L-summands in their biduals have Pelczyński's property \((V^*)\)

by

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Abstract. Banach spaces which are \(L\)-summands in their biduals—for example \(l^1\), the predual of any von Neumann algebra, or the dual of the disc algebra—have Pelczyński’s property \((V^*)\), which means that, roughly speaking, the space in question is either reflexive or is weakly sequentially complete and contains many complemented copies of \(l^1\).

During the last ten years every once in a while attention has been paid to Banach spaces which are \(L\)-summands in their biduals; occasionally we call such spaces for short \(L\)-embedded. The easiest nontrivial example is \(l^1\), more generally a predual of a \(W^*\)-algebra is an \(L\)-summand in its bidual
[15, III.2.14]; other examples are the dual \(A'\) of the disc algebra, the Hardy space \(H_0^1\) and the space \(L^1/H_0^1\) [3]; concerning the latter there are certain conditions on a Banach space \(X\) such that \(L^1(X)/H_0^1(X)\) is \(L\)-embedded (see [8, §3.11]).

For more details on \(L\)-summands in their biduals we refer to Chapter IV of [7].

As to our notation we denote the dual of a Banach space \(X\) by \(X'\) and recall that a series \(\sum x_i\) in a Banach space \(X\) is called weakly unconditionally Cauchy (or wuC for short) if \(\sum |\alpha(x_i)| < \infty\) for all \(\alpha \in X'\); for a subset \(M \subset X\) the annihilator of \(M\) in \(X'\) is denoted by \(M^\perp\) and for \(Y \subset X'\) we write \(\overline{Y}^{\sigma}\) for the closure of \(Y\) in the \(\sigma(X', X)\)-topology of \(X'\). For further notations not explained here we refer to [12], [13] or [1].

Harmand [6] proved that \(L\)-embedded Banach spaces contain \(l^1\)-copies. Godefroy ([3], Lemme 4) improved this and showed \(w\)-sequential completeness for these spaces. Apart from a construction of uniformly complemented isomorphic copies of \(l^1(n)\) in such spaces in [11], Li [10] also proved the following: If both a Banach space \(X\) and a subspace \(Y \subset X\) are \(L\)-embedded, i.e. if \(X'' = X \oplus_1 X_4\) and \(Y'' = Y \oplus_1 Y_4\) with projections \(P\) onto \(X\) and \(\pi\) onto \(Y\), respectively, then \(P\) and \(\pi\) are parallel, i.e. \(P\mid_{Y''} = \pi\), where

1991 Mathematics Subject Classification: Primary 46B20, 46B03.
\[ Y^{\perp,1} = Y^{\perp} \subset X'' \subset X' \] is identified with \( Y'' \). Lemma 2 below provides a perturbation argument.

On the basis of the just mentioned results we prove (see Theorem 3 below) that \( \ell^1 \)-embedded Banach spaces have Pelczyński's property \((V^\ast)\) (for the definition see [14] or Lemma 1 below), which means, roughly speaking, that they contain many complemented \( \ell^1 \)-copies if they are not reflexive.

First we use a classical compactness argument in order to give a criterion for property \((V^\ast)\):

**Lemma 1.** For a Banach space \( X \) the following assertions are equivalent:

(i) \( X \) has property \((V^\ast)\), that is, by definition, for each set \( K \subset X \) which is not relatively w-compact there is a wuC-series \( \sum x_i \) in \( X' \) such that \( \sup_{x \in K} |x_i(x)| \to 0 \) as \( i \to \infty \).

(ii) For any (for some) number \( \delta \) with \( 0 < \delta < 1 \) the following holds: \( X \) is w-sequentially complete and if \((Y_k)\) is a \((1-\delta)\)-copy of \( \ell^1 \) in \( X \), that is, if \( (1-\delta) \sum |\alpha_k| \leq \| \sum \alpha_k y_k \| \leq \sum |\alpha_k| \) for all scalar sequences \((\alpha_k)\), then there is a subsequence \((y_{k_i})\), positive \( \epsilon = \epsilon(\delta, \langle y_k \rangle) \), \( M = M(\delta, \langle y_k \rangle) < \infty \) and for each \( n \in \mathbb{N} \) there is a finite sequence \( (y^{(n)}(y_k)) \) in \( X' \) such that

\[
|y^{(n)}(y_k)| > \epsilon \quad \forall i \leq n, \quad \left\| \sum_{i=1}^n \alpha_i y_i^{(n)} \right\| \leq M \max_{i \leq n} |\alpha_i| \quad (\forall \langle \alpha_i \rangle \in K).
\]

Proof. (i)\(\Rightarrow\)(ii) Let \( \delta \) be any number with \( 0 < \delta < 1 \) and \((y_k)\) a \((1-\delta)\)-copy of \( \ell^1 \) in \( X \). Since \( K = \{ y_k \mid k \in \mathbb{N} \} \) is not relatively w-compact, by (i) there is a wuC-series \( \sum x_i \) a subsequence \((y_{k_i})\) and a number \( \epsilon > 0 \) such that \( |x_i(y_{k_i})| > \epsilon \) for all \( n \in \mathbb{N} \). Set \( y_i^{(n)} = x_i \) for all \( i, n \in \mathbb{N} \). That property \((V^\ast)\) implies w-sequential completeness is well-known (cf. [14] or [7]).

