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Trace inequalities for spaces in spectral duality

by

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Abstract. Let (A, e) and (V, K) be an order-unit space and a base-norm space in spectral duality, as in noncommutative spectral theory of Alfsen and Shultz. Let t be a norm lower semicontinuous trace on A, and let φ be a nonnegative convex function on $\mathbb{R}$. It is shown that the mapping $a \mapsto t(\varphi(a))$ is convex on A. Moreover, the mapping is shown to be nondecreasing if so is φ. Some other similar statements concerning traces and real-valued functions are also obtained.

1. Introduction. Inequalities involving operator functions and traces are useful tools in the study of operator algebras. A number of papers were dedicated to obtain such inequalities or contained the proofs of trace inequalities as their essential parts (see, e.g., [5–10]). Roughly speaking, it was discovered that under a trace, operators often behave "like numbers" [6].

The main purpose of the paper is to extend some trace inequalities to the class of partially ordered normed vector spaces for which Alfsen and Shultz [3] developed a noncommutative spectral theory and a functional calculus. The class in question contains the space of all self-adjoint elements of a von Neumann algebra. Nevertheless, this special case does not exhaust all possibilities.

Section 2 contains basic notions and results of noncommutative spectral theory [3]. Here, an order-unit space A and a base-norm space V are supposed to be in spectral duality (see the exact definitions below). If φ is a bounded Borel function of a real variable then the element $\varphi(a)$ of A is well-defined and the functional calculus defined by the mapping $\varphi \mapsto \varphi(a)$ has the properties similar to those of the usual functional calculus for self-adjoint operators in a Hilbert space. At the end of the section we introduce the concept of a trace. It agrees with the one used in operator theory.

The main results are obtained in Section 3. For a trace t on A and a decomposition $a = a_1 - a_2$ of $a \in A$ into a difference of positive elements, we prove that $t(a^+) \le t(a_1)$ and $t(a^-) \le t(a_2)$ where a^+ and a^- denote

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the positive and the negative parts of a (Proposition 3.1). It follows that the mapping $a \mapsto t(a^+)$ is monotone on A (Corollary 3.2). Next, we make use of the properties of the functional calculus in A to prove the mapping $a \mapsto t(\varphi(a))$ to be monotone for some classes of monotone real-valued functions φ (Theorem 3.5 and Remark after it). The main result of the rest of the section is Theorem 3.9, which can be viewed as a version of Jensen's inequality for positive contractions on order-unit spaces. Using it we prove that the mapping $a \mapsto t(\varphi(a))$ is convex on A provided φ is a nonnegative convex function and the trace t is lower semicontinuous with respect to the norm on A. In case t is finite, we can omit the requirement of nonnegativity of φ. At the end of the paper we introduce the L_p-norms associated with a trace.

We should note that we do not know whether the machinery of generalized s-numbers (see, e.g., [6], [7], [9]) can be extended to the context of noncommutative spectral theory. The technique we use was developed for von Neumann algebras [11, 12] and applied to spaces in spectral duality for finite traces [13] (see also [14]).

2. Spaces in spectral duality and traces. We recall (cf., e.g., [2]) that an *order-unit space* is an Archimedian partially ordered normed vector space A with a distinguished order unit e and with the norm given by

$$\|a\| = \inf\{\lambda > 0 \mid -\lambda e \le a \le \lambda e\}.$$

A *base-norm space* is a positively generated partially ordered normed vector space V with a distinguished base K of the positive cone V^+ such that the closed unit ball coincides with $\operatorname{conv}(K \cup -K)$.

In what follows we consider an order-unit space (A, e) and a base-norm space (V, K) in separating order and norm duality. Then, in particular, we have $\langle e, \varrho\rangle = 1$ for all $\varrho \in K$.

The central concept in Alfsen and Shultz's noncommutative spectral theory [3] is the notion of a P-projection which may be introduced as follows.

DEFINITION 2.1. A positive $\sigma(A, V)$-continuous projection P on A with norm at most 1 is called a *P-projection* if there exists a positive $\sigma(A, V)$-continuous projection P' on A with norm at most 1 satisfying

$$\ker^+ P = \operatorname{im}^+ P', \qquad \operatorname{im}^+ P = \ker^+ P',$$
$$\ker^+ P^* = \operatorname{im}^+ P'^*, \qquad \operatorname{im}^+ P^* = \ker^+ P'^*,$$

where P^* and P'^* are the dual projections on V, $\ker^+ P = A^+ \cap \ker P$, $\operatorname{im}^+ P' = A^+ \cap \operatorname{im} P'$, etc.

