tfrac12|B|^{-1}\int_{\gamma B}|B|^{-1}\int_{\gamma B}|b(\zeta)-b(\eta)|^2 d\zeta\,d\eta\Big)^{1/2}.\end{aligned} \tag{2.12}$$

For $\{\gamma_k\}$ a δ,κ lattice, $u>0$, $B=D_{\delta+u}$ and $0<p,q<\infty$,

$$\|b\|_{B^{pq}}=\|\{\operatorname{osc}_B b(\gamma_i)\}_i\|_{l^{pq}}.$$

For a nonnegative function F with $\|F\|_{L^1(G)}=1$ and for b a function on G such that for any $\gamma\in G$,

$$\int_G|b(\zeta)|^2F(\gamma^{-1}\zeta)\,d\zeta<\infty,$$

we define the *F-oscillation* of b by

$$\begin{aligned}\operatorname{osc}_F(\gamma)&=\Big(\int_G\Big|b(\zeta)-\int_G b(\eta)F(\gamma^{-1}\eta)\,d\eta\Big|^2F(\gamma^{-1}\zeta)\,d\zeta\Big)^{1/2}\\ &=\Big(\tfrac12\int_G\int_G|b(\zeta)-b(\eta)|^2F(\gamma^{-1}\zeta)F(\gamma^{-1}\eta)\,d\zeta\,d\eta\Big)^{1/2}.\end{aligned} \tag{2.13}$$

For $\{\gamma_i\}$ a lattice we set

$$\|b\|_{B^{pq}(F,\{\gamma_i\})}=\|\{\operatorname{osc}_F b(\gamma_i)\}_i\|_{l^{pq}}.$$

If $F=|B|^{-1}\chi_B$, then $\operatorname{osc}_F(\gamma)=\operatorname{osc}_B b(\gamma)$.

PROPOSITION 2.4. (i) *Different lattices and parameters give equivalent* $\mathcal{B}^{pq}$ *norms (quasi-norms).*

(ii) *If* $F(\zeta) \le ce^{-Md(\zeta,e)}$ *and* $M > 2d/\min\{1,p\}$, *then*

$$\|b\|_{\mathcal{B}^{pq}(F,\{\gamma_i\})} \le c\|b\|_{\mathcal{B}^{pq}}.$$

Proof. We omit the elementary but tedious proof of this fact.

For $m = (m_1, \ldots, m_d)$, $m_1, \ldots, m_d$, k integers, $a > 1$, $b > 0$ let U^{ba}_{mk} be a modified version of the standard partition of G,

$$U^{ba}_{mk} = [a^k bm_1, a^k b(m_1+1)) \times \ldots \times [a^k bm_d, a^k b(m_d+1)) \times [a^{k-1}, a^k). \tag{2.14}$$

For a function b defined on G, $a > 1$, $b > 0$ and U^{ba}_{mk} given in (2.14) we define

$$\|b\|_{l^p(L^2_{b,a})} = \Big(\sum_{m,k}\Big(\int_{U^{ba}_{mk}} |b(\zeta)|^2 d\zeta\Big)^{p/2}\Big)^{1/p}.$$

We refer to the papers by Feichtinger [F] and Fournier–Stewart [FS] for the background on the mixed norm spaces. The following two propositions are standard.

PROPOSITION 2.5. *For different* a, b *the norms* $\|\ \|_{l^p(L^2_{b,a})}$ *are equivalent. For* $\{\gamma_i\}$ *a lattice in* G *and* D *a ball centered at* e, *if* $\{\gamma_i D\}$ *cover* G, *then the norm*

$$\Big(\sum_i\Big(\int_{\gamma_i D} |b(\zeta)|^2 d\zeta\Big)^{p/2}\Big)^{1/p}$$

is equivalent to $\|\ \|_{l^p(L^2_{b,a})}$.

PROPOSITION 2.6. (i) *Let* $1 \le p_0, p_1 \le \infty$, $0 \le \theta \le 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$. *If* T *is a bounded operator on* $l^{p_0}(L^2_{b,a})$ *and on* $l^{p_1}(L^2_{b,a})$, *then it is bounded on* $l^p(L^2_{b,a})$ *and*

$$\|T\|_{l^p(L^2_{b,a})} \le \|T\|^{1-\theta}_{l^{p_0}(L^2_{b,a})}\|T\|^{\theta}_{l^{p_1}(L^2_{b,a})}. \tag{2.15}$$

(ii) *The dual space of* $l^p(L^2_{b,a})$, $1 \le p < \infty$, *is* $l^{p'}(L^2_{b,a})$ *and the pairing is given by*

$$\int_G f(\zeta)\bar{g}(\zeta)\,d\zeta.$$

By $A(G)$ we denote the *Fourier algebra* on G, i.e.

$$A(G) = \{\widetilde{H} * K : H, K \in L^2(G)\}$$

(see [E]), where $\widetilde{H}(\zeta) = \overline{H(\zeta^{-1})}$ and the convolution on G is defined as

$$F * G(\eta) = \int_G F(\zeta^{-1}\gamma)G(\gamma)\,d\gamma.$$

PROPOSITION 2.7. *If* $F(\zeta) = f(d(\zeta,e))$, *where* f *is a* C^∞ *function compactly supported in* $[0,\infty)$, *then* $F \in A(G)$.

