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Markov's property of the Cantor ternary set

by

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Abstract. We prove that the Cantor ternary set E satisfies the classical Markov inequality (see [Ma]): for each polynomial p of degree at most n ($n = 0, 1, 2, \ldots$)

$$\text{(M)} \qquad |p'(x)| \le Mn^m \sup_E |p| \quad \text{for } x \in E,$$

where M and m are positive constants depending only on E.

0. Introduction. In 1889 A. A. Markov proved (M) for $E = [-1;1]$. Since that time it has become the object of extensive research (see e.g. [R-S] in the one-dimensional case and [Pa-Pl 1] for $\mathbb{R}^n$). In particular, it has appeared that the inequality plays an important role in the approximation and extension of C^∞ functions defined on compact subsets of $\mathbb{R}^n$ to C^∞ functions on the whole space (see [Pa-Pl 1], [Pa-Pl 2] and [Pl 4]).

The question about Markov's property for the Cantor ternary set has remained unanswered for many years. J. Siciak [Si 3] showed that there exists a Cantor type set $E \subset \mathbb{R}$ such that Leja's extremal function L_E (see [Lj 2], p. 261) has the following Hölder continuity property:

$$\text{(HCP)} \qquad L_E(z) \le 1 + M\delta^m \quad \text{if } \operatorname{dist}(z, E) \le \delta \le 1,$$

with some positive constants $M > 0$ and $m > 0$ depending only on E, which, by Cauchy's integral formula, is sufficient for E to preserve Markov's inequality (M) (see [Si 2], Remark after Lemma 1 and [Pl 1], Lemma 3.1). On the other hand, Pleśniak [Pl 2] constructed a Cantor type set E such that Leja's extremal function is continuous on $\mathbb{C}$ but E does not satisfy (M).

These results have given no answer to the question of whether the Cantor ternary set has Markov's property. In this paper we prove that the answer is affirmative. Actually, we show that this set even has the (HCP) property.

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1. Leja's extremal function. Let K be a compact subset of the complex plane $\mathbb{C}$. For each $n \in \mathbb{N}$ and $(z_0^n, z_1^n, \ldots, z_n^n) \in K^{n+1} \subset \mathbb{C}^{n+1}$ we put

$$V(z_0^n, z_1^n, \ldots, z_n^n) = \prod_{0 \le j < k \le n} |z_j^n - z_k^n|,$$

$$\Delta^j(z_0^n, z_1^n, \ldots, z_n^n) = \prod_{k=0, k \ne j}^{n} |z_j^n - z_k^n| \quad \text{for } j = 0, 1, \ldots, n.$$

There exist $n+1$ points $y_0^n, y_1^n, \ldots, y_n^n$ in K such that

$$V(y_0^n, y_1^n, \ldots, y_n^n) = \max\{V(z_0^n, z_1^n, \ldots, z_n^n) : z_0^n, z_1^n, \ldots, z_n^n \in K\}$$

and

$$\Delta^0(y_0^n, y_1^n, \ldots, y_n^n) \le \min\{\Delta^j(y_0^n, y_1^n, \ldots, y_n^n) : j \in \{1, \ldots, n\}\}.$$

F. Leja proved (see e.g. [Lj 2], p. 258) that for each compact set K the limit of the sequence $[\Delta^0(y_0^n, y_1^n, \ldots, y_n^n)]^{1/n}$ is equal to the transfinite diameter of K, introduced by M. Fekete [Fe]:

$$d(K) = \lim_{n\to\infty} [\Delta^0(y_0^n, y_1^n, \ldots, y_n^n)]^{1/n}.$$

One can prove that for each compact subset of the complex plane the transfinite diameter is equal to the logarithmic capacity (see [La], Ch. II, Sec. 4).

