Pointwise estimates for densities of stable semigroups of measures

by

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Abstract. Let \( \{ \mu_t \} \) be a symmetric \( \alpha \)-stable semigroup of probability measures on a homogeneous group \( N \), where \( 0 < \alpha < 2 \). Assume that \( \mu_t \) are absolutely continuous with respect to Haar measure and denote by \( h_t \) the corresponding densities. We show that the estimate

\[
h_t(x) \leq t \Omega(|x|) |x|^{-\alpha}, \quad x \neq 0,
\]

holds true with some integrable function \( \Omega \) on the unit sphere \( S \) if and only if the density of the Lévy measure of the semigroup belongs locally to the Zygmund class \( \mathcal{L} \log L(N \setminus \{0\}) \).

The problem turns out to be related to the properties of the maximal function

\[
\mathcal{M}f(x) = \sup_{r>0} \frac{1}{r} \int_0^r h_{t-s} * f * h_s(x) \, ds
\]

which, as is proved here, is of weak type \((1,1)\).

Introduction. Let \( \{ \mu_t \} \) be a symmetric \( \alpha \)-stable semigroup of probability measures on \( \mathbb{R}^n \), where \( 0 < \alpha < 2 \). (Note that the Gaussian semigroups are excluded.) Assume that \( \mu_t \) are absolutely continuous with respect to Lebesgue measure and denote by \( h_t \) the corresponding densities. It is known that the \( h_t \) are automatically continuous functions on \( \mathbb{R}^n \) and vanish at infinity (cf. Glowacki [9]).

The question considered here is that of the rate of decay of the densities at infinity. Let us remark that in the ideal case when the Lévy measure is smooth away from the origin we have

\[
\lim_{r \to \infty} r^{n+\alpha} h_t(r\bar{x}) = m(\bar{x}),
\]

for \( \bar{x} \in \Sigma = \{ y \in \mathbb{R}^n : |y| = 1 \} \), where \( m \) is the density of the Lévy measure (cf. Dziubański [8]). It is easily seen that, by homogeneity, (1) implies

\[
h_t(x) \leq C m(\bar{x}) |x|^{-n-\alpha}, \quad x \neq 0, \ t > 0,
\]

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where $\hat{z}$ denotes the unit vector $x/|x|$ and $C$ is a positive constant.

However, as simple examples show, the densities $h_t$ may very well exist even when the Lévy measure is singular. In addition, they always satisfy the estimate

$$h_t(x) \leq t^{\beta}|x|^{-n-\beta}, \quad x \neq 0, \quad t > 0,$$

for any $0 < \beta < \alpha$, where $\Omega_\beta$ is a function on the unit sphere satisfying an integral Lipschitz condition (cf. Glovacki [9]). Therefore, the right question to ask seems to be: How is the estimate

$$h_t(x) \leq t^{\beta}|x|^{-n-\alpha}, \quad x \neq 0, \quad t > 0,$$

with $\Omega$ enjoying certain regularity properties related to the regularity of the Lévy measure itself? Note that in (4) the exponent assumes the critical value $\alpha$.

In this paper we consider this question in the general context of a nilpotent homogeneous group $\mathcal{N}$ of which $\mathbb{R}^n$ with the usual dilations is the simplest example. We show that for every symmetric $\alpha$-stable semigroup $\mu_t$ with densities $h_t$ the estimate (4) holds true with some $\Omega \in L^1(\Sigma)$ if and only if the density $m$ of the Lévy measure locally belongs to the Zygmund class $L^p(\mathcal{N} \setminus \{e\})$. If $1 < p \leq \infty$ and $\Omega \in L^p(\mathcal{N} \setminus \{e\})$, this is so if and only if $m \in L^p(\mathcal{N} \setminus \{e\})$. (In the above formulas $n$ is understood to be the homogeneous dimension of the group which in most cases is different from the topological dimension.) We also prove that if the Lévy measure is absolutely continuous relative to Haar measure, then (1) holds for a.e. $\hat{z}$ on the unit sphere $\Sigma$ with $m$ being the density of the Lévy measure restricted to $\Sigma$.

Our method is based on a perturbation formula. The problem turns out to be related to the properties of the maximal function

$$Mf(x) = \sup_{t > 0} \frac{1}{t} \int_0^t h_{t-s} * f * h_s(x) \, ds,$$

which, as is easily seen, coincides with the usual maximal function associated with the semigroup provided $\mathcal{N}$ is commutative. It is proved here that $M$ is of weak type $(1,1)$ in the general case.

1. Preliminaries and notation. Let $\mathcal{N}$ be a homogeneous Lie group endowed with a family of dilations $\{\delta_t\}$ and a homogeneous norm $x \mapsto |x|$ which is subadditive, that is, satisfies

$$|xy| \leq |x| + |y|$$

for $x, y \in \mathcal{N}$. Such norms always exist, as is shown by Hebisch and Sikora [12]. Let $dx$ denote Haar measure on $\mathcal{N}$ and $Q$ the homogeneous dimension of $\mathcal{N}$. Let $B(r) = \{x \in \mathcal{N} : |x| < r\}$ denote the ball of radius $r > 0$ and let $\Sigma = \{x \in \mathcal{N} : |x| = 1\}$ be the unit sphere relative to the homogeneous norm. For $x \in \mathcal{N} \setminus \{e\}$, let

$$\hat{z} = \delta_{|x|-1}x.$$

There exists a unique Radon measure $d\hat{z}$ on $\Sigma$ such that for all continuous functions $f$ on $\mathcal{N}$ with compact support

$$\int_N f(x) \, dx = \int_0^\infty t^{Q-1} \int_{\Sigma} f(\delta_t \hat{z}) \, d\hat{z} \, dt.$$

We denote by $C_0(\mathcal{N})$ the space of compactly supported $C_0$ functions on $\mathcal{N}$.