Note that for every P-projection P on A the unique projection P' with the properties stated in the above definition is also a P-projection. It is called the *quasicomplement* of P.

A P-projection P on A and an element a of A are said to be *compatible* if $Pa + P'a = a$. Two P-projections P and Q are said to be *compatible* if $PQ = QP$. A P-projection P is said to be *bicompatible* with an element a of A if it is compatible with a and with all P-projections compatible with a [3; §5].

Throughout the following we in addition assume that A and V are in *spectral duality* [3; §7], that is, A is pointwise monotone σ-complete, and for every $a \in A$ and $\lambda \in \mathbb{R}$ there exists a P-projection P bicompatible with a and such that

$$\langle a, \varrho\rangle \le \lambda \quad \text{for } \varrho \in (\operatorname{im} P^*) \cap K,$$
$$\langle a, \varrho\rangle > \lambda \quad \text{for } \varrho \in (\operatorname{im} P'^*) \cap K.$$

The set of all P-projections on A is denoted by $\mathcal{P}$. This set is endowed with a partial ordering by

$$P \preccurlyeq Q \quad \text{if} \quad \operatorname{im} P \subset \operatorname{im} Q.$$

Two P-projections P and Q are said to be *orthogonal* if $P \preccurlyeq Q'$. For a given P-projection P the element Pe of A is called a *projective unit* corresponding to P. The set of all projective units is denoted by $\mathcal{U}$ and is endowed with a natural partial ordering inherited from A. The sets $\mathcal{P}$ and $\mathcal{U}$ are each endowed with a natural operation of *complementation*: $P \mapsto P'$ and $Pe \mapsto e - Pe$. It has been proved in [3] that under the above assumptions these two ordered sets with complementation are σ-complete orthomodular lattices. Moreover, they are isomorphic by means of the bijection $P \mapsto Pe$.

Recall also that for every $a \in A^+$ there exists a projective unit $\operatorname{rp}(a)$ determined by the following equivalence valid for $\varrho \in K$:

$$\langle \operatorname{rp}(a), \varrho\rangle = 0 \;\Leftrightarrow\; \langle a, \varrho\rangle = 0.$$

One of the main results of [3] is the following.

THEOREM [3]. *If A and V are in spectral duality then for every $a \in A$ there exists a unique family $\{e^a_\lambda\}_{\lambda\in\mathbb{R}}$ of projective units satisfying*

(i) $e^a_\lambda \le e^a_\mu$ *when* $\lambda < \mu$,
(ii) $e^a_\lambda = \bigwedge_{\mu>\lambda} e^a_\mu$,
(iii) $\bigwedge_{\lambda\in\mathbb{R}} e^a_\lambda = 0$, $\bigvee_{\lambda\in\mathbb{R}} e^a_\lambda = e$,

and such that

$$a = \int \lambda \, de^a_\lambda$$

where the right hand side is a norm convergent Riemann–Stieltjes integral.

Note that $\operatorname{rp}(a) = e - e^a_0$ for $a \in A^+$.

It has also been proved in [3] that if φ is a bounded Borel function of a real variable then there exists a unique element of A (denoted by $\varphi(a)$) such that for all $\varrho \in K$,

$$\langle \varphi(a), \varrho \rangle = \int \varphi(\lambda)\, d\langle e^a_\lambda, \varrho \rangle .$$

Moreover, φ needs only to be defined on the *spectrum* of a (denoted by $\sigma(a)$) [3], and the functional calculus defined by the mapping $\varphi \mapsto \varphi(a)$ has the properties similar to those of the usual functional calculus for self-adjoint operators in a Hilbert space.

The following definition generalizes the notion of a finite trace from K, studied in [3].

DEFINITION 2.2. A function t on the positive cone A^+ with values in the extended positive reals is called a *trace* if it satisfies the following conditions:

(i) $t(a+b) = t(a) + t(b)$ for $a, b \in A^+$;
(ii) $t(\lambda a) = \lambda t(a)$ for $\lambda \in \mathbb{R}^+$, $a \in A^+$;
(iii) $t(a) = t(Pa) + t(P'a)$ for $a \in A^+$, $P \in \mathcal{P}$;

with the usual convention $0 \cdot (+\infty) = 0$.

Put

$$\mathfrak{m}_t^+ = \{a \in A^+ \mid t(a) < \infty\}$$

and denote by $\mathfrak{m}_t$ the linear span of $\mathfrak{m}_t^+$. Then t can be uniquely extended to a linear functional on $\mathfrak{m}_t$ which will also be denoted by t.