Let the polynomials L_j^n be defined by

$$L_j^n(z) = \prod_{k=0, k\ne j}^{n} \frac{z - y_k^n}{y_j^n - y_k^n}.$$

They are called the *Lagrange extremal polynomials* corresponding to the nodes $y_0^n, y_1^n, \ldots, y_n^n$. Suppose that $d(K) > 0$. Then the limit

$$L_K(z) = \lim_{n\to\infty} |L_0^n(z)|^{1/n}$$

exists for $z \in \mathbb{C}$ (see [Lj 2], p. 261, 265, 267), and is called *Leja's extremal function* associated with K.

Denote by $D_\infty(K)$ the unbounded component of the complement $\widehat{\mathbb{C}} \setminus E$ of E. In 1933 F. Leja [Lj 1] proved the following

LEJA'S THEOREM 1.1. *If K is a compact subset of $\mathbb{C}$, regular in the sense of Dirichlet, then*

$$\ln L_K(z) = g(z, \infty) \quad \textit{for } z \in D_\infty(K),$$

where $g(\cdot, \infty)$ is Green's function of $D_\infty(K)$ with pole at ∞.

We recall that Green's function of a regular domain D with pole at $z_0 \in D$ is the unique function $g(\cdot, z_0) : D \setminus \{z_0\} \to (0, \infty)$ such that

(i) $g(\cdot, z_0)$ is harmonic in $D \setminus \{z_0\}$,
(ii) $g(z, z_0)$ tends to zero as z tends to the boundary of D,

(iii) $g(z, z_0)$ has a logarithmic pole at z_0, i.e.

$$\lim_{z\to z_0} \left[g(z, z_0) - \ln \frac{1}{|z - z_0|}\right] < \infty \quad \text{for } z_0 \ne \infty,$$

and

$$\lim_{z\to z_0} [g(z, z_0) - \ln|z|] < \infty \quad \text{for } z_0 = \infty.$$

It is known (see e.g. [La], Ch. IV, Sec. 2) that Green's function is symmetric in the sense that

$$g(z, z_0) = g(z_0, z) \tag{1.1}$$

for every z, z_0 in D.

2. Constructing a Cantor set. Given a sequence $(l_k)_{k=0,1,2,\ldots}$ such that for every k,

$$l_{k+1} < \tfrac{1}{2} l_k \quad \text{and} \quad l_0 = 1,$$

let $\{I_k\}_{k=0,1,2,\ldots}$ be a family of subsets of $[0;1]$ such that $I_0 = [0;1]$ and I_{k+1} is obtained from I_k by deleting the open concentric subinterval of length $l_k - 2l_{k+1}$ from each interval (component) of I_k. We call

$$E = \bigcap_{k=0}^{\infty} I_k$$

a *generalized Cantor set.*

Consider the sequence $(l_k)_{k=0,1,2,\ldots}$ such that

$$l_k = q^k, \quad \text{where } q \in (0; 1/2). \tag{2.1}$$

Then every set I_k consists of 2^k subintervals $I_{k,1}, \ldots, I_{k,2^k}$ of length q^k each. We denote by $Q_{k,n}$ the closed ball with center at the middle point of $I_{k,n}$ and radius $\frac{1}{2}(l_{k-1} - l_k)$ (we let $l_{-1} = q^{-1}$) and we set

$$B_k = \bigcup_{n=1}^{2^k} Q_{k,n} \quad \text{and} \quad A_k = B_k - B_{k+1}.$$

By Wiener's criterion (see e.g. [La], Th. 5.6 or [Ts], Th. III.62), the Cantor set E corresponding to the sequence (2.1) is regular in the sense of Dirichlet.