Let $\{\mu_t\}$ be a continuous semigroup of probability measures on $\mathcal{N}$. It is assumed throughout the paper that $\{\mu_t\}$ is symmetric and $\alpha$-stable with exponent $0 < \alpha < 2$, that is,

$$f(\mu_t) = (f \circ \delta_{t^{1/\alpha}}, \mu_t), \quad t > 0.$$

It is also assumed that the measures $\mu_t$ have densities $h_t$ relative to Haar measure $dx$ on $\mathcal{N}$. The densities $h_t$ are automatically square-integrable, hence continuous and bounded (cf. Glovacki [9]). In terms of the densities, (1.1) is equivalent to

$$h_t(x) = t^{-Q/\alpha} h_{t^{-1/\alpha}}(x), \quad x \in \mathcal{N}, \quad t > 0,$$

where $h = h_1$. It is well known that the generating functional $P$ of such a semigroup is a symmetric real random on $\mathcal{N}$ homogeneous of degree $-Q - \alpha$. $P$ is also dissipative, that is to say,

$$\langle f, P \rangle \leq 0$$

for every real $f \in C_0(\mathcal{N})$ which takes on its largest value at the identity $e$. Any real dissipative functional is a generating functional for some (uniquely determined) continuous semigroup of measures.

Since $P$ is dissipative, it coincides on $\mathcal{N} \setminus \{e\}$ with a Radon measure $\nu$. This measure is positive and bounded outside any neighbourhood of the identity. We shall refer to $\nu$ as the Lévy measure of the semigroup $\{\mu_t\}$. We have

$$\langle f, \nu \rangle = \lim_{t \to 0} \frac{1}{t} \langle f, h_t \rangle, \quad f \in C_0(\mathcal{N} \setminus \{e\}).$$

If the Lévy measure $\nu$ has a density $k$ relative to Haar measure, then, by homogeneity, there exists a function $N \in L^1(\Sigma)$ such that

$$k(x) = N(\hat{z})|x|^{-Q-\alpha} \quad \text{for a.e. } x \in \mathcal{N} \setminus \{e\}.$$

Given a symmetric and bounded neighbourhood $W$ of the identity, let $\psi = \psi_W$ be a nonnegative smooth function supported in $W$ and equal to $1$
in a neighbourhood of \( e \). Then the distribution \( \tilde{P} = \psi P \) supported in \( W \) is dissipative as well. We also have

\[
P = \tilde{P} + v_0,
\]

where \( v_0 \) is a bounded positive measure equal to \( \nu \) outside \( W \). We choose and fix once for all a neighbourhood \( W \) such that \( W^{2q} \subseteq B(1) \), where \( q \geq Q/\alpha + 1 \) is a fixed integer.

Denote by \( \{ \tilde{\mu}_t \} \) the continuous semigroup of measures generated by \( \tilde{P} \). It is easy to see that \( \tilde{\mu}_t \geq 0 \) and \( \tilde{\mu}_t(N) \leq 1 \). In addition, we have the following perturbation formula:

\[
\mu_t = \tilde{\mu}_t + \int_0^t \mu_{t-s} \ast v_0 \ast \mu_s \, ds, \quad t > 0,
\]

where the integral is understood to be convergent in the weak sense (see, e.g., Kato [15]). As is easily seen, (1.3) implies \( \tilde{\mu}_t \leq \mu_t \) so that the \( \tilde{\mu}_t \) are also absolutely continuous. Their densities will be denoted by \( \tilde{h}_t \). By homogeneity and the semigroup property,

\[
\|\tilde{h}_t\|_\infty \leq \|h_t\|_\infty \leq \|h_{t/2}\|_2^2 = C t^{-Q/\alpha}, \quad t > 0,
\]

where

\[
\|f\|_\infty = \sup_{x \in N} |f(x)| \quad \text{and} \quad C = 2^Q \|h\|_2^2 = 2^Q \int_N h(x)^2 \, dx.
\]

(1.5) LEMMA. For every positive integer \( j \), there exists a constant \( C_j > 0 \) such that

\[
\tilde{\mu}_t(N \setminus W^j) \leq C_j t^j, \quad t > 0.
\]

Proof. This is a direct application of Dufio [4]. \( \blacksquare \)

(1.6) COROLLARY. There exists a constant \( M \) such that

\[
\sup_{|x| \geq 1} \tilde{h}_t(x) \leq Mt, \quad t > 0.
\]