Proof. This follows by a standard application of the spherical transform to bi-invariant functions. For details about the spherical transform see [H].

The following fact is included in the paper [E] of Eymard.

LEMMA 2.8. *The spectrum of the commutative Banach algebra* $A(G) \oplus \{\lambda e\}$ *is* $G \cup \{\infty\}$.

3. Direct estimates. Let ψ be an admissible wavelet, and b a locally square integrable function defined on G. A commutator $C_b = [M_b, P]$ is well defined on $C_c(G)$, the compactly supported continuous functions on G, and for $F, H \in C_c(G)$,

$$(C_b F, H) = \int_G\int_G (b(\zeta) - b(\eta))\langle\psi_\eta, \psi_\zeta\rangle F(\eta)\overline{H(\zeta)}\,d\eta\,d\zeta.$$

We need the following discrete version of the Calderón reproducing formula:

PROPOSITION 3.1 ([FJ1]). *There is a Schwartz class function* Ψ *such that* $\widehat{\Psi}$ *is compactly supported in* $\mathbb{R}^d \setminus \{0\}$ *and for* $f, g \in L^2(\mathbb{R}^d)$,

$$\langle f, g\rangle = \sum_n \langle f, \Psi_{\eta_n}\rangle\langle\Psi_{\eta_n}, g\rangle. \tag{3.1}$$

The next theorem provides the estimate on singular values in terms of the oscillation numbers.

THEOREM 3.2. *Suppose that a Schwartz class function* Ψ *satisfies* (3.1), *and*

$$|\langle\psi_\zeta, \Psi\rangle| \le ce^{-Md(\zeta,e)}, \quad \text{where } M > d.$$

Let

$$F(\zeta) = \Big(\int_G |\langle\psi_\zeta, \Psi\rangle|\,d\zeta\Big)^{-1}|\langle\psi_\zeta, \Psi\rangle|.$$

If $\operatorname{osc}_F b(\eta_n) \to 0$ *as* $d(\eta_n, e) \to \infty$, *then* C_b *may be decomposed as a sum of two compact operators and after relabeling the lattice points so that the numbers* $\operatorname{osc}_F b(\eta_n)$ *are written in nonincreasing order, each operator satisfies the following estimate on its singular values:*

$$s_n \le c\operatorname{osc}_F b(\eta_n). \tag{3.2}$$

Proof. Let $\widehat{b}(\eta_i)=\int_G b(\gamma)F(\eta_i^{-1}\gamma)\,d\gamma$. An application of (3.1) gives

$$\begin{aligned}(b(\zeta)-b(\eta))\langle\psi_\eta,\psi_\zeta\rangle &= (b(\zeta)-b(\eta))\sum_i\langle\psi_\eta,\Psi_{\eta_i}\rangle\langle\Psi_{\eta_i},\psi_\zeta\rangle\\ &=\sum_i(b(\zeta)-\widehat{b}(\eta_i))\langle\psi_\eta,\Psi_{\eta_i}\rangle\langle\Psi_{\eta_i},\psi_\zeta\rangle\\ &\quad-\sum_i(b(\eta)-\widehat{b}(\eta_i))\langle\psi_\eta,\Psi_{\eta_i}\rangle\langle\Psi_{\eta_i},\psi_\zeta\rangle\\ &=\sum_i \operatorname{osc}_F b(\eta_i)\langle\psi_\eta,\Psi_{\eta_i}\rangle(b(\zeta)-\widehat{b}(\eta_i))(\operatorname{osc}_F b(\eta_i))^{-1}\langle\Psi_{\eta_i},\psi_\zeta\rangle\\ &\quad-\sum_i \operatorname{osc}_F b(\eta_i)(b(\eta)-\widehat{b}(\eta_i))(\operatorname{osc}_F b(\eta_i))^{-1}\langle\psi_\eta,\Psi_{\eta_i}\rangle\langle\Psi_{\eta_i},\psi_\zeta\rangle\\ &=K_1(\zeta,\eta)-K_2(\zeta,\eta).\end{aligned}$$

To prove (3.2) for K_1 it is enough to show that

$$\Big\|\sum_{i=n}^{\infty}\operatorname{osc}_F b(\eta_i)\langle\psi_\eta,\Psi_{\eta_i}\rangle(b(\zeta)-\widehat{b}(\eta_i))\times(\operatorname{osc}_F b(\eta_i))^{-1}\langle\Psi_{\eta_i},\psi_\zeta\rangle\Big\|_{S^\infty}\le c\operatorname{osc}_F b(\eta_n).$$

A similar estimate holds for K_2.