3. Harmonic measure of a Cantor set. If $k \in \mathbb{N}_0$ and $n \in \{1, \ldots, 2^k\}$, then the characteristic function $\chi_{I_{k,n}}$ of the interval $I_{k,n}$ is continuous on the Cantor set E. Since E is regular in the sense of Dirichlet, there exists a unique function h harmonic in $\widehat{\mathbb{C}} \setminus E$ and continuous in $\widehat{\mathbb{C}}$ such that

$$h(z) = \chi_{I_{k,n}}(z) \quad \text{for } z \in E.$$

We call this function h the *harmonic measure* of the set $I_{k,n} \cap E$. Let us introduce the following notations:

$$\omega(I_{k,n} \cap E, \cdot) = h(\cdot), \quad \omega(I_{k,n} \cap E) = h(\infty).$$

By the maximum principle for harmonic functions, h takes values only from $[0;1]$. Furthermore,

$$z \notin E \Rightarrow 0 < h(z) < 1 \tag{3.1}$$

for $k \neq 0$ (if $k = 0$ then $h \equiv 1$).

We shall need the following property of the harmonic measure. Let $k \in \mathbb{N}_0$, $l \in \{1, \ldots, 2^{k+1}\}$ and $n \in \{1, \ldots, 2^k\}$ be such that

$$I_{k+1,l} \cup I_{k+1,l+1} \subset I_{k,n}. \tag{3.2}$$

Then

$$\omega(I_{k+1,l} \cap E) + \omega(I_{k+1,l+1} \cap E) = \omega(I_{k,n} \cap E). \tag{3.3}$$

To see this, it is sufficient to notice that the function

$$\omega(I_{k+1,l} \cap E, \cdot) + \omega(I_{k+1,l+1} \cap E, \cdot)$$

is the harmonic measure of $I_{k,n} \cap E$ and so

$$\omega(I_{k+1,l} \cap E, z) + \omega(I_{k+1,l+1} \cap E, z) = \omega(I_{k,n} \cap E, z)$$

for all $z \in \widehat{\mathbb{C}}$, in particular for $z = \infty$.

4. Harmonic measure and Green's function. For k, l, n such that condition (3.2) holds, we set

$$A_{k,n} = Q_{k,n} \setminus (Q_{k+1,l} \cup Q_{k+1,l+1}).$$

We shall need the following

LEMMA 4.1 ([M-V], Lemma 3.4). *There exist positive constants c_1, c_2 such that for every $k \in \mathbb{N}_0$ and $n \in \{1, \ldots, 2^k\}$, and every $z \in A_{k,n}$, we have*

$$c_1 \omega(I_{k,n} \cap E) \le g(z, \infty) \le c_2 \omega(I_{k,n} \cap E), \tag{4.1}$$

where $g(\cdot, \infty)$ is Green's function of $\widehat{\mathbb{C}} \setminus E$ with pole at ∞.

Proof. Fix $k \in \mathbb{N}_0$ and $n \in \{1, \ldots, 2^k\}$. Let $\gamma_{k,n}$ be the circle with center at the middle point of $I_{k,n}$ and of radius

$$\frac{1}{3} q^k (1 - q)\left(1 + \frac{1}{q}\right).$$

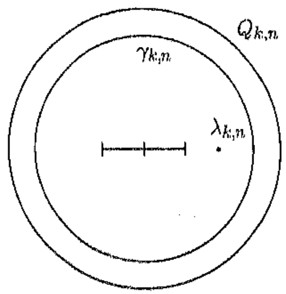

Fig. 1

Let $\lambda_{k,n}$ (see Fig. 1) lie on the diameter of $\gamma_{k,n}$ including $I_{k,n}$ and satisfy

$$\operatorname{dist}(\lambda_{k,n}, \gamma_{k,n}) = \frac{1}{3} q^k (1 - q)\left(\frac{1}{2q} - 1\right)$$

(there are two such points; we take the one which is closer to $(1, 0)$).