Proof. If \( |x| \geq 1 \), then

\[
\tilde{h}_t(x) = \int_N \tilde{h}_{t/2}(y^{-1}x) \tilde{h}_{t/2}(y) \, dy
\]

\[
\leq \int \tilde{h}_{t/2}(y^{-1}x) \tilde{h}_{t/2}(y) \, dy + \int \tilde{h}_{t/2}(y^{-1}x) \tilde{h}_{t/2}(y) \, dy
\]

\[
\leq 2C_q(t/2)^{-Q/\alpha} \leq Mt,
\]

where we have made use of Lemma (1.5) and (1.4). \( \blacksquare \)

We conclude this section with

(1.7) PROPOSITION. There exists a constant \( A \) such that

\[
|h(x) - h(e)| \leq A|x|^{\alpha}, \quad x \in N.
\]

Moreover, there exists \( \Omega_0 \in L^1(\Sigma) \) such that

\[
h(x) \leq \Omega_0(x)|x|^{-Q}, \quad x \in N \setminus \{e\}.
\]

Proof. This has been proved in Glowacki [9]. \( \blacksquare \)

For a general account of homogeneous groups, see Folland and Stein [7]. As for an introduction to the theory of continuous semigroups of measures on Lie groups, we recommend Hulanicki [12] and Dufio [4]. See also Pazy [16].

2. Statement of the theorem. Let \( \mathcal{M} \) be a measure space. As usual, we denote by \( L^P(\mathcal{M}) \) the totality of measurable functions \( f \) on \( \mathcal{M} \) such that \( |f|^p \) is integrable. The Zygmund class \( L^\log L(\mathcal{M}) \) consists of all measurable functions \( f \) such that

\[
\int |f| (1 + \log^+ |f|) < \infty.
\]

It is a proper subspace of \( L^1(\mathcal{M}) \).

The following theorem collects together the results of Proposition (3.1), Theorem (3.4), Theorem (5.1), and Theorem (7.6) below.

(2.1) THEOREM. The estimate

\[
h(x) \leq \Omega(x)|x|^{-Q-\alpha} \quad \text{for a.e.} \quad x \in N \setminus \{e\}
\]

with \( \Omega \in L^P(\Sigma) \) holds true if and only if the Lévy measure \( \nu \) is absolutely continuous with respect to Haar measure on \( N \setminus \{e\} \) and the corresponding density belongs locally to \( L^p(N \setminus \{e\}) \) if \( 1 < p \leq \infty \) or to \( L^1(N \setminus \{e\}) \) if \( p = 1 \).

As a by-product, we get information on the a.e. pointwise asymptotic behaviour of the densities (cf. Dziubański [5], formula (4)).

(2.2) THEOREM. If the Lévy measure \( \nu \) is absolutely continuous with respect to Haar measure on \( N \setminus \{e\} \), then

\[
\lim_{t \to \infty} t^{Q+\alpha} h(t^\alpha \tilde{x}) = N(\tilde{x}) \quad \text{for a.e.} \quad \tilde{x} \in \Sigma,
\]

where \( N \in L^1(\Sigma) \) and \( N(\tilde{x})/|x|^{Q+\alpha} \) is the density of \( \nu \).

Proof. This follows by homogeneity from Corollary (4.5) below. \( \blacksquare \)
3. The case \( p > 1 \). In this section a complete account is given of the easier case when \( p > 1 \).

(3.1) Proposition. Let \( 1 \leq p \leq \infty \). If there exists \( \Omega \in L^p(\Sigma) \) such that

\[
h(x) \leq \Omega(\bar{x})|x|^{-Q-\alpha} \quad \text{for a.e.} \ x \in \mathcal{N} \setminus \{e\},
\]

then the Lévy measure \( \nu \) of \( \{h_t\} \) is absolutely continuous with respect to Haar measure on \( \mathcal{N} \setminus \{e\} \) with a density in \( L^p_{\text{loc}}(\mathcal{N} \setminus \{e\}) \).

Proof. We have

\[
\langle f, \nu \rangle = \lim_{t \to 0} \frac{1}{t} \langle f, h_t \rangle
\]

for \( f \in C_c^\infty(\mathcal{N} \setminus \{e\}) \). By hypothesis,

\[
h_t(x) \leq t\Omega(\bar{x})|x|^{-Q-\alpha}, \quad t > 0,
\]

for a.e. \( x \in \mathcal{N} \setminus \{e\} \), whence

\[
\langle f, \nu \rangle \leq \int |f(x)|\Omega(\bar{x})|x|^{-Q-\alpha} \, dx,
\]

which immediately implies our claim. \( \blacksquare \)

Let us define

\[
M_R f(x) = \sup_{t > 0} |f \ast h_t(x)| \quad \text{and} \quad M_L f(x) = \sup_{t > 0} |h_t \ast f(x)|
\]

for \( f \in C_c^\infty(\mathcal{N}) \) and \( x \in \mathcal{N} \). Recall from Glowacki [9] that both \( M_R \) and \( M_L \) are of type \( (p, p) \) for every \( 1 < p \leq \infty \). Therefore, the maximal function

\[
Mf(x) = \sup_{t > 0} \frac{1}{t} \left| \int_0^t h_{t-s} \ast f \ast h_s(x) \, ds \right|
\]

is also of type \( (p, p) \) as

\[
Mf(x) \leq M_L(M_R f)(x), \quad f \in C_c^\infty(\mathcal{N}), \quad x \in \mathcal{N}.
\]