Let $f,g\in L^2(G)$. Then

$$\begin{aligned}&\Big|\int_G\int_G\sum_{i=n}^{\infty}\operatorname{osc}_F b(\eta_i)\langle\psi_\eta,\Psi_{\eta_i}\rangle f(\eta)\\ &\qquad\times(b(\zeta)-\widehat{b}(\eta_i))(\operatorname{osc}_F b(\eta_i))^{-1}\langle\Psi_{\eta_i},\psi_\zeta\rangle g(\zeta)\,d\eta\,d\zeta\Big|\\ &\le\operatorname{osc}_F b(\eta_n)\sum_{i=n}^{\infty}\int_G|\langle\psi_\eta,\Psi_{\eta_i}\rangle|\,|f(\eta)|\,d\eta\\ &\qquad\times\int_G|b(\zeta)-\widehat{b}(\eta_i)|\,(\operatorname{osc}_F b(\eta_i))^{-1}|\langle\Psi_{\eta_i},\psi_\zeta\rangle|\,|g(\zeta)|\,d\zeta\\ &\le\operatorname{osc}_F b(\eta_n)\Big(\sum_i\int_G e^{-Md(\eta_i,\eta)}d\eta\int_G e^{-Md(\eta_i,\eta)}|f(\eta)|^2d\eta\Big)^{1/2}\\ &\qquad\times\Big(\sum_i\int_G|b(\zeta)-\widehat{b}(\eta_i)|^2\,|\langle\Psi_{\eta_i},\psi_\zeta\rangle|\,d\zeta(\operatorname{osc}_F b(\eta_i))^{-2}\\ &\qquad\times\int_G e^{-Md(\eta_i,\zeta)}|g(\zeta)|^2d\zeta\Big)^{1/2}\\ &\le c\operatorname{osc}_F b(\eta_n)\|f\|\,\|g\|.\end{aligned}$$

We may conclude the estimate of the Schatten–Lorentz norm in the following corollary.

COROLLARY 3.3. *Suppose that there is a Schwartz class function Ψ satisfying* (3.1) *for which*

$$|\langle\psi_\zeta,\Psi\rangle|\le ce^{-Md(\zeta,e)}.$$

If $M>2d/\min\{1,p\}$, $p<\infty$, then

$$\|C_b\|_{S^{pq}}\le c\|b\|_{B^{pq}}.$$

Proof. This follows directly from Proposition 2.4 and Theorem 3.2.

4. Converse estimates.

In the first part we study the case $p>1$, and we obtain converse estimates for commutators for all admissible wavelets. The proof relies on the Fourier algebra and Schur multipliers. However, these techniques are not necessary and a modification of the proof of Theorem 4.4 also gives such a result. But for the method of Theorem 4.4 to work we need to assume sufficient decay of the reproducing kernel. In the second part we discuss the case $0<p\le 1$.

We recall the notion of a Schur multiplier, and we state some basic facts about Schur multipliers related to our context.

A function $M(\zeta,\eta)$ defined on $G\times G$ is called a *Schur multiplier* on S^{pq} if for any kernel $K(\zeta,\eta)$ representing an S^{pq} operator on $L^2(G)$, the product $M(\zeta,\eta)K(\zeta,\eta)$ also represents an S^{pq} operator, and the above operation is bounded.

PROPOSITION 4.1. (i) *If ψ is an admissible wavelet, $H(\zeta)=h(d(\zeta,e))$, where h is a compactly supported C^∞ function, and* $\operatorname{supp}H\subset\{\zeta:\operatorname{Re}\langle\psi_\zeta,\psi\rangle>\|\psi\|^2/2\}$, *then*

$$\frac{H(\zeta)}{\langle\psi_\zeta,\psi\rangle}\in A(G).\tag{4.1}$$

(ii) *If $F\in A(G)$ then $F(\zeta^{-1}\eta)$ is a Schur multiplier on S^p, $1\le p\le\infty$, on compact operators and on S^{pq}, $1<p<\infty$, $0<q\le\infty$.*

(iii) *If for some $r>0$, $\{\gamma_iD_r\}$ are pairwise disjoint then for any hyperbolic ball D centered at e,*

$$\sum_i\chi_{\gamma_iD}(\eta)\chi_{\gamma_iD}(\zeta)$$

is a Schur multiplier on S^p, $1\le p\le\infty$, and on S^{pq}, $1<p<\infty$, $0<q\le\infty$.