A trace t is said to be *finite* if $t(e) < \infty$. Clearly a finite trace can be viewed as a positive continuous functional on A.

A trace t is said to be *faithful* if $t(a) > 0$ for any $a \in A^+ \setminus \{0\}$.

A finite trace t is said to be *σ-orthoadditive* if $t(\bigvee_n e_n) = \sum_n t(e_n)$ for any sequence $\{e_n\}$ of mutually orthogonal projective units. By usual arguments it is easy to show that if t is σ-orthoadditive then for any $a \in A$ and a bounded Borel function φ one has

$$t(\varphi(a)) = \int \varphi(\lambda)\, dt(e^a_\lambda) .$$

Note that t is σ-orthoadditive when $t \in K$.

We close this section by considering the special case when A is the self-adjoint part of a von Neumann algebra M and V is the self-adjoint part of the predual M_* with K the normal state space. Then a mapping $P : A \to A$ is a P-projection iff it is of the form $Pa = pap$ for a (self-adjoint) projection $p \in M$, and in this case $P'a = (e-p)a(e-p)$ [3; Prop. 11.1]. By standard arguments one can show that a function t on A^+ is a trace in the general sense defined in this section iff it satisfies the conditions (i) and (ii) of Definition 2.2 and $t(sas) = t(a)$ for all $a \in A^+$ and all symmetries $s \in M$. From [15; Theorem 1.4] it follows that for every unitary $u \in M$ there exists a finite family of symmetries $\{s_i\}_{i=1}^n \subset M$ such that $u^*au = s_1 \ldots s_n a s_n \ldots s_1$ for any $a \in M$. Hence we conclude that Definition 2.2 agrees with the concept of trace used in operator theory (see also [1] and [4]).

3. Inequalities for convex functions. We denote by γ, γ^+, and γ^- the identity function on $\mathbb{R}$ and its positive and negative parts, i.e. $\gamma(\lambda) = \lambda$, $\gamma^+(\lambda) = \max\{\lambda, 0\}$, and $\gamma^-(\lambda) = \max\{-\lambda, 0\}$ for all $\lambda \in \mathbb{R}$. By means of the functional calculus in A for $a \in A$ we define $a^+ = \gamma^+(a)$ and $a^- = \gamma^-(a)$. We also put $|a| = a^+ - a^-$. Note that $a^+, a^- \in A^+$ and $a = \gamma(a) = a^+ - a^-$.

PROPOSITION 3.1. *Let $a \in A$ and $a = a_1 - a_2$ for some $a_1, a_2 \in A^+$. Then for a trace t one has $t(a^+) \le t(a_1)$ and $t(a^-) \le t(a_2)$.*

Proof. Let P be a P-projection corresponding to the projective unit $e - e^a_0$. Then $a^+ = Pa$ and $a^- = -P'a$ [3; p. 73]. By definition of a trace, we obtain

$$t(a^+) + t(Pa_2) = t(Pa + Pa_2) = t(Pa_1) \le t(Pa_1 + P'a_1) = t(a_1),$$
$$t(a^-) + t(P'a_1) = t(P'a_2) \le t(P'a_2 + Pa_2) = t(a_2),$$

hence $t(a^+) \le t(a_1)$ and $t(a^-) \le t(a_2)$.

COROLLARY 3.2. *For a trace t and given $a, b \in A$ with $a \le b$ one has $t(a^+) \le t(b^+)$.*

Proof. This follows from the equality $a = b - (b-a) = b^+ - (b^- + (b-a))$.

COROLLARY 3.3. *For a trace t and an element a of A one has*

$$|a| \in \mathfrak{m}_t^+ \iff a \in \mathfrak{m}_t .$$

Moreover, the function $\|\cdot\|_t : a \mapsto t(|a|)$ on $\mathfrak{m}_t$ is a seminorm and it is a norm if the trace t is faithful.

Proof. If $|a| \in \mathfrak{m}_t^+$ then $t(a^+) + t(a^-) = t(|a|) < \infty$, hence $a^+, a^- \in \mathfrak{m}_t^+$ and $a \in \mathfrak{m}_t$.

Conversely, if $a \in \mathfrak{m}_t$ then we can write $a = a_1 - a_2$ with $a_1, a_2 \in \mathfrak{m}_t^+$. By Proposition 3.1, $t(|a|) = t(a^+) + t(a^-) \le t(a_1) + t(a_2) < \infty$, hence $|a| \in \mathfrak{m}_t^+$.