(3.4) Theorem. Let \( 1 < p \leq \infty \). If the Lévy measure \( \nu \) is absolutely continuous and its density belongs locally to \( L^p(\mathcal{N} \setminus \{e\}) \), then there exists an \( \Omega \in L^p(\Sigma) \) such that

\[
h(x) \leq \Omega(\bar{x})|x|^{-Q-\alpha} \quad \text{for a.e.} \ x \in \mathcal{N}.
\]

Proof. By homogeneity, it is sufficient to show that for some \( r > 0 \) there exists \( \Omega \in L^p(\Sigma) \) such that

\[
h_t(r\bar{z}) \leq r\Omega(z) \quad \text{for a.e.} \ z \in \Sigma \text{ and } t > 0.
\]

To this end, note that, by hypothesis, the measure \( \nu_0 \) in (1.2) has a density \( k_0 \in L^p(\mathcal{N}) \). By the perturbation formula (1.3),

\[
h_t(x) \leq \hat{h}_t(x) + \int_0^t h_{t-s} \ast k_0 \ast h_s(x) \, ds
\]

\[
\leq \hat{h}_t(x) + tMk_0(x).
\]

Therefore, by Corollary (1.6),

\[
h_t(r\bar{z}) \leq t(M + Mk_0(\bar{z}, z)), \quad r > 1, \quad |z| = 1.
\]

Since \( \mathcal{M} \) is of type \( (p, p) \), \( Mk_0 \in L^p(\mathcal{N}) \) and so, by Fubini's theorem, the function \( \bar{z} \to Mk_0(\bar{z}, \bar{z}) \) is in \( L^p(\Sigma) \) for a.e. \( r \geq 1 \). Pick one such \( r \) and let

\[
\Omega(\bar{x}) = \tilde{M} + Mk_0(\bar{z}, \bar{z}) \quad \text{as } \bar{z} \in \Sigma.
\]

Obviously, \( \Omega \) satisfies (3.5). \( \blacksquare \)

4. A maximal theorem. The Zo norm of an \( L^1 \) function \( \varphi \) on a homogeneous group \( \mathcal{N} \) is defined by

\[
\|\varphi\|_0 = \sup_{x \in \mathcal{N}} \int_{|\tilde{x}| > 2|\tilde{x}|} \sup_{|\tilde{y}| > 0} |\delta_0 \varphi(x \tilde{y}) - \delta_0 \varphi(y)| \, dy,
\]

where, by definition,

\[
\delta_0 \varphi(x) = t^{-Q} \varphi(t^{-1} x), \quad x \in \mathcal{N}, \quad t > 0.
\]

The following result is well known as Zo's lemma (cf. Zo [18] as well as Stein [17], pp. 71-73).

(4.1) Lemma. Let \( k \in L^1(\mathcal{N}) \). If \( \|k\|_0 < \infty \), then the maximal operator \( K \) defined by

\[
Kf(x) = \sup_{t > 0} |f \ast h_t(x)|
\]

is of weak type \((1, 1)\).

The main result of this section is

(4.2) Proposition. The maximal operator \( M \) as defined by (3.2) is of weak type \((1, 1)\).

Proof. By a simple change of variable, for \( 0 \leq f \in L^1(\mathcal{N}) \),

\[
Mf = \sup_{t > 0} \int_0^1 h_{t\theta} \ast f \ast h_{t(1-\theta)} \, d\theta \leq M_0 f + M_{\infty} f,
\]
where
\[
M_0 f = \sup_{t > 0} \int_0^{1/2} h_{(1-\theta)t} * f * h_{\theta t} \, d\theta,
\]
\[
M_\infty f = \sup_{t > 0} \int_0^{1/2} h_{\theta t} * f * h_{(1-\theta)t} \, d\theta.
\]

We are going to show that \(M_\infty\) is of weak type \((1,1)\). The proof for \(M_0\) is entirely “symmetric”. To this end, let \(G = \mathcal{N} \times \mathcal{N}\) be the direct product of \(\mathcal{N}\) by itself. \(G\) acts on \(\mathcal{N}\) by
\[
(x,y)z = x^{-1}zy, \quad (x,y) \in G, \quad z \in \mathcal{N},
\]
and \(M_\infty\) is obtained by transference by means of this action from the maximal function on \(G\)
\[
MF(x,y) = \sup_{t > 0} F * H_t, \quad 0 \leq F \in L^1(G),
\]
where
\[
H_t(x,y) = \int_0^{1/2} h_{\theta t}(x)h_{(1-\theta)t}(y) \, d\theta
\]
is an integrable function on \(G\) for every \(t > 0\). By the transference principle (Emerson [6], see also Calderón [1] as well as Coifman and Weiss [2], [3]), it is therefore sufficient to show that \(M\) is of weak type \((1,1)\). Let us make \(G\) into a homogeneous group by introducing the product dilations
\[
A_t(x,y) = (\delta tx, \delta ty),
\]
and the homogeneous norm
\[
[(x,y)] = [x] + [y]
\]
for \((x,y) \in G\) and \(t > 0\). Then
\[
H_t = A_t(A_t(H_0)), \quad t > 0,
\]
and, by Zo’s lemma (Lemma (4.1)), it is enough to show that the Zo norm of \(H_t \in L^1(G)\) is finite.