Proof. (i) We consider the commutative Banach algebra with identity $A(G)\oplus\{\lambda e\}$. By Lemma 2.8 its spectrum is $G\cup\{\infty\}$. Take $G(\zeta)=g(d(\zeta,e))$, $g\in C_c^\infty([0,\infty))$, such that $c-G(\zeta)\equiv0$ on the support of H and

$$\operatorname{Re}(c-G(\zeta)+\langle\psi_\zeta,\psi\rangle)>\|\psi\|^2/2\qquad\text{for all }\zeta\in G.$$

By Proposition 2.7, $G(\zeta) \in A(G)$. Clearly $\langle \psi_\zeta, \psi \rangle$ is an idempotent with respect to convolution on G and is in $L^2(G)$, thus also in $A(G)$. So

$$c - G(\zeta) + \langle \psi_\zeta, \psi \rangle \in A(G) \oplus \{\lambda e\}.$$

Since it is different from zero on $G \cup \{\infty\}$ it is invertible in $A(G) \oplus \{\lambda e\}$. Thus

$$\frac{H(\zeta)}{c - G(\zeta) + \langle \psi_\zeta, \psi \rangle} \in A(G) \oplus \{\lambda e\}.$$

Clearly

$$\frac{H(\zeta)}{c - G(\zeta) + \langle \psi_\zeta, \psi \rangle} = \frac{H(\zeta)}{\langle \psi_\zeta, \psi \rangle},$$

and $H(\zeta)/\langle \psi_\zeta, \psi \rangle = 0$ if $d(\zeta, e)$ is large enough. Thus $H(\zeta)/\langle \psi_\zeta, \psi \rangle \in A(G)$.

The statements (ii), (iii) follow directly for S^1 and S^∞, and then by interpolation for the other indices.

The next lemma provides the converse estimate for singular values of the perturbed commutator.

LEMMA 4.2. *Let b be a locally square integrable function, and D a hyperbolic disc with center e. Assume that the operator C given by the kernel*

$$\sum_i (b(\zeta) - b(\eta)) \chi_{\gamma_i D}(\eta) \chi_{\gamma_i D}(\zeta), \tag{4.2}$$

is compact and that the sets $\{\gamma_i D\}$ are pairwise disjoint. Then there is a constant c depending only on D such that the singular values $\{s_n\}$ of C satisfy

$$s_n \geq c \operatorname{osc}_D b(\gamma_n) \quad \textit{for all } n. \tag{4.3}$$

The lattice points are relabeled so that $\{\operatorname{osc}_D b(\gamma_n)\}$ form a nonincreasing sequence.

Proof. Let C_i be given by the kernel $(b(\zeta) - b(\eta)) \chi_{\gamma_i D}(\eta) \chi_{\gamma_i D}(\zeta)$. Clearly

$$\|C_i \chi_{\gamma_i D}\| = c \operatorname{osc}_D b(\gamma_i),$$

thus C_i has a singular value s_i such that

$$s_i \geq c \operatorname{osc}_D b(\gamma_i),$$

and an eigenvector F_i corresponding to the eigenvalue s_i^2 of $C_i^* C_i$ is in $L^2(\gamma_i D)$. We obtain

$$C^* C = \sum_i C_i^* C_i \quad \text{and} \quad C^* C F_k = \sum_i C_i^* C_i F_k = s_k^2 F_k,$$

so s_k are distinct singular values of C and (4.3) follows.

In the next theorem we get the converse estimate for the Schatten–Lorentz norm of the commutator C_b for the case $p > 1$.

THEOREM 4.3. *Let b be a locally square integrable function, and ψ any admissible wavelet. If $C_b \in S^{pq}$, $p, q = 1$, or $1 < p < \infty$, $0 < q \leq \infty$, then $b \in \mathcal{B}^{pq}$ and*

$$\|b\|_{\mathcal{B}^{pq}} \leq c \|C_b\|_{S^{pq}}.$$

Proof. We take a hyperbolic ball D_0 with center $(0, \ldots, 0, 1)$ and a function H, $H \equiv 1$ on D_0, so that the requirements of Proposition 4.1(i) are satisfied. By Proposition 4.1(i), (ii),

$$\frac{H(\zeta^{-1}\eta)}{\langle \psi_\eta, \psi_\zeta \rangle}$$

is a Schur multiplier on all the spaces in question. Let D be a hyperbolic ball with center $(0, \ldots, 0, 1)$ and radius smaller than half of the radius of D_0. We take a lattice $\{\tau_k^i\}_k^{i=1,\ldots,N}$ such that $\{\tau_k^i D'\}_k^i$ cover G, $\{\tau_k^i D\}_k$ are pairwise disjoint for fixed i, D' is centered at e and has radius slightly smaller than the radius of D.

Since $\sum_k \chi_{\tau_k^i D}(\eta) \chi_{\tau_k^i D}(\zeta) \neq 0$ only if $\zeta^{-1}\eta \in D_0$, we have

$$\frac{\sum_k \chi_{\tau_k^i D}(\eta) \chi_{\tau_k^i D}(\zeta)}{\langle \psi_\eta, \psi_\zeta \rangle} = \frac{H(\zeta^{-1}\eta) \sum_k \chi_{\tau_k^i D}(\eta) \chi_{\tau_k^i D}(\zeta)}{\langle \psi_\eta, \psi_\zeta \rangle}$$

and this function is a Schur multiplier on the trace class and on S^{pq} by Proposition 4.1(iii). Thus if C_b is trace class (in S^{pq}) then also

$$\sum_k (b(\zeta) - b(\eta)) \chi_{\tau_k^i D}(\zeta) \chi_{\tau_k^i D}(\eta)$$

is trace class (in S^{pq}). Lemma 4.2 finishes the proof.