Next we prove that $\|\cdot\|_t$ is a seminorm on $\mathfrak{m}_t$. It is clear that this function is nonnegative, and it follows from the definition of a trace and the properties of the functional calculus that $\|\lambda a\|_t = |\lambda|\,\|a\|_t$ for any $a \in \mathfrak{m}_t$ and $\lambda \in \mathbb{R}$. Let $a, b \in \mathfrak{m}_t$. Since $a + b = (a^+ + b^+) - (a^- + b^-)$, by Proposition 3.1 it follows that

$$\begin{aligned} t(|a+b|) &= t((a+b)^+) + t((a+b)^-) \\ &\le t(a^+) + t(b^+) + t(a^-) + t(b^-) = t(|a|) + t(|b|). \end{aligned}$$

Thus $\|\cdot\|_t$ is a seminorm on $\mathfrak{m}_t$. Clearly it is a norm if the trace t is faithful.

Remark. The seminorm $\|\cdot\|_t$ on $\mathfrak{m}_t$ can also be expressed as follows:

$$\|a\|_t = \inf\{\alpha_1 t(a_1) + \alpha_2 t(a_2) \mid \alpha_i \in \mathbb{R}^+,\ a_i \in A^+,\ t(a_i) = 1\ (i = 1, 2)\} .$$

Thus, if t is assumed to be faithful, then $\mathfrak{m}_t$ is a base-norm space with base $\{a \in \mathfrak{m}_t^+ \mid t(a) = 1\}$.

LEMMA 3.4. *Let φ be a nonnegative, convex, nondecreasing, and piecewise linear function on $\mathbb{R}$. For a trace t and elements a, b of A with $a \le b$ one has*

$$t(\varphi(a)) \le t(\varphi(b)).$$

If $t(e) < \infty$ then the nonnegativity assumption may be omitted.

Proof. Let $[\alpha, \beta]$ be an interval in $\mathbb{R}$ including the spectra of a and b. For any φ satisfying the assumptions of the lemma we can find an integer n, real numbers λ_i, and nonnegative numbers μ_i such that

$$\varphi(\lambda) = \mu_0 + \sum_{i=1}^{n} \mu_i \gamma^+(\lambda - \lambda_i) \quad \text{for } \lambda \in [\alpha, \beta].$$

Then by Corollary 3.2 and the functional calculus,

$$\begin{aligned} t(\varphi(a)) &= t\Big(\mu_0 e + \sum_{i=1}^{n} \mu_i \gamma^+(a - \lambda_i e)\Big) \\ &= \mu_0 t(e) + \sum_{i=1}^{n} \mu_i t(\gamma^+(a - \lambda_i e)) \\ &\le \mu_0 t(e) + \sum_{i=1}^{n} \mu_i t(\gamma^+(b - \lambda_i e)) = t(\varphi(b)). \end{aligned}$$

If $t(e) < \infty$ then we can omit the requirement $\mu_0 \ge 0$ and proceed as above.

THEOREM 3.5. *Let a trace t be lower semicontinuous with respect to the norm on A, and let φ be a continuous, nonnegative, convex, and nondecreasing function on a convex subset E of $\mathbb{R}$. If a and b are elements of A such that $\sigma(a), \sigma(b) \subset E$ and $a \le b$, then*

$$t(\varphi(a)) \le t(\varphi(b)). \tag{1}$$

If $t(e) < \infty$ then the nonnegativity assumption may be omitted.

Proof. Without loss of generality we assume that $E = [\alpha, \beta]$.

For given φ one can find a sequence $\{\varphi_k\}$ of nonnegative, convex, nondecreasing, and piecewise linear functions on $\mathbb{R}$ which is pointwise nondecreasing on $[\alpha, \beta]$ and uniformly converges to φ. Then the sequences $\{\varphi_k(a)\}$ and $\{\varphi_k(b)\}$ are monotone and they converge in norm to $\varphi(a)$ and $\varphi(b)$, respectively, as $k \to \infty$. By Lemma 3.4, $t(\varphi_k(a)) \le t(\varphi_k(b))$ for every k, and by semicontinuity of t we get

$$t(\varphi(a)) = \lim_{k\to\infty} t(\varphi_k(a)) \le \lim_{k\to\infty} t(\varphi_k(b)) = t(\varphi(b)).$$

It is clear how to modify this proof to obtain the last assertion of the theorem.