We have
\[
\|H_t\|_{L^1} \leq \int_0^{1/2} \|h_\theta \otimes h_{1-\theta}\|_{L^1} d\theta = \|H_t^\theta\|_{L^1} d\theta,
\]
where
\[
H_t^\theta = h_t \otimes h_{(1-\theta)t}, \quad 0 < \theta \leq 1/2, \quad t > 0.
\]
The last equality is due to the invariance of the Zo norm under dilations.

It is clear that for each \(\theta \in (0,1/2), \{H_t^\theta\}\) is an \(\alpha\)-stable symmetric semigroup of measures on the homogeneous group \(G\) with \(P_\theta = P \otimes \delta + \delta \otimes (1/\theta - 1)P\) being its infinitesimal generator. It has been shown in Glowacki [9] that for every symmetric \(\alpha\)-stable semigroup of measures with absolutely continuous densities \(h_t\), the Zo norm of \(h_t\) is finite. Therefore, in particular,
\[
\|H_t^{1/2}\|_{L^1} = C_t < \infty.
\]

For \(X \in \mathcal{G}\) put
\[
|X|_\theta = |\varrho_\theta X|,
\]
where \(\varrho_\theta(x,y) = (x, \delta t^{-1}y), \) and \(\vartheta = (1/\theta - 1)^{1/\alpha}\), so that
\[
H_t^\theta(X) = \vartheta \cdot H_t^{1/2}(\varrho_\theta X),
\]
and, by our choice of the homogeneous norm (4.3),
\[
\vartheta^{-1}|X| \leq |X|_\theta \leq |X|, \quad X \in \mathcal{G}.
\]

For fixed \(X \in \mathcal{G}\) and \(2^{1/\alpha} \leq \vartheta < \infty\) let
\[
I(X) = \int_{|Y| \geq 2|X|} \sup_{t>0} |H_t^\theta(XY) - H_t^\theta(Y)| \, dY.
\]
Then
\[
I(X) = \int_{2\vartheta|X| \geq |Y| \geq 2|X|} \sup_{t>0} |H_t^\theta(XY) - H_t^\theta(Y)| \, dY
\]
\[
+ \int_{|Y| \geq 2|X|} \sup_{t>0} |H_t^\theta(XY) - H_t^\theta(Y)| \, dY
\]
\[
\leq 2 \int_{4\vartheta^2|X| \geq |Y| \geq |X|} \sup_{t>0} H_t^\theta(Y) \, dY
\]
\[
+ \int_{|Y| \geq 2\vartheta|X|} \sup_{t>0} |H_t^\theta(XY) - H_t^\theta(Y)| \, dY
\]
\[
\leq 2 \int_{4\vartheta^2|X| \geq |Y||X|} \sup_{t>0} H_t^\theta(Y) \, dY
\]
\[
+ \int_{|Y| \geq 2|X|} \sup_{t>0} |H_t^\theta(XY) - H_t^\theta(Y)| \, dY.
\]

Let us remark that we have taken advantage of the fact that the homogeneous norm on \(\mathcal{N}\), and consequently that on \(G\), is subadditive. By the change of variable \(V = \varrho_\theta Y\),
\[ I(\varphi_{\theta^{-1}}X) \leq 2 \int \sup_{|V| \geq \delta^{-1}|X|} H^{1/2}_t(V) dV \]

\[ + \int \left| V \right| \sup_{|V| \geq 2|X|} \left| H^{1/2}_t(XV) - H^{1/2}_t(V) \right| dV , \]

where \( \delta > 0 \).

The second term is of course bounded by \( C_1 \left\| H^{1/2}_t \right\|_{L_2} \), By Proposition (1.7) and homogeneity,

\[ \left| H^{1/2}_t(V) \right| \leq \Omega_0 \left( \frac{|V|}{|X|} \right)^{-2Q}, \quad V \neq (e, e), t > 0, \]

for some \( \Omega_0 \) integrable on the unit sphere in \( \mathbb{G} \), where \( 2Q \) is the homogeneous dimension of \( \mathbb{G} \). Therefore the first term of (4.4) is less than or equal to

\[ \int \Omega_0(\frac{|V|}{|X|}) \left| V \right|^{-2Q} dV \leq C_2 \left| \log |X| \right| \]

which implies that

\[ \left\| H^{1/2}_t \right\|_{L_2} = \sup_{X \in \mathbb{G}} I(X) = \sup_{X \in \mathbb{G}} I(\varphi_{\theta^{-1}}X) \leq C \left| \log |X| \right| , \]

and, consequently,

\[ \left\| H^{1/2}_t \right\|_{L_2} \leq \int_0^{1/2} \left\| H^{1/2}_t \right\|_{L_2} d\theta \leq C \int_0^{1/2} \left| \log |X| \right| d\theta < \infty , \]

which completes the proof of Proposition (4.2).