Now we formulate the converse result for $0 < p \leq 1$ and $0 < q \leq \infty$. In this case we need to assume sufficient decay of the reproducing kernel. We do not know what is the optimal condition on the wavelet which gives this converse result.

THEOREM 4.4. *Let $0 < p \leq 1$, $0 < q \leq \infty$. Suppose there is a Schwartz class function Ψ satisfying (3.1) and for some $M > d(p+2)/(2p)$,*

$$|\langle \psi_\zeta, \Psi \rangle| \leq c e^{-M d(\zeta, e)} \quad \textit{for all } \zeta \in G.$$

If $C_b \in S^{pq}$, then $b \in \mathcal{B}^{pq}$ and

$$\|b\|_{\mathcal{B}^{pq}} \leq c \|C_b\|_{S^{pq}}.$$

Proof. We sketch the proof of this result by indicating the main steps.

Step 1. We pick a hyperbolic lattice $\{\tau_k\}$ and a hyperbolic ball D centered at e such that $\operatorname{Re}\langle \psi_\zeta, \psi \rangle \geq \|\psi\|^2/2$ if $\zeta \in 3D$ and for every suitably large r if $\tau_k, \tau_l \in D_r$ then there are

$$\tau_{kl}(1), \ldots, \tau_{kl}(P) \in \{\tau_k\} \cap D_r, \qquad P \le cd(\tau_k, \tau_l),$$

the distance of each $\tau_{kl}(j)$ to the geodesic joining τ_k and τ_l is smaller than δ, each τ_k may appear at most K times in the sequence $\{\tau_{kl}(j)\}$,

$$|\tau_{kl}(j)D \cap \tau_{kl}(j+1)D| > \varepsilon \quad \text{for } i = 0, \ldots, P, \quad \tau_{kl}(0) = \tau_k, \quad \tau_{kl}(P+1) = \tau_l.$$

The constants $K, c, \varepsilon, \delta$ depend only on the parameters of the lattice $\{\tau_k\}$.

Next we divide $\{\tau_k\}$ into L pieces

$$\{\tau_k^1\}_k, \ldots, \{\tau_k^L\}_k$$

so that for every $o = 1, \ldots, L$, $\{\tau_k^o D\}_k$ are pairwise disjoint. For a large positive number N each $\{\tau_k^o\}_k$ is divided into $C(N)$ pieces

$$\{\tau_k^{o1}\}_k, \ldots, \{\tau_k^{oC(N)}\}_k$$

so that $d(\tau_k^{om} D, \tau_l^{om} D) > N$ whenever $k \ne l$.

Step 2. For fixed r, o and N we define

$$C_N^o = \sum_{m=1}^{C(N)} (b(\zeta) - b(\eta)) \langle \psi_\eta, \psi_\zeta \rangle \sum_{\substack{k \\ \tau_k^{om} \in B(e,r)}} \chi_{\tau_k^{om} D}(\zeta) \sum_{\substack{l \\ \tau_l^{om} \in B(e,r)}} \chi_{\tau_l^{om} D}(\eta),$$

$$D^o = \sum_{m=1}^{C(N)} (b(\zeta) - b(\eta)) \langle \psi_\eta, \psi_\zeta \rangle \sum_{\substack{k \\ \tau_k^{om} \in B(e,r)}} \chi_{\tau_k^{om} D}(\zeta) \chi_{\tau_k^{om} D}(\eta),$$

$$R_N^o = \sum_{m=1}^{C(N)} (b(\zeta) - b(\eta)) \langle \psi_\eta, \psi_\zeta \rangle \sum_{\substack{k \ne l \\ \tau_k^{om}, \tau_l^{om} \in B(e,r)}} \chi_{\tau_k^{om} D}(\zeta) \chi_{\tau_l^{om} D}(\eta).$$

Clearly $C_N^o = D^o + R_N^o$ and $\|C_N^o\|_{S^{pq}} \le C_N \|C_b\|_{S^{pq}}$.

Step 3. We prove that for some $\beta > 0$,

$$\|R_N^o\|_{S^{pq}} \le ce^{-\beta N} \|\chi_{K_r}(j) \operatorname{osc}_D b(\tau_j)\|_{l^{pq}},$$
$$o = 1, \ldots, L, \ K_r = \{j : \tau_j \in D_r\}.$$

Step 4. We show that

$$\|\chi_{K_r}(j) \operatorname{osc}_D b(\tau_j)\|_{l^{pq}} \le c \sum_{o=1}^{L} \|D^o\|_{S^{pq}}.$$

Step 5. We are ready to finish the proof. We get

$$\|\chi_{K_r}(j) \operatorname{osc}_D b(\tau_j)\|_{l^{pq}} \le c \sum_{o=1}^{L} \|D^o\|_{S^{pq}} \le c \sum_{o=1}^{L} \|C_N^o\|_{S^{pq}} + c \sum_{o=1}^{L} \|R_N^o\|_{S^{pq}}$$
$$\le c_N \|C_b\|_{S^{pq}} + ce^{-\beta N} \|\chi_{K_r}(j) \operatorname{osc}_D b(\tau_j)\|_{l^{pq}}.$$

Now we take N large enough and get

$$\|\chi_{K_r}(j) \operatorname{osc}_D b(\tau_j)\|_{l^{pq}} \le c_N \|C_b\|_{S^{pq}}$$

for all r large enough. This concludes the proof.