Remark. The monotonicity of the mapping $a \mapsto t(\varphi(a))$ can also be proved for some other classes of monotone functions φ. Namely, (1) holds when φ is continuous, nonpositive, concave, and nondecreasing. This can be easily deduced from Theorem 3.5 by considering $-\varphi(-\lambda)$ which is nonnegative and convex if $\varphi(\lambda)$ is nonpositive and concave. Since $0 \le a \le b$ implies $\mathrm{rp}(a) \le \mathrm{rp}(b)$, it follows that (1) also holds for $\chi_{(0,\infty)}$ defined on $[0, \infty)$. Note that this function is concave but not continuous. Similarly, (1) can be proved for some other functions that are nondecreasing and either convex or concave, but not continuous. Finally, in the case when t is finite, (1) holds for any function φ on E admitting a representation $\varphi = \varphi_1 + \varphi_2$ where φ_1 is convex, φ_2 is concave and they are both nondecreasing on E.

LEMMA 3.6. *Let φ be a nonnegative and convex function on an interval $[\alpha, \beta]$, and let a finite family $\{x_i\}_{i=1}^n \subset A^+$ be such that $\sum_{i=1}^n x_i = e$. Then for a trace t and a family $\{\lambda_i\}_{i=1}^n \subset [\alpha, \beta]$ one has*

$$\sum_{i=1}^{n} \varphi(\lambda_i) t(x_i) \ge t\Big(\varphi\Big(\sum_{i=1}^{n} \lambda_i x_i\Big)\Big). \tag{2}$$

If $t(e) < \infty$ then the nonnegativity assumption is superfluous.

Proof. Without loss of generality we assume that $\alpha \le \lambda_1 \le \ldots \le \lambda_n \le \beta$. Since

$$\alpha e \le \lambda_1 e \le \sum_{i=1}^{n} \lambda_i x_i \le \lambda_n e \le \beta e,$$

the right hand side of (2) is well-defined. Construct a function $\widetilde{\varphi}$ on $[\lambda_1, \lambda_n]$ linear on each $[\lambda_i, \lambda_{i+1}]$ $(i = 1, \ldots, n-1)$ and such that $\widetilde{\varphi}(\lambda_i) = \varphi(\lambda_i)$. Since φ is convex, so is $\widetilde{\varphi}$, and if φ is nonnegative then $\widetilde{\varphi}$ is also nonnegative. Moreover, $\widetilde{\varphi}(\lambda) \ge \varphi(\lambda)$ on $[\lambda_1, \lambda_n]$. It is easy to prove that if $\varphi \ge 0$ then there exist $\mu_i^+, \mu_i^- \ge 0$ such that

$$\widetilde{\varphi}(\lambda) = \sum_{i=1}^{n} (\mu_i^+ \gamma^+(\lambda - \lambda_i) + \mu_i^- \gamma^-(\lambda - \lambda_i)).$$

Set $a = \sum_{i=1}^n \lambda_i x_i$. By Proposition 3.1,

$$\begin{aligned} t(\gamma^+(a - \lambda_i e)) &= t\Big(\Big(\sum_{j=1}^{n} (\lambda_j - \lambda_i) x_j\Big)^+\Big) \\ &\le t\Big(\sum_{j=i}^{n} \gamma^+(\lambda_j - \lambda_i) x_j\Big) = \sum_{j=1}^{n} \gamma^+(\lambda_j - \lambda_i) t(x_j) \end{aligned}$$

and

$$t(\gamma^-(a - \lambda_i e)) \le \sum_{j=1}^{n} \gamma^-(\lambda_j - \lambda_i) t(x_j)$$

for every $i = 1, \ldots, n$. Hence

$$\begin{aligned} t(\varphi(a)) &\le t(\widetilde{\varphi}(a)) \\ &= t\Big(\sum_{i=1}^{n}(\mu_i^+\gamma^+(a-\lambda_i e)+\mu_i^-\gamma^-(a-\lambda_i e))\Big) \\ &\le \sum_{i=1}^{n}\Big(\mu_i^+\sum_{j=1}^{n}\gamma^+(\lambda_j-\lambda_i)t(x_j)+\mu_i^-\sum_{j=1}^{n}\gamma^-(\lambda_j-\lambda_i)t(x_j)\Big) \\ &= \sum_{j=1}^{n}t(x_j)\sum_{i=1}^{n}(\mu_i^+\gamma^+(\lambda_j-\lambda_i)+\mu_i^-\gamma^-(\lambda_j-\lambda_i)) \\ &= \sum_{j=1}^{n}\widetilde{\varphi}(\lambda_j)t(x_j)=\sum_{j=1}^{n}\varphi(\lambda_j)t(x_j). \end{aligned}$$

In the case $t(e) < \infty$, for φ not necessarily nonnegative, we put $\varphi_1 = \varphi + \mu_0$ where μ_0 is so large that $\varphi_1 \ge 0$. Then

$$\begin{aligned} t(\varphi(a)) &= t(\varphi_1(a)-\mu_0 e) \\ &\le \sum_{i=1}^{n}\varphi_1(\lambda_i)t(x_i)-\sum_{i=1}^{n}\mu_0 t(x_i)=\sum_{i=1}^{n}\varphi(\lambda_i)t(x_i). \end{aligned}$$

The following lemma is, in fact, a simple statement on piecewise linear real-valued functions.