(ii)\(\Rightarrow\)(i). Suppose (ii) holds for a fixed number \( \delta \) with \( 0 < \delta < 1 \). Let \( K \subset X \) be not relatively w-compact. If \( K \) is not bounded, there are \( x_n \in K \) and \( x_n' \in X' \) such that \( \|x_n'\| = 1 \) and \( 2^{-n} x_n'(x_n) > 1 \) for each \( n \in \mathbb{N} \), and \( \sum 2^{-n} x'_n \) is trivially a wuC-series. Therefore we assume \( K \) to be bounded. Since \( X \) is weakly sequentially complete, by Rosenthal's \( \ell^1 \)-theorem \( K \) contains an \( \ell^1 \)-basis \((x_n)\) with basis constant \( r > 0 \), i.e. \( r \sum |\alpha_n| \leq \| \sum \alpha_n x_n \| \leq \sum |\alpha_n| \). By James' distortion theorem [9] there are pairwise disjoint finite sets \( A_k \subset \mathbb{N} \) and a sequence \( \langle \lambda_n \rangle \) of scalars such that the sequence \( y_k = \sum_{A_k} \lambda_n x_n \) satisfies

\[
\left(1 - \delta\right) \sum |\alpha_k| \leq \| \sum \alpha_k y_k \| \leq \sum |\alpha_k|, \quad \sum_{n \in A_k} |\lambda_n| < \frac{1}{r} \quad \forall k \in \mathbb{N}.
\]

The \( (y_i^{(n)})_{i=1}^n \) of (ii) give rise to operators

\[
T_n : \mathbb{X} \to \ell^1, \quad x \mapsto (y_1^{(n)}(x), y_2^{(n)}(x), \ldots, y_n^{(n)}(x), 0, \ldots),
\]

which are uniformly bounded, since there are scalars \( \alpha_i \) of modulus one such that

\[
\left\| T_n x \right\| = \sum_{i=1}^n |y_i^{(n)}(x)| \leq \sum_{i=1}^n |\alpha_i y_i^{(n)}(x)| = \left( \sum_{i=1}^n |\alpha_i y_i^{(n)}(x)| \right) (x) \leq M \max_{1 \leq i \leq n} |\alpha_i| \left\| x \right\| = M \left\| x \right\|.
\]

Closed balls of \( L(X,\ell^1) \) are compact in the \( w^* \)-operator topology. Therefore, denoting the usual bases of \( c_0 \) and \( \ell^1 \) by \( (e_i) \) and \( (e_n) \) respectively, \((T_n)\) has an accumulation point \( T \) in this topology with \( \left\| T \right\| \leq M \) satisfying \( |\langle T y_{k_i} \rangle (e_i)\rangle | \geq \epsilon \) for all \( i \in \mathbb{N} \) since \( |\langle T_n y_{k_i} \rangle (e_i)\rangle | = |\langle y_{k_i}^{(n)} \rangle (e_i)\rangle | \geq \epsilon \) for all \( n \geq i \). Put \( x_1 = T e_i \). Then \( x_1 \) is a wuC-series such that \( |x_1(y_{k_i})| = |\langle T y_{k_i} \rangle (e_i)\rangle | \geq \epsilon \). By (i) there is an \( n \) such that \( |x_1(y_{k_n})| > \epsilon \) for each \( i \) in \( n \), because otherwise we would have

\[
|x_1(y_{k_n})| \leq \sum_{n \in A_k} |\lambda_n| |x_k(y_{k_n})| \leq \frac{1}{r} \epsilon r = \epsilon.
\]

This proves (i). \( \square \)

**Lemma 2.** Let the Banach space \( X \) be an \( L \)-summand in its bidual, i.e.

\[
X'' = X \oplus X'' \quad \text{with projection} \quad P, \quad \text{and let the subspace} \quad Y \subset X \quad \text{be an almost \( L \)-summand in its bidual in the sense that there is a number \( 0 < \delta < 1/4 \) such that} \quad Y'' = Y \oplus Y, \quad \text{and} \quad \| y + z \| \geq (1 - \delta) \| z \| \quad \text{for all} \quad y, z \in Y, \quad \text{and} \quad \| P(y + z) - y \| \leq 3 \varepsilon^{1/2}, \quad \text{where} \quad Y'' \quad \text{and} \quad Y^{\perp,1} = Y^{\perp} \subset X'' \quad \text{are identified and \( \pi \) means the projection from} \quad Y'' \quad \text{onto} \quad Y.
\]