We now replace $\mathbb{C} \setminus E$ by the smaller domain $\operatorname{int} Q_{k,n} \setminus E$, which is also regular in the sense of Dirichlet (notice that regularity is a local property of the boundary and every continuum is regular). Let $\widetilde{\omega}_{k,n}(I_{k,n} \cap E, \cdot)$ be the unique harmonic function in $\operatorname{int} Q_{k,n} \setminus E$ continuous in $Q_{k,n}$ such that

$$\widetilde{\omega}_{k,n}(I_{k,n} \cap E, z) = \chi_{I_{k,n}}(z)$$

for all z on the boundary of $\operatorname{int} Q_{k,n} \setminus E$. Consider the function $\widetilde{\omega}_{0,1}(I_{0,1} \cap E, \cdot)$. By (3.1) there exists a positive constant d_1 such that for every $z \in \gamma_{0,1}$,

$$d_1 \le \widetilde{\omega}_{0,1}(I_{0,1} \cap E, z).$$

For each $k \in \mathbb{N}$ and $n \in \{1, \ldots, 2^k\}$ the function $\widetilde{\omega}_{k,n}(I_{k,n} \cap E, \cdot)$ can be recovered from $\widetilde{\omega}_{0,1}(I_{0,1} \cap E, \cdot)$ by a translation and homothety of ratio q^k, because the Dirichlet problem has a unique solution. The circle $\gamma_{0,1}$ can be obtained from $\gamma_{k,n}$ by the same translation and homothety. Hence for every k, n and $z \in \gamma_{k,n}$,

$$d_1 \le \widetilde{\omega}_{k,n}(I_{k,n} \cap E, z).$$

By the maximum principle, we obtain

$$\widetilde{\omega}_{k,n}(I_{k,n} \cap E, \cdot) \le \omega(I_{k,n} \cap E, \cdot)$$

for every $z \in Q_{k,n}$, in particular for $z \in \gamma_{k,n}$. Thus by (3.1) we have

$$d_1 \le \omega(I_{k,n} \cap E, z) < 1 \tag{4.2}$$

for each $k \in \mathbb{N}_0$, $n \in \{1, \ldots, 2^k\}$ and $z \in \gamma_{k,n}$. Analogously, there exists $d_2 > 0$ such that for each k, n and $z \in \gamma_{k,n}$,

$$d_2 \le g(z, \lambda_{k,n}). \tag{4.3}$$

Fix $k \in \mathbb{N}_0$ and $n \in \{1, \ldots, 2^k\}$. Let $\widehat{g}_{k,n}(\cdot, \alpha)$ be Green's function of $\widehat{\mathbb{C}} \setminus (I_{k,n} \cap E)$ with pole at α. Let $u_{k,n}$ be the composition of some translation and homothety such that $u_{k,n}(\widehat{\mathbb{C}} \setminus (I_{0,1} \cap E)) = \widehat{\mathbb{C}} \setminus (I_{k,n} \cap E)$. It is easily seen that

$$\widehat{g}_{k,n}(\lambda_{k,n}, \infty) = \widehat{g}_{k,n}(u_{k,n}(\lambda_{0,1}), \infty) = \widehat{g}_{0,1}(\lambda_{0,1}, \infty) \tag{4.4}$$

for all k and n. By the maximum principle we have

$$g(z, \lambda_{k,n}) \leq \widehat{g}_{k,n}(z, \lambda_{k,n})$$

for each k, n and $z \in \mathbb{C}$. Hence by (4.4) and (1.1), the estimate

$$g(z, \lambda_{k,n}) \leq M\,, \tag{4.5}$$

with some positive constant M independent of k, n and $z \in \gamma_{k,n}$, will be proved if we show that there is a $b > 0$ such that for every $k \in \mathbb{N}_0$, $n \in \{1, \ldots, 2^k\}$ and $z \in \gamma_{k,n}$,

$$\widehat{g}_{k,n}(z, \lambda_{k,n}) \leq b\widehat{g}_{k,n}(\infty, \lambda_{k,n})\,.$$

Let $a_{k,n}$ be the middle point of the interval $I_{k,n}$ (see Fig. 2).