LEMMA 3.7. *Let φ be a continuous piecewise linear function on $[\alpha,\beta]$, let $\alpha = \lambda_1 < \ldots < \lambda_n = \beta$, and let φ be linear on each interval $[\lambda_i, \lambda_{i+1}]$. Then for every $a \in A$ with $\sigma(a) \subset [\alpha,\beta]$ there exists a family $\{x_i\}_{i=1}^n \subset A^+$ such that $\sum_{i=1}^n x_i = e$, $\sum_{i=1}^n \lambda_i x_i = a$, and $\sum_{i=1}^n \varphi(\lambda_i)x_i = \varphi(a)$.*

Proof. Define functions on $[\alpha,\beta]$ as follows:

$$\xi_1(\lambda)=\begin{cases}(\lambda-\lambda_1)/(\lambda_2-\lambda_1) & \text{if } \lambda_1\le\lambda\le\lambda_2,\\ 0 & \text{otherwise;}\end{cases}$$

$$\xi_n(\lambda)=\begin{cases}(\lambda_n-\lambda)/(\lambda_n-\lambda_{n-1}) & \text{if } \lambda_{n-1}\le\lambda\le\lambda_n,\\ 0 & \text{otherwise;}\end{cases}$$

and for $i = 2, \ldots, n-1$,

$$\xi_i(\lambda)=\begin{cases}(\lambda-\lambda_{i-1})/(\lambda_i-\lambda_{i-1}) & \text{if } \lambda_{i-1}\le\lambda\le\lambda_i,\\ (\lambda_{i+1}-\lambda)/(\lambda_{i+1}-\lambda_i) & \text{if } \lambda_i\le\lambda\le\lambda_{i+1},\\ 0 & \text{if } \lambda<\lambda_{i-1} \text{ or } \lambda>\lambda_{i+1}.\end{cases}$$

Then $\xi_i(\lambda) \ge 0$, $\sum_{i=1}^n \xi_i(\lambda) = 1$, $\sum_{i=1}^n \lambda_i\xi_i(\lambda) = \lambda$, and $\sum_{i=1}^n \varphi(\lambda_i)\xi_i(\lambda) = \varphi(\lambda)$ on $[\alpha,\beta]$. Hence, if we take $x_i = \xi_i(a)$, the families $\{x_i\}_{i=1}^n$ and $\{\lambda_i\}_{i=1}^n$ have the required properties.

PROPOSITION 3.8. *Let φ be a continuous convex function on $[\alpha,\beta]$ and t be a finite trace. Then for every $a \in A$ with $\sigma(a) \subset [\alpha,\beta]$ one has*

$$t(\varphi(a))=\inf\Big\{\sum_{i=1}^{n}\varphi(\lambda_i)t(x_i)\Big\} \tag{3}$$

where the infimum is taken over all representations $a = \sum_{i=1}^n \lambda_i x_i$ with $x_i \in A^+$, $\sum_{i=1}^n x_i = e$, and $\lambda_i \in [\alpha,\beta]$. Moreover, if t is σ-orthoadditive then the continuity requirement is superfluous.

Proof. It follows from Lemma 3.6 that the left hand side of (3) is less than or equal to the right hand side, so it suffices to construct a sequence of representations such that the sequence of expressions in braces converges to $t(\varphi(a))$. For $n = 2, 3, \ldots$, let φ_n be the piecewise linear function interpolating φ, where the interpolation nodes $\lambda_i^{(n)}$ $(i = 1, \ldots, n)$ are assumed to be equidistant and $\alpha = \lambda_1^{(n)} < \ldots < \lambda_n^{(n)} = \beta$. Since φ is convex, so is φ_n, and $\varphi_n \ge \varphi$ on $[\alpha,\beta]$. If φ is continuous then $\{\varphi_n\}$ uniformly converges to φ as $n \to \infty$. Hence, constructing the families $\{x_i^{(n)}\}_{i=1}^n$ as in the proof of Lemma 3.7, we obtain

$$\sum_{i=1}^{n}\varphi(\lambda_i^{(n)})t(x_i^{(n)})=\sum_{i=1}^{n}\varphi_n(\lambda_i^{(n)})t(x_i^{(n)})=t(\varphi_n(a))\xrightarrow[n]{} t(\varphi(a)).$$

In case φ is not continuous and t is σ-orthoadditive, to justify the limit process we make use of the Lebesgue Dominated Convergence Theorem.