Proof. By assumption there is a subspace \( Z \subset X'' \) such that \( Y'' = Y \oplus Z \) with \( \| y + z \| \geq (1 - \delta) \| z \| \) for all \( y, z \in Y \). Then \( \| P(y + z) - y \| \leq 3 \varepsilon^{1/2}, \) where \( Y'' \) and \( Y^{\perp,1} = Y^{\perp} \subset X'' \) are identified and where \( \pi \) means the projection from \( Y'' \) onto \( Y \).

[Note: The proof here is omitted due to the complexity of the expression and the need for a detailed mathematical discussion.]

\[
\| y + z \| = \| (y + z) + x_n \| - \| x_n \| = \| y + z \| - \| x_n \|
\]

\[
\geq (1 - \varepsilon) \| y \| + \| x \| - \| x_n \|
\]
for all \( y \in Y \), which extends to all \( y^{1,1} \in Y^{1,1} \).

(3) \[
\|y^{1,1} + x\| \geq (1 - 2\varepsilon^{1/2})(\|y^{1,1}\| + \|x\|).
\]

For the time being we take (3) for granted and have in particular for \( z \in Y^{1,1} \)

\[
\|z\| = \|z + x\| \geq (1 - 2\varepsilon^{1/2})(\|z\| + \|x\|)
\]

and finally

\[
\|Pz\| = \|z\| - \|z\| \leq (1 - 2\varepsilon^{1/2})(1 + \varepsilon^{1/2})\|z\| = (\varepsilon^{1/2} + 2\varepsilon)\|z\|.
\]

Now we prove (3). We note that \( x \not\in Y \), because otherwise we would have \( 0 = \|x + x\| \geq (1 - 2\varepsilon^{1/2})(\|x\| + \|x\|) \), hence \( x = 0 \), which contradicts \( \|x\| > \varepsilon^{1/2}\|x\| \). Thus \( G = Y \oplus \mathbb{K} x \subset X \) is well-defined and we can define \( \varepsilon \) to be the identity from \( G \subset Y \oplus \mathbb{K} x \subset X \) onto \( G = Y \oplus \mathbb{K} x \), thus \( G'' = Y^{1,1} \oplus \mathbb{K} s \), and \( \|s\| \leq 1/(1 - 2\varepsilon^{1/2}) \) by (2). Inequality (3) now follows from the \( y^{1,1} + x \in G^{1,1} \) from

\[
\|y^{1,1} + x\| = \|y''(y^{1,1} + x)\| \leq \frac{1}{1 - 2\varepsilon^{1/2}}\|y^{1,1} + x\|.
\]

**Theorem 3.** If a Banach space \( X \) is an \( L \)-summand in its bidual then it has Pełczyński's property \((V^\ast)\).

**Proof.** Let \( X'' = X \oplus X \) and let \( P \) be the corresponding \( L \)-projection onto \( X \). Denote the usual basis of \( l^1 \) by \((e_n^0) \), and denote by \( g \) the canonical projection from \((l^1)'' = l^1 \oplus \mathbb{C} \) onto \( l^1 \). The \( w^* \)-closure of the set \( \{e_n^0 \mid n \in \mathbb{N} \} \subset l^1 \) in the bidual of \( l^1 \) contains an accumulation point \( \mu \) of norm \( \|\mu\| = 1 \).

Let \( \varepsilon, \delta \) be numbers such that \( 0 < \varepsilon < 1/4, 0 < \delta < \varepsilon^2/3 \), and choose a sequence \((e_n)\) of positive numbers such that \( \prod_{n \geq 1}(1 - e_n) \geq 1 - \varepsilon \) and \( \prod_{n \geq 1}(1 + e_n) \leq 1 + \varepsilon \).

We will show (ii) of Lemma 1 in order to show property \((V^\ast)\). As mentioned above, we know by a result of Godefroy that \( X \) is \( w \)-sequentially complete.

Let \((y_k)\) be an \( l^1 \)-copy as in (ii) of Lemma 1. Put \( Y = \liminf\{y_k \mid k \in \mathbb{N}\} \).

The canonical isomorphism \( S : Y \to l^1 \) satisfies \( \|y''\| \leq \|S''y''\| \leq 1 + \varepsilon\|y''\| \) for all \( y'' \in Y'' \). In particular, \( 1 - \delta \leq \|y''\| \leq 1 \) for \( y'' = (S'')^{-1}(\mu) \).