Remark 4.5. Results similar to those for C_b are true for $C_b M_\omega$, where M_ω denotes multiplication by $\omega(\zeta) = s^\alpha$, $\alpha \in \mathbb{R}$, $\zeta = (u, s)$. Namely, if the wavelet ψ satisfies the condition

$$|\langle \psi_\zeta, \Psi \rangle| \le ce^{-Md(\zeta, \eta)},$$

where Ψ is a Schwartz class function for which (3.1) holds, and M is large enough (how large depends on α, p and d), then

$$\|C_b M_\omega\|_{S^{pq}} \cong \|\omega(\gamma_i) \operatorname{osc}_D b(\gamma_i)\|_{l^{pq}}.$$

The lattice $\{\gamma_i\}$ and the ball D satisfy the usual conditions.

Remark 4.6. For $\{\gamma_k\}$ a δ, κ lattice, $u > 0$, and $D = D_{\delta+u}$, the norm in the space $\mathcal{BMO}$ is defined as

$$\|b\|_{\mathcal{BMO}} = \|\{\operatorname{osc}_D b(\gamma_i)\}_i\|_{l^\infty},$$

and $\mathcal{VMO}$ is the subspace of $\mathcal{BMO}$ consisting of those functions b for which

$$\operatorname{osc}_D b(\gamma_i) \to 0 \qquad \text{as } d(\gamma_i, e) \to \infty.$$

It is not hard to check that C_b is bounded if and only if $b \in \mathcal{BMO}$, and it is compact if and only if $b \in \mathcal{VMO}$.

Using the characterization of compact commutators we describe the essential spectrum of a Toeplitz operator

$$T_b = PM_bP$$

with bounded continuous symbol in $\mathcal{VMO}$, where P is the orthogonal projection from $L^2(G)$ onto the space of Calderón transforms of square integrable functions. Toeplitz operators and Hankel operators are related in the following way:

$$T_{b_1 b_2} - T_{b_1} T_{b_2} = H_{b_1}^* H_{b_2}.$$

The ideas presented here are based on [Str] and [Z2]. In 1–3 below we assume that b is a bounded function uniformly continuous with respect to the hyperbolic metric.

1. Let $\mathcal{T}$ denote the set of all possible limits of nets $\{\tau_{\zeta_\alpha}\} \subset (\beta G)^G$, such that $d(\zeta_\alpha, e) \to \infty$, where $\tau_\zeta(\gamma) = \zeta\gamma$ and βG denotes the Čech–Stone

compactification of G. If $\tau_{\zeta_\alpha} \to \tau \in \mathcal{T}$, then $b \circ \tau_{\zeta_\alpha} \to b \circ \tau$ uniformly on compact sets.

2. It follows from the comments above that C_b is compact if and only if for every $\tau \in \mathcal{T}$, $b \circ \tau$ is constant.

3. If C_b is compact, then

$$\sigma_e(T_b) = \bigcap_{r>0} \overline{b(G \setminus D_r)} = \{c(\tau) : b \circ \tau \equiv c(\tau),\ \tau \in \mathcal{T}\},$$

where σ_e denotes the essential spectrum.

4. If C_b is compact, then $\widetilde{b}$ is uniformly continuous with respect to the hyperbolic metric, and $T_{b-\widetilde{b}}$ is compact.

5. Summarizing, for every bounded continuous function b for which C_b is compact,

$$\sigma_e(T_b) = \bigcap_{r>0} \overline{\widetilde{b}(G \setminus D_r)} = \{c(\tau) : \widetilde{b} \circ \tau \equiv c(\tau),\ \tau \in \mathcal{T}\}.$$

5. A decomposition of oscillation spaces. In the following section we show a decomposition theorem for the space $\mathcal{B}^p = \mathcal{B}^{pp}$ for $1 < p < \infty$.

For a hyperbolic ball D centered at e we define the averaging operator A_D with respect to D as

$$A_D b(\zeta) = |D|^{-1} \chi_D * b(\gamma), \tag{5.1}$$

where b is a locally integrable function on G.

We want to prove that the operator norm of A_D on $l^p(L^2_{b,a})$ is smaller than 1 if a is close to 1 and b is close to 0. To be able to do that we need the following two lemmas.