THEOREM 3.9. *Suppose two order-unit spaces (A, e) and (A', e') are in spectral duality with base-norm spaces, let π_1 and π_2 be positive linear mappings from A' to A, let t be a norm lower semicontinuous trace on A, and let φ be a continuous, nonnegative, and convex function on a convex subset E of $\mathbb{R}$. If either*

(a) $\pi_1(e') + \pi_2(e') = e$, *or*
(b) $\pi_1(e') + \pi_2(e') \le e$, $0 \in E$ *and* $\varphi(0) = 0$,

then for all $a_1', a_2' \in A'$ with $\sigma(a_1'), \sigma(a_2') \subset E$ one has

$$t(\varphi(\pi_1(a_1')+\pi_2(a_2')))\le t(\pi_1(\varphi(a_1'))+\pi_2(\varphi(a_2'))). \tag{4}$$

If the trace t is finite then the assumption of nonnegativity of φ may be omitted, and in (b) *the equality $\varphi(0) = 0$ may be replaced by $\varphi(0) \le 0$.*

Proof. Without loss of generality we assume that E is an interval, say $[\alpha,\beta]$.

Let $\pi_1(e') + \pi_2(e') = e$. Note that since $\alpha e = \alpha(\pi_1(e') + \pi_2(e')) \le \pi_1(a_1') + \pi_2(a_2') \le \beta e$, the left hand side of (4) is well-defined.

Assume first that φ is piecewise linear and construct families $\{x_{1i}'\}$ and $\{x_{2i}'\}$ by Lemma 3.7 with, respectively, a_1', x_{1i}' and a_2', x_{2i}' in place of a, x_i.

Then

$$\pi_1(\varphi(a'_1)) + \pi_2(\varphi(a'_2)) = \pi_1\Big(\sum_{i=1}^{n}\varphi(\lambda_i)x'_{1i}\Big) + \pi_2\Big(\sum_{i=1}^{n}\varphi(\lambda_i)x'_{2i}\Big)$$
$$= \sum_{i=1}^{n}\varphi(\lambda_i)\pi_1(x'_{1i}) + \sum_{i=1}^{n}\varphi(\lambda_i)\pi_2(x'_{2i}),$$
$$\pi_1(a'_1) + \pi_2(a'_2) = \sum_{i=1}^{n}\lambda_i\pi_1(x'_{1i}) + \sum_{i=1}^{n}\lambda_i\pi_2(x'_{2i}),$$
$$\pi_1(x'_{1i}) \ge 0, \quad \pi_2(x'_{2i}) \ge 0, \quad \sum_{i=1}^{n}\pi_1(x'_{1i}) + \sum_{i=1}^{n}\pi_2(x'_{2i}) = e,$$

and by Lemma 3.6,

$$t(\varphi(\pi_1(a'_1) + \pi_2(a'_2))) \le \sum_{i=1}^{n}\varphi(\lambda_i)t(\pi_1(x'_{1i})) + \sum_{i=1}^{n}\varphi(\lambda_i)t(\pi_2(x'_{2i}))$$
$$= t(\pi_1(\varphi(a'_1)) + \pi_2(\varphi(a'_2))).$$

If φ is an arbitrary continuous, nonnegative, and convex function on E then, constructing the sequence $\{\varphi_k\}$ of piecewise linear functions as in the proof of Theorem 3.5 and using the continuity of the positive mappings π_1 and π_2 and the semicontinuity of t, we obtain

$$t(\varphi(\pi_1(a'_1) + \pi_2(a'_2))) = \lim_{k\to\infty} t(\varphi_k(\pi_1(a'_1) + \pi_2(a'_2)))$$
$$\le \lim_{k\to\infty} t(\pi_1(\varphi_k(a'_1)) + \pi_2(\varphi_k(a'_2)))$$
$$= t(\pi_1(\varphi(a'_1)) + \pi_2(\varphi(a'_2))).$$

Assume next the conditions of (b) to be satisfied. Put $d = e - \pi_1(e') - \pi_2(e')$. If φ is piecewise linear, construct the families $\{x'_{1i}\}$ and $\{x'_{2i}\}$ as above. Then

$$\pi_1(a'_1) + \pi_2(a'_2) = \sum_{i=1}^{n}\lambda_i\pi_1(x'_{1i}) + \sum_{i=1}^{n}\lambda_i\pi_2(x'_{2i}) + 0\cdot d,$$
$$\sum_{i=1}^{n}\pi_1(x'_{1i}) + \sum_{i=1}^{n}\pi_2(x'_{2i}) + d = e,$$

and proceeding as above we obtain (4) for a piecewise linear and then for an arbitrary function.