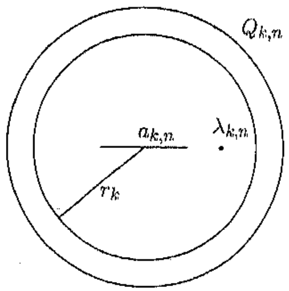

Fig. 2

Let $D_{k,n}$ denote the complement in $\widehat{\mathbb{C}}$ of $B(a_{k,n}, r_k)$ (the closed ball with center $a_{k,n}$ and radius r_k), where

$$r_k = \frac{1}{4}q^k(1-q)\left(\frac{1}{q} + 2\right).$$

We consider the function

$$h_{k,n}(z) = \frac{r_k}{z} + a_{k,n}\,.$$

It is easily seen that

$$h_{k,n}(\operatorname{int} B(0,1)) = D_{k,n}.$$

The composition of $h_{k,n}$ and $\widehat{g}_{k,n}(\cdot, \lambda_{k,n})$ defined on $\operatorname{int} B(0,1)$ is harmonic and takes only positive values. Thus, by Harnack's inequality (see e.g. [H-K], Th. 1.18),

$$\widehat{g}_{k,n}(h_{k,n}(w), \lambda_{k,n}) \leq \frac{1+r}{1-r}\widehat{g}_{k,n}(h_{k,n}(0), \lambda_{k,n})$$

where $|w| = r = (3q^{-1} + 6)/(4q^{-1} + 4)$. Observe that the image under $h_{k,n}$ of the circle with center at 0 and radius r is the circle $\gamma_{k,n}$. Hence if $z \in \gamma_{k,n}$, we have

$$\widehat{g}_{k,n}(z, \lambda_{k,n}) \leq \frac{1+r}{1-r}\widehat{g}_{k,n}(\infty, \lambda_{k,n})\,,$$

which gives inequality (4.5).

It follows from (4.2), (4.3) and (4.5) that there exist positive constants b_1, b_2 independent of k, n and $z \in \gamma_{k,n}$ such that

$$b_1\omega(I_{k,n} \cap E, z) \leq g(z, \lambda_{k,n}) \leq b_2\omega(I_{k,n} \cap E, z)\,.$$

By the maximum principle, these inequalities also hold for $z = \infty$, i.e.

$$b_1\omega(I_{k,n} \cap E) \leq g(\infty, \lambda_{k,n}) \leq b_2\omega(I_{k,n} \cap E)\,. \tag{4.6}$$

Now we consider the set $A_{0,1}$. By Harnack's inequality we see that for every $w \in A_{0,1}$,

$$m_1 u(\lambda_{0,1}) \leq u(w) \leq m_2 u(\lambda_{0,1})$$

for every harmonic positive function u defined in some open neighborhood of $A_{0,1}$. Let $u_{k,n}$ be the composition of some translation and homothety such that $u_{k,n}(A_{0,1}) = A_{k,n}$. Then

$$m_1 g(u_{k,n}(\lambda_{0,1}), \infty) \leq g(u_{k,n}(w), \infty) \leq m_2 g(u_{k,n}(\lambda_{0,1}), \infty)$$

for every k, n and $w \in A_{0,1}$, because the composition of $u_{k,n}$ and $g(\cdot, \infty)$ is harmonic. Thus

$$m_1 g(\lambda_{k,n}, \infty) \leq g(z, \infty) \leq m_2 g(\lambda_{k,n}, \infty)\,.$$

Hence by (4.6) and (1.1), we obtain inequality (4.1) and the proof of Lemma 4.1 is complete.

COROLLARY 4.2. *There exist positive constants C_1 and C_2 such that for every k and n, if $A_{k,n} \cap A_{k,n+1} \neq \emptyset$ then*

$$C_1\omega(I_{k,n} \cap E) \leq \omega(I_{k,n+1} \cap E) \leq C_2\omega(I_{k,n} \cap E)\,. \tag{4.7}$$

Proof. By Lemma 4.1 applied to $z \in A_{k,n} \cap A_{k,n+1}$ we get

$$c_1\omega(I_{k,n} \cap E) \leq g(z, \infty) \leq c_2\omega(I_{k,n} \cap E)$$

and

$$c_1\omega(I_{k,n+1} \cap E) \leq g(z, \infty) \leq c_2\omega(I_{k,n+1} \cap E)\,,$$

which gives (4.7).