**Proof.**

This is derived by a routine argument, from (1.3), Corollary (1.6), and Proposition (4.2). \( \square \)

**5. The case \( p = 1 \): sufficiency**

**Theorem.** If \( \nu \) has a density which is locally in the Zygmund class on \( \mathbb{N} \setminus \{e\} \), then there exists \( \Omega \in L^1(\mathbb{G}) \) such that

\[ h(x) \leq \Omega(\frac{|x|}{\delta^{-1}})^{-Q - \alpha}, \quad x \in \mathbb{N} \setminus \{e\}. \]

**Proof.**

The argument is quite analogous to that in the proof of Proposition (3.1). The only difference is that now \( k_0 \) is in the Zygmund class \( L^2(N) \), which, by Corollary (4.6), implies that for some \( r \geq 1, \)

\[ \Omega(\frac{x}{\delta}) = M + M k_0(\delta, x), \quad x \in \mathbb{G}, \]

is integrable on \( \mathbb{G} \) and satisfies the desired inequality. \( \square \)

The remaining part of the paper is devoted to the demonstration of the inverse implication.

**6. A space of homogeneous type.** Given \( x, y \in \mathbb{N} \), we denote by \( d(x, y) \) the greatest lower bound of the subset of all real \( r > 0 \) such that

\[ x, y \in B_r, \]

for some \( z \) in the ball \( B(r) \). It is not hard to see that, due to the subadditivity of the homogeneous norm, \( d \) is a metric on \( \mathbb{N} \).

Let

\[ A_x(r) = \{ y \in \mathbb{N} : d(x, y) < r \} \]

be the open ball of radius \( r \) relative to \( d \) centred at \( x \in \mathbb{N} \). Let also

\[ A_x(r) = \chi_{B(r)} \ast \delta_x \ast \chi_{B(r)}, \]

where \( \delta_x \) stands for the Dirac point mass located at \( x \in \mathbb{N} \).

** Lemma.** For every \( r > 0 \) and every \( x \in \mathbb{N} \),

\[ \Phi^{(x)}(y) \leq \Phi^{(x)}(z) \quad \text{if} \quad d(x, z) \leq r . \]

**Proof.**

If \( \Phi^{(x)}(y) > 0 \), then \( d(x, y) < r \) and there exist \( x_1, x_2 \in B(r) \) such that

\[ \Phi^{(x)}(y) = \Phi^{(x)}(x_1^{-1}x_2) \]

\[ = \left( \delta_{x_1} \ast \chi_{B(r)} \right) \ast \left( \delta_x \ast \chi_{B(r)} \right) \]

\[ \leq \chi_{B(2r)} \ast \delta_x \ast \chi_{B(2r)}(x) = \Phi^{(x)}(x) . \]

In a similar fashion it is shown that \( d(x, z) \leq r \) implies \( \Phi^{(z)}(x) \leq \Phi^{(z)}(z) \), which combined with (6.2) completes the proof. \( \square \)

**Lemma.** For every \( r > 0 \) and every \( x \in \mathbb{N} \),

\[ |B(r)|^2 \leq \left\| \Phi^{(x)} \right\|_{L_1} \cdot A_x(r) \leq |B(2r)|^2 . \]

**Proof.**

By Lemma (6.1), we have

\[ \left\| \Phi^{(x)} \right\|_{L_1} \cdot A_x(r) \leq \int A_x(r) \Phi^{(y)}(y) dy . \]
Consequently,
\[
|B(r)|^2 = \int \Phi_{a(r)}^{(2)}(y) \, dy \leq \|\Phi_{a(r)}^{(2)}\|_{\infty} |A_a(r)|
\]
\[
\leq \int \Phi_{a(r)}^{(2)}(y) \, dy \leq |B(3r)|^2,
\]
which completes the proof. \(\blacksquare\)

(6.4) Proposition. The measure space \((\mathcal{N}, dx)\) endowed with the metric \(d\) is a space of homogeneous type.

Proof. It is sufficient to show that there exists a constant \(C > 0\) such that for every \(r > 0\) and every \(x \in \mathcal{N}\),
\[
|A_a(2r)| \leq C |A_a(r)|
\]
(see Coifman and Weiss [2]). In fact, by Lemma (6.3),
\[
|A_a(2r)| \leq \frac{|B(6r)|^2}{\|\Phi_{a(2r)}^{(2)}\|_{\infty}} \leq 6^{2Q} \frac{|B(r)|^2}{\|\Phi_{a(r)}^{(2)}\|_{\infty}} \leq 6^{2Q} |A_a(r)|. \quad \blacksquare
\]

For \(r > 0\), let
\[
\chi_{r}(x) = \frac{1}{|B(r)|} \chi_{B(3r)}(x), \quad x \in \mathcal{N}.
\]

(6.5) Lemma. For every \(r > 0\) and every \(x \in \mathcal{N}\),
\[
\frac{1}{|A_a(r)|} \int_{A_a(r)} f(z) \, dz \leq 6^{2Q} \chi_{r} * f * \chi_{r}(x), \quad f \geq 0.
\]

Proof. In fact, by Proposition (6.4),
\[
\frac{1}{|A_a(r)|} \int_{A_a(r)} f(z) \, dz \leq \frac{6^{2Q} |A_a(2r)|}{|A_a(2r)|} \int_{A_a(2r)} f(z) \, dz
\]
and, by Lemma (6.3),
\[
\frac{1}{|A_a(2r)|} \int_{A_a(2r)} f(z) \, dz \leq \frac{\int_{A_a(2r)} f(z) \, dz}{|A_a(r)|}
\]
\[
\leq \frac{1}{|B(r)|^2} \int_{A_a(r)} f(z) \|\Phi_{a(r)}^{(2)}\|_{\infty} \, dz
\]
\[
\leq \chi_{r} * f * \chi_{r}(x)
\]
for \(r > 0, x \in \mathcal{N}\. \quad \blacksquare
\]