LEMMA 5.1. *For every $\varepsilon > 0$ there are $a > 1$, $b > 0$ such that the operator norms of A_D on $l^\infty(L^2_{b,a})$ and on $l^1(L^2_{b,a})$ are bounded by $1 + \varepsilon$.*

Proof. We prove the case $l^\infty(L^2_{b,a})$; the other case follows by duality. To simplify notation we write $\{U_j\}$ instead of $\{U^{ba}_{mk}\}$. Let

$$F_{ij}(\zeta,\eta) = |D|^{-1} \chi_{U_i}(\zeta) \chi_D(\zeta^{-1}\eta) \chi_{U_j}(\eta).$$

We have

$$\begin{aligned}\|A_D b\|_{l^\infty(L^2_{b,a})} &= \sup_i \Big(\int_{U_i} |A_D b|^2 d\zeta \Big)^{1/2} \\ &\le \sup_i \sum_j \|F_{ij}\|_{L^2(U_i), L^2(U_j)} \|b\|_{l^\infty(L^2_{b,a})}.\end{aligned}$$

By the Schur lemma it is enough to show that there are $a > 1$, $b > 0$ such that

$$\sup_i \sum_j \sup_{\zeta \in U_i} \int_{U_j} F_{ij}(\zeta,\eta)\, d\eta \le 1 + \varepsilon, \tag{5.2}$$

$$\sup_i \sum_j \sup_{\eta \in U_j} \int_{U_i} F_{ij}(\zeta,\eta)\, d\zeta \le 1 + \varepsilon. \tag{5.3}$$

We prove (5.2). The proof of (5.3) is the same. Clearly

$$\sup_{\zeta \in U_i} \int_{U_j} F_{ij}(\zeta,\eta)\, d\eta \le |D|^{-1} |U_j|,$$

and

$$\sup_{\zeta \in U_i} \int_{U_j} F_{ij}(\zeta,\eta)\, d\eta = 0 \quad \text{if } d(U_i,U_j) > R,$$

where R is the radius of the ball D. We obtain

$$\sum_j \sup_{\zeta \in U_i} \int_{U_j} F_{ij}(\zeta,\eta)\, d\eta \le |D|^{-1} \sum_{\substack{j \\ d(U_i,U_j) \le R}} |U_j| \le |D|^{-1} |D_{R+2\delta}|,$$

where $\delta = \sup_i \operatorname{diam} U_i$. Since $\delta \to 0$ as $a \to 1$ and $b \to 0$, the lemma follows.

LEMMA 5.2. *The operator norm of A_D on $L^2(G)$ is strictly smaller than* 1.

Proof. The operator A_D is given by integration against the kernel

$$|D|^{-1} \chi_D(\zeta^{-1}\eta).$$

Let $\omega(\zeta) = s^{d/2}$, where $\zeta = (u,s)$. By the Schur lemma it is enough to prove that

$$|D|^{-1} \int_G \chi_D(\zeta^{-1}\eta) \omega(\eta)\, d\eta \le c\omega(\zeta) \tag{5.4}$$

for some $c < 1$. Substituting variables in (5.4) we reduce it to

$$|D|^{-1} \int_D \omega(\eta)\, d\eta < 1. \tag{5.5}$$

To prove (5.5) we show that the function

$$|D|^{-1} \int_D s^\beta d\zeta$$

attains a strict minimum at $\beta = d/2$. This gives (5.5) since the above function attains value 1 at 0.

It is not hard to check that for $D = D_R$,

$$D = \left\{\zeta = (u,s) \in G : \sum_{i=0}^{d} u_i^2 + (s - \cosh R)^2 \le \sinh^2 R\right\}$$

(see [B]). We obtain

$$\int_D s^\beta d\zeta = c \int_{-R}^{R} e^{(\beta - d/2)x}(\cosh R - \cosh x)^{d/2}\, dx\,,$$

where the constant c does not depend on β. We compute the first two derivatives of $\int_D s^\beta d\zeta$. We have

$$\frac{d}{d\beta}\int_D s^\beta d\zeta = c \int_{-R}^{R} x e^{(\beta - d/2)x}(\cosh R - \cosh x)^{d/2}\, dx\,,$$

$$\frac{d^2}{d\beta^2}\int_D s^\beta d\zeta = c \int_{-R}^{R} x^2 e^{(\beta - d/2)x}(\cosh R - \cosh x)^{d/2}\, dx\,.$$

Since the first derivative vanishes at $\beta = d/2$ and the second is positive, the function $\int_D s^\beta d\zeta$ attains a strict minimum at $\beta = d/2$, and (5.5) follows.

THEOREM 5.3. *Let $1 < p < \infty$. If a is sufficiently close to 1 and b is sufficiently close to 0, then the operator norm of A_D on $l^p(L^2_{b,a})$ is smaller than* 1.

Proof. This statement follows immediately from Proposition 2.6 and Lemmas 5.1, 5.2.

PROPOSITION 5.4. *If $b \in \mathcal{B}^p$, then*

(i) $\|A_D b\|_{\mathcal{B}^p} \le c\|b\|_{\mathcal{B}^p}$,
(ii) $\|b - A_D b\|_{l^p(L^2)} \le c\|b\|_{\mathcal{B}^p}$.

Proof. We omit the easy proof.