5. (HCP) property of a Cantor set. The main result of this paper is the following

PROPOSITION 5.1. *The Cantor set E constructed in Section 2 has the (HCP) property, i.e. there exist positive constants M and m such that for every $\delta \in (0;1]$ we have*

$$L_E(z) \le 1 + M\delta^m \quad \text{if } \operatorname{dist}(z,E) \le \delta \le 1 .$$

Proof. By Theorem 1.1, it is sufficient to prove that

$$g(z,\infty) \le M_1\delta^m \quad \text{if } \operatorname{dist}(z,E) = \delta \le 1 , \tag{5.1}$$

where M_1 and m are positive constants independent of δ and z.

Fix $z \in \mathbb{C}$ such that $\operatorname{dist}(z,E) = \delta > 0$. If $z \notin B_0$, then $\delta > \frac{1}{2}q^{-1} - 1$ and we obtain inequality (5.1) with

$$M_1 = \frac{2q}{1-2q}\max\left\{g(w,\infty) : \frac{1}{2q} - 1 < \operatorname{dist}(w,E) \le 1\right\}, \qquad m = 1 .$$

If $z \in B_0$, then there is exactly one $k \in \mathbb{N}_0$ such that $z \in A_k$. Hence

$$\delta = \operatorname{dist}(z,E) \ge \operatorname{dist}(A_k,E) = \tfrac{1}{2}q^k - q^{k+1} ,$$

whence

$$k \ge \frac{\ln\delta - \ln(1/2-q)}{\ln q} .$$

By (4.7), there exists a positive constant c such that for every k, l and n, if $l \ne n$ and $A_{k,n} \cap A_{k,l} \ne \emptyset$ then

$$c\omega(I_{k,n} \cap E) \le \omega(I_{k,l} \cap E) . \tag{5.2}$$

Choose $n \in \{1,\ldots,2^k\}$ such that $z \in A_{k,n}$. Let l and n_1 be positive integers such that $l \ne n$ and $I_{k,n} \cup I_{k,l} \subset I_{k-1,n_1}$. Then, by (5.2) and (3.3), we have

$$\omega(I_{k,n} \cap E) + c\omega(I_{k,n} \cap E) \le \omega(I_{k,n} \cap E) + \omega(I_{k,l} \cap E) = \omega(I_{k-1,n_1} \cap E) .$$

Hence

$$\omega(I_{k,n} \cap E) \le \frac{1}{1+c}\omega(I_{k-1,n_1} \cap E) .$$

Analogously, for some n_2,

$$\omega(I_{k-1,n_1} \cap E) \le \frac{1}{1+c}\omega(I_{k-2,n_2} \cap E) .$$

Consequently, by Lemma 4.1 we get

$$g(z,\infty) \le c_2\omega(I_{k,n} \cap E) \le c_2\frac{1}{1+c}\omega(I_{k-1,n_1} \cap E) \le \ldots$$
$$\le c_2\frac{1}{(1+c)^k}\omega(I_{0,1} \cap E) = c_2\frac{1}{(1+c)^k}\omega(E)$$

$$= c_2\frac{1}{(1+c)^k} \le c_2\frac{1}{(1+c)^{\frac{\ln\delta - \ln(1/2-q)}{\ln q}}} = M_1\delta^m ,$$

with

$$M_1 = c_2(1+c)^{\frac{\ln(1/2-q)}{\ln q}} \quad \text{and} \quad m = -\frac{\ln(1+c)}{\ln q} .$$

The proof of Theorem 5.1 is complete.

Repeating the previous proof one can get the (HCP) property for a Cantor set in $\mathbb{C}^N$.