Denote by \(\text{Mes}^+(\mathcal{N})\) the space of positive measurable functions on \(\mathcal{N}\). For \(f \in \text{Mes}^+(\mathcal{N})\), let
\[
\Phi^* f(x) = \sup_{0 < r \leq 1} \chi_r * f * \chi_r(x),
\]
\[
f^*(x) = \sup_{r > 0} \frac{1}{|A_a(r)|} \int_{A_a(r)} f(z) \, dz, \quad x \in \mathcal{N}.
\]

Of course \(f^*\) is the usual Hardy–Littlewood maximal function for \(f\) on the space \((\mathcal{N}, dx, d)\) of homogeneous type.

(6.7) Lemma. For every nonnegative \(f \in L^1(\mathcal{N})\) with \(\|f\|_1 \leq |B(1)|\) and every \(\lambda \geq 1\),
\[
\{x \in \mathcal{N} : f^*(x) > \lambda\} \subseteq \{x \in \mathcal{N} : \Phi^* f(x) > 6^{-2Q} \lambda\}.
\]

Proof. This is an immediate consequence of (6.6) and the fact that
\[
\frac{1}{|A_a(r)|} \int f(y) \, dy \leq \frac{|B(1)|}{|B(r)|} \leq 1
\]
for \(r \geq 1\). The first inequality holds because \(x B(r) \subset A_a(r)\. \quad \blacksquare
\]

Recall that the Hardy–Littlewood maximal function \(f^*\) on a space of homogeneous type satisfies the inverse Kolmogorov inequality
\[
\frac{1}{\lambda} \int f^*(y) \, dy \leq C \{x \in \mathcal{N} : f^* > c \lambda\}
\]
for every \(f \in \text{Mes}^+(\mathcal{N})\), every \(\lambda > 0\), and some constants \(C, c > 0\). We may assume that \(c \leq 1\).

(6.9) Proposition. For every \(f \in \text{Mes}^+(\mathcal{N})\) with \(\|f\|_1 \leq |B(1)|\),
\[
\int f(x) \log^+ f(x) \, dx \leq \frac{6^{2Q} C}{c} \int \Phi^* f(x) \, dx + \log \frac{1}{c} \int f(x) \, dx.
\]

Proof. In fact,
\[
\int f(x) \log^+ f(x) \, dx = \int f(x) \log f(x) \, dx
\]
\[
= \int \frac{1}{\lambda} \int_{f(x) > \lambda} f(x) \, dx \, d\lambda
\]
\[
\leq \log \frac{1}{c} \|f\|_1 + \int \frac{1}{\lambda} \int \Phi^* f(x) \, dx \, d\lambda.
\]

The last term is, by (6.8) and Lemma (6.7), estimated by
\[
C \int_{1/e}^{\infty} \left| \{x \in \mathcal{N} : f^*(x) > c\lambda \} \right| d\lambda \leq C \int_{1/e}^{\infty} \left| \{x \in \mathcal{N} : \Phi^* f(x) > 6^{-2Q} c\lambda \} \right| d\lambda
\]
\[
\leq \frac{6^{-2Q} C}{c} \int_{\mathcal{N}} \Phi^* f(x) \, dx .
\]

For \( \eta > 0 \) and \( f \in \text{Mes}^+(\mathcal{N}) \), let
\[
\Phi^*_\eta f(x) = \sup_{0 < \epsilon < \eta} \chi_{\epsilon} \ast f \ast \chi_{\epsilon}(x), \quad x \in \mathcal{N} .
\]

(8.11) Corollary. Let \( f \in \text{Mes}^+(\mathcal{N}) \). If \( \Phi^*_\eta f \in L^1(\mathcal{N}) \) for some \( \eta \), then \( f \in L \log L(\mathcal{N}) \).

Proof. If \( \eta = 1 \), then \( \Phi^*_1 = \Phi^* \) and our claim is a simple consequence of (6.10). In the general case
\[
\Phi^*_\eta f(x) = [\Phi^*(f \circ \delta_\eta)](\delta_{\eta^{-1}}x), \quad x \in \mathcal{N} ,
\]
which takes us back to the case \( \eta = 1 \).

7. The case \( p = 1 \): necessity

(7.1) Lemma. There exist constants \( C, \varepsilon, \delta > 0 \) such that
\[
\overline{h}_t(x) \geq C\chi_{(x,t]}(x), \quad x \in \mathcal{N}, \quad 0 < t < \delta .
\]

Proof. By Proposition (1.7),
\[
|h_t(x) - h_\varepsilon(x)| \leq A|t|^{-Q/\alpha - 1} |x|^\alpha , \quad x \in \mathcal{N}, \quad 0 < t \leq 1 .
\]
By the perturbation formula (1.3) and (1.4),
\[
\|h_t - \overline{h}_t\|_\infty \leq B|t|^{-Q/\alpha + 1}, \quad t > 0 .
\]
We also have
\[
h_\varepsilon(x) = D|\varepsilon|^{-Q/\alpha}, \quad t > 0 .
\]

In the above formulas, \( A, B, \) and \( D \) are positive constants independent of \( x \) and \( t \).