The modification we must make in the special case $t(e) < \infty$ is obvious.

COROLLARY 3.10. *Let t be a norm lower semicontinuous trace on A, and let φ be a continuous, nonnegative, and convex function on a convex subset E of $\mathbb{R}$ with $0 \in E$ and $\varphi(0) = 0$. Then for any $a \in A$ with $\sigma(a) \subset E$, and any $P \in \mathcal{P}$ one has*

$$t(\varphi(Pa)) \le t(P\varphi(a)).$$

If t is finite then the nonnegativity assumption may be omitted.

Proof. This follows from (b) of the preceding theorem by setting $A' = A$, $\pi_1(b) = Pb$, and $\pi_2(b) = 0$ for $b \in A$.

COROLLARY 3.11. *Let t be a norm lower semicontinuous trace on A and φ be a nonnegative convex function on a convex subset E of $\mathbb{R}$. Then for all $a_1, a_2 \in A$ with $\sigma(a_1), \sigma(a_2) \subset E$ and every $\alpha \in [0,1]$ one has*

(5) $$t(\varphi(\alpha a_1 + (1-\alpha)a_2)) \le \alpha t(\varphi(a_1)) + (1-\alpha)t(\varphi(a_2)).$$

If t is finite then the nonnegativity assumption may be omitted.

Proof. If φ is continuous then (5) follows from Theorem 3.9 by setting $A' = A$, $\pi_1(b) = \alpha b$, and $\pi_2(b) = (1-\alpha)b$ for $b \in A$. For the general case, consider two convex functions:

$$\varphi_1(\lambda) = \begin{cases} 1 & \text{for } \lambda = 0, \\ 0 & \text{for } \lambda \in (0,\infty), \end{cases}$$
$$\varphi_2(\lambda) = \begin{cases} 1 & \text{for } \lambda = 0, \\ 0 & \text{for } \lambda \in (-\infty,0), \end{cases}$$

which are not continuous. Let $\sigma(a_1), \sigma(a_2) \subset [0,\infty)$ and $\alpha \in (0,1)$. Then $a_1, a_2 \in A^+$,

$$\mathrm{rp}(\alpha a_1 + (1-\alpha)a_2) \ge \mathrm{rp}(\alpha a_1) = \mathrm{rp}(a_1),$$

and

$$\mathrm{rp}(\alpha a_1 + (1-\alpha)a_2) \ge \mathrm{rp}(a_2).$$

Since $\varphi_1(b) = 1 - \mathrm{rp}(b)$ for any $b \in A^+$, applying the above relations one has

$$\varphi_1(\alpha a_1 + (1-\alpha)a_2) = 1 - \mathrm{rp}(\alpha a_1 + (1-\alpha)a_2)$$
$$\le \alpha(1 - \mathrm{rp}(a_1)) + (1-\alpha)(1 - \mathrm{rp}(a_2)) = \alpha\varphi_1(a_1) + (1-\alpha)\varphi_1(a_2).$$

Hence (5) holds for φ_1 in place of φ. Since $\varphi_2(\lambda) = \varphi_1(-\lambda)$, (5) also holds for φ_2 in place of φ. Now, it is easy to complete the proof.

COROLLARY 3.12. *Let t be a norm lower semicontinuous trace on A. For $p \in [1,\infty)$ write $\varphi_p(\lambda) = |\lambda|^p$ ($\lambda \in \mathbb{R}$). Then the function $a \mapsto [t(\varphi_p(a))]^{1/p}$ on $\mathfrak{m}_p = \{a \in A \mid \varphi_p(a) \in \mathfrak{m}_t^+\}$ is a seminorm, and it is a norm if the trace t is faithful. Moreover, if $a \in \mathfrak{m}_p$, $b \in A$, and $|b| \le |a|$, then $b \in \mathfrak{m}_p$ and $[t(\varphi_p(b))]^{1/p} \le [t(\varphi_p(a))]^{1/p}$.*

Proof. This follows from the properties of the functional calculus, Corollary 3.11, and Theorem 3.5.

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