THEOREM 5.5. *Let $1 < p < \infty$ and $b \in \mathcal{B}^p$. There are $b_{\mathrm{h}} \in \mathcal{B}^p$ and $b_{\mathrm{c}} \in l^p(L^2)$ such that b_{h} is harmonic, $\|b_{\mathrm{h}}\|_{\mathcal{B}^p} \le c\|b\|_{\mathcal{B}^p}$, $\|b_{\mathrm{c}}\|_{l^p(L^2)} \le c\|b\|_{\mathcal{B}^p}$ and*

$$b = b_{\mathrm{h}} + b_{\mathrm{c}}\,.$$

Moreover, this decomposition is unique.

Proof. We show first the decomposition relative to a ball D, i.e. for $b \in \mathcal{B}^p$ there are b_{h}^D, b_{c}^D such that $A_D b_{\mathrm{h}}^D = b_{\mathrm{h}}^D$, $\|b_{\mathrm{h}}^D\|_{\mathcal{B}^p} \le c\|b\|_{\mathcal{B}^p}$, $\|b_{\mathrm{c}}^D\|_{l^p(L^2)} \le c\|b\|_{\mathcal{B}^p}$, and

$$b = b_{\mathrm{h}}^D + b_{\mathrm{c}}^D\,. \tag{5.6}$$

Later we observe that b_{h}^D does not depend on D.

By Proposition 5.4, $b - A_D b \in l^p(L^2)$. We write

$$b - A_D^n b = \sum_{k=0}^{n-1} A_D^k (b - A_D b)\,.$$

Theorem 5.3 shows that the series

$$\sum_k A_D^k (b - A_D b)$$

is absolutely convergent in $l^p(L^2_{b,a})$ if a and b are properly chosen. This shows that $b - A_D^n b$ converges in $l^p(L^2)$ to some function $b_{\mathrm{c}}^D \in l^p(L^2)$. Clearly $b - A_D^n b$ also converges in $\mathcal{B}^p$, thus $A_D^n b$ converges to some function b_{h}^D. Taking the limit with respect to n in the expression

$$b = A_D^n b + b - A_D^n b$$

gives (5.6). The operator A_D is bounded on $\mathcal{B}^p$, so $A_D b_{\mathrm{h}}^D = b_{\mathrm{h}}^D$. The uniqueness of the decomposition follows by an application of A_D^n to both sides of the equation and passing to the limit.

We show that the decomposition $b = b_{\mathrm{h}}^D + b_{\mathrm{c}}^D$ does not depend on D. Take two hyperbolic discs D_1, D_2 centered at e. The operators A_{D_1}, A_{D_2} commute since the corresponding functions are bi-invariant (see Corollary 5.2 in [H]). Let

$$b = b_{\mathrm{h}}^1 + b_{\mathrm{c}}^1$$

be the decomposition corresponding to D_1. We write

$$b_{\mathrm{h}}^1 = b_{\mathrm{h}}^{1,2} + b_{\mathrm{h,c}}^{1,2}$$

for the decomposition of b_{h}^1 with respect to D_2. Since

$$A_{D_1} b_{\mathrm{h}}^{1,2} = A_{D_1} \lim_n A_{D_2}^n b_{\mathrm{h}}^1 = \lim_n A_{D_1} A_{D_2}^n b_{\mathrm{h}}^1$$
$$= \lim_n A_{D_2}^n A_{D_1} b_{\mathrm{h}}^1 = \lim_n A_{D_2}^n b_{\mathrm{h}}^1 = b_{\mathrm{h}}^{1,2}$$

we have $A_{D_1} b_{\mathrm{h}}^{1,2} = b_{\mathrm{h}}^{1,2}$ and $A_{D_2} b_{\mathrm{h}}^{1,2} = b_{\mathrm{h}}^{1,2}$. We may write

$$b = b_{\mathrm{h}}^{1,2} + b_{\mathrm{c}}^{1,2}, \quad \text{where } b_{\mathrm{c}}^{1,2} \in l^p(L^2)\,.$$

By the uniqueness of the decomposition we obtain

$$b_{\mathrm{h}}^1 = b_{\mathrm{h}}^{1,2}.$$

Similarly we get $b_{\mathrm{h}}^2 = b_{\mathrm{h}}^{1,2}$. This shows $b_{\mathrm{h}}^1 = b_{\mathrm{h}}^2$, and we may drop the dependence on D in the decomposition. Since b_{h} has the mean value property with respect to all hyperbolic discs, it is harmonic.

Summarizing, if ψ satisfies the assumption in Corollary 3.3 with $M > 2d$, then for $1 < p < \infty$, $C_b \in S^p$ if and only if $b \in \mathcal{B}^p$, and this happens if and only if

$$b = b_{\mathrm{h}} + b_{\mathrm{c}}$$

with b_{h} a $\mathcal{B}^p$ harmonic function and b_{c} in $l^p(L^2)$.

In particular, for Hankel operators on the classical weighted Bergman spaces A^β the above is true if $\beta > 2$.

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