Now, let \( |x| \leq (\varepsilon t)^{1/\alpha} \), where \( \varepsilon \) is to be chosen later. Then, by (7.3) (7.5),
\[
\overline{h}_t(x) \geq h_\varepsilon(x) - Bt^{-Q/\alpha + 1} \geq h_\varepsilon(x) - A|x|^{\varepsilon t^{-Q/\alpha - 1}} - Bt^{-Q/\alpha + 1}
\]
\[
\geq D|\varepsilon t|^{-Q/\alpha} - (Ae + Bt)|\varepsilon t|^{-Q/\alpha} \geq \frac{1}{2} D|\varepsilon t|^{-Q/\alpha} \geq C|\varepsilon t|^{-Q/\alpha}
\]
for sufficiently small \( \varepsilon \) and \( t \), where \( C = \frac{1}{2} D \). Thus our assertion is proved.

(7.6) Theorem. If \( h \) satisfies the estimate
\[
h(x) \leq O(\varepsilon)|x|^{-Q/\alpha} \quad \text{for a.e.} \ x \in \mathcal{N} \setminus \{e\}
\]
with \( \Omega \in L^1(\mathcal{N}) \), then the Lévy measure \( \nu \) of \( \{\mu_t\} \) is absolutely continuous and its density belongs to the Zygmund class \( L \log L \) locally on \( \mathcal{N} \setminus \{e\} \).

Proof. It is sufficient to show that the measure \( \nu \), as defined in (1.2), has a density \( \log L \). By Proposition (3.1), \( \nu \) does have a density \( k_0 \) in \( L^1(\mathcal{N}) \). To prove that \( k_0 \in L \log L \), we shall show that \( \Phi^*_\eta k_0 \in L^1(\mathcal{N}) \) for some \( 0 < \eta \leq 1 \) and then invoke Corollary (6.11). Note that, by homogeneity, we may assume that the support of \( k_0 \) is far away from the identity so that \( \chi_t \ast k_0 \ast \chi_1 \) vanishes in a neighbourhood of \( e \).

Now, by Lemma (7.1) and the perturbation formula (1.3), there exist constants \( A, \varepsilon, \delta > 0 \)
\[
\chi_{(\varepsilon t)^{1/\alpha}} \ast k_0 \ast \chi_{(\varepsilon t)^{1/\alpha}}(x) \leq A \int_{\varepsilon t}^\frac{1}{t} h_{t-x} \ast k_0 \ast h_{t}(x) \, ds
\]
\[
\leq \frac{1}{t} A h_t(x) \leq A \log(\varepsilon t) |x|^{-Q/\alpha}
\]
for \( x \neq e \) and \( 0 < t < \delta \). Taking into account the fact that the supports of all the functions \( \chi_t \ast k_0 \ast \chi_1 \), where \( 0 < \eta \leq 1 \), are uniformly cut off from the identity \( e \), we conclude that \( \Phi^*_\eta k_0 \) is integrable for some \( 0 < \eta \leq 1 \), which completes the proof.

References

Markov’s property of the Cantor ternary set

by

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Abstract. We prove that the Cantor ternary set \( E \) satisfies the classical Markov inequality (see [Ma]): for each polynomial \( p \) of degree at most \( n \) \( n = 0, 1, 2, \ldots \)
\[
|p'(x)| \leq M n^m \sup_E |p| \quad \text{for } x \in E,
\]
where \( M \) and \( m \) are positive constants depending only on \( E \).

0. Introduction. In 1889 A. A. Markov proved (M) for \( E = [-1; 1] \).
Since then it has become the object of extensive research (see e.g. [R-S] in the one-dimensional case and [Pa-Pi 1] for \( \mathbb{R}^n \)). In particular, it has appeared that the inequality plays an important role in the approximation and extension of \( C^\infty \) functions defined on compact subsets of \( \mathbb{R}^n \) to \( C^\infty \) functions on the whole space (see [Pa-Pi 1], [Pa-Pi 2] and [Pi 4]).

The question about Markov’s property for the Cantor ternary set has remained unanswered for many years. J. Siciak [Si 3] showed that there exists a Cantor type set \( E \subset \mathbb{R} \) such that Leja’s extremal function \( L_E \) (see [Lj 2], p. 261) has the following Hölder continuity property:
\[
L_E(x) \leq 1 + M \delta^m \quad \text{if dist}(x, E) \leq \delta \leq 1,
\]
with some positive constants \( M > 0 \) and \( m > 0 \) depending only on \( E \), which, by Cauchy’s integral formula, is sufficient for \( E \) to preserve Markov’s inequality (M) (see [Si 2], Remark after Lemma 1 and [Pi 1], Lemma 3.1). On the other hand, Plisniak [Pi 2] constructed a Cantor type set \( E \) such that Leja’s extremal function is continuous on \( C \) but \( E \) does not satisfy (M).

These results have given no answer to the question of whether the Cantor ternary set has Markov’s property. In this paper we prove that the answer is affirmative. Actually, we show that this set even has the (HCP) property.

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