6. Generalizations. The obtained result can be generalized to higher dimensions. Then Leja's extremal function is replaced with *Siciak's extremal function* defined for a compact subset K of $\mathbb{C}^N$ by

$$\Phi_K(z) = \sup\{|p(z)|^{1/\deg p} : p \text{ is a nonconstant polynomial such that } \sup_K |p| \le 1\} .$$

If $N = 1$ then the two extremal functions are equal (see [Si 1]). Siciak proved (see [Si 1], Sec. 8, Th. 1) that if $K_1,\ldots,K_n$ are compact subsets of $\mathbb{C}$ then

$$\Phi_{K_1\times\ldots\times K_n}(z_1,\ldots,z_n) = \max\{\Phi_{K_1}(z_1),\ldots,\Phi_{K_n}(z_n)\}$$

for $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$. This implies that if $K_1,\ldots,K_n$ have the (HCP) property, then so does the set $K_1 \times \ldots \times K_n$.

Pleśniak proved [Pl 3] that the (HCP) property is invariant under regular holomorphic mappings, viz.

If K is a polynomially convex (HCP) compact subset of $\mathbb{C}^n$ and $h = (h_1,\ldots,h_k)$ is an analytic mapping defined in an open neighborhood of K, with values in $\mathbb{C}^k$ ($k \le n$), such that for some permutation σ on k letters we have

$$\det\left[\frac{\partial h_i}{\partial z_{\sigma(j)}}(z)\right]_{i,j=1,\ldots,k} \ne 0$$

for each $z \in K$ (the differential is onto), then $h(E)$ is an (HCP) subset of $\mathbb{C}^k$.

Using these two results and starting from the Cantor set constructed in Section 2 one can provide various examples of sets in $\mathbb{C}^k$ with the (HCP) property and consequently, preserving an analogue of Markov's inequality (M).

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Representations of bimeasures

by

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Abstract. Separately σ-additive and separately finitely additive complex functions on the Cartesian product of two algebras of sets are represented in terms of spectral measures and their finitely additive counterparts. Applications of the techniques include a bounded joint convergence theorem for bimeasure integration, characterizations of positive-definite bimeasures, and a theorem on decomposing a bimeasure into a linear combination of positive-definite ones.

1. Introduction and notation. Throughout this paper, S_i is a nonempty set and Σ_i an algebra (field) of subsets of S_i for $i = 1, 2$. Unless specified otherwise, $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ is an arbitrary bounded separately (finitely) additive function. In case β is separately σ-additive (i.e., $\beta(X, \cdot)$ and $\beta(\cdot, Y)$ are countably additive for all $X \in \Sigma_1$, $Y \in \Sigma_2$), β will be called a (complex) *bimeasure*. For the basic theory of bimeasures defined on products of σ-algebras we refer to [1] and [13]. The C^*-algebra theory we need may be found e.g. in [12].

All vector spaces will be complex. For any Hilbert space H, $(\cdot \mid \cdot)$ or $(\cdot \mid \cdot)_H$ denotes its inner product, and $L(H)$ the space of bounded linear operators on H.

Our main results depend on the Grothendieck inequality, "the fundamental theorem in the metric theory of tensor products" of Grothendieck [7]: For the spaces $C(\Omega_i)$ of continuous complex functions on compact Hausdorff spaces Ω_i, $i = 1, 2$, and any bounded bilinear form $B : C(\Omega_1) \times C(\Omega_2) \to \mathbb{C}$ there are positive linear forms $\phi : C(\Omega_1) \to \mathbb{C}$ and $\psi : C(\Omega_2) \to \mathbb{C}$ such that

$$|B(f,g)|^2 \le \phi(|f|^2)\psi(|g|^2)$$

for all $f \in C(\Omega_1)$, $g \in C(\Omega_2)$. (We do not normalize ϕ and ψ, and do not display the Grothendieck constant.) As noted by several authors (see e.g. [5], [6], [10]), Grothendieck's theorem implies that B can be expressed in terms of Hilbert space representations of the commutative C^*-algebras $C(\Omega_i)$: There

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