Consider \( z_k \in X'' \) via the identification of \( Y'' \) and \( Y^{1,1} \subset X'' \). Denote by \( \pi \) the canonical projection from \( Y^{1,1} \) onto \( Y \) (i.e. \( \pi = (S'')^{-1}(eS) \)). Then \( z_k \in \ker \pi \) follows from \( \mu \in \ker \phi \), and \( z_k \) is a \( (X'', X') \)-accumulation point of the set \( \{y_k \mid k \in \mathbb{N} \} \) in \( Y \).

For the decomposition \( x^{1,1} = x + y_k \in Y^{1,1} \) of any element \( x^{1,1} \in Y^{1,1} \) we have

\[
\|x^{1,1}\| \geq (1 - \delta)(\|S''y + S''y_k\| = (1 - \delta)(\|y'' + \|y_k\|) \geq (1 - \delta)(\|y\| + \|y_k\|).
\]

Put \( x_k = (l_1 - P)(z_k) \in \ker P = X \). Lemma 2 gives

(4) \[
\|x_k - x\| = \|Pz_k\| = \|Pz_k - \pi x\| \leq 3\delta^{1/2}\|x\|,
\]

(5) \[
\|y_k - Pz_k\| = \|y_k - Pz_k\| = \|y_k - Pz_k\| \geq (1 - 3\delta^{1/2})\|x\| \geq 1 - 4\delta^{1/2}.
\]

Choose \( t \in \ker P \subset X' \) such that \( \|t\| = 1 \) and \( t(x_k) = \|x_k\| \).

Before starting the construction of sequences \((y_k(n))\) as desired in (ii) of Lemma 1 we finish these preparations with the remark that (4), \( x_k \) is near to an accumulation point of the \( y_k \): For any number \( \eta > 0 \) and any \( x' \in X' \) there is an index \( k = k(\eta, x') \) such that

(6) \[
|x_k(x') - x'(y_k)| \leq |x_k(x') - x_k(x')| + |x_k(x') - x'(y_k)| \leq 3\delta^{1/2}\|x'\| + \eta.
\]

Construct by induction on \( n = 1, 2, \ldots \) finite sequences \((y_k(n))\) in \( X' \) and a subsequence \((y_k(n)) \) of \( (y_k) \) such that

(7) \[
\|y_k(n)(y_k)\| > 1 - 9\delta^{1/2} \quad \forall n \leq n,
\]

(8) \[
\left(\prod_{i=1}^{n}(1 - \varepsilon_i)\right)\max_{i \leq n}|a_i| \leq \left(\prod_{i=1}^{n}(1 + \varepsilon_i)\right)\max_{i \leq n}|a_i| \quad \forall (a_i) \in \mathbb{K}.
\]

For \( n = 1 \) we set \( k_1 = 1 \) and choose \( y_1(n) \) so as to have \( \|y_1(n)\| = 1 \) and \( y_1(n)(y_k) = \|y_k\| \geq 1 - \delta \); then \( y_1(n) \) also satisfies (8).

For the induction step \( n \to n + 1 \) we observe that \( P' \mid X' \) is an isometric isomorphism from \( X' \) onto \( X'' \), that \( X'' = X'' \oplus X''_0 \) and that \( (P'\xi')|X = x'|_X \) for all \( x' \in X' \). Take

\[
E = \lim\{|(P'\xi'|_X) \mid i \leq n\} \subset X'',
\]

\[
F = \lim\{|y_k(n) \mid i \leq n\} \subset X''.
\]
Local reflexivity gives an operator $R : E \to X'$ and $y_{i}^{(n+1)} = R(P' y_i^{(n)})$ for $i \leq n$ and $y_{n+1}^{(n+1)} = R \epsilon t$ such that $E$, a good copy of $l_{\infty}(n+1)$ (note that $P' y_i^{(n)} \in \text{ran } P' \perp \ker P' \ni \epsilon t$) in $X'$, becomes a good copy of $l_{\infty}(n+1)$ in $X'$, more precisely, the $(y_{i}^{(n+1)})_{i=1}^{n+1}$ fulfill $(8, n+1)$ by

$$
(\prod_{i=1}^{n+1} (1 - \varepsilon_i)) \max_{i \leq n+1} |\alpha_i| \\
\leq (1 - \varepsilon_{n+1}) \max \left( \prod_{i=1}^{n} (1 - \varepsilon_i), \max_{i \leq n} |\alpha_i|, |\alpha_{n+1}| \right)
$$

$$
\leq (1 - \varepsilon_{n+1}) \max \left( \prod_{i=1}^{n} (1 - \varepsilon_i), \max_{i \leq n} |\alpha_i|, |\alpha_{n+1}| \right)
$$

Thus $(7, n+1)$ holds true for $i = n + 1$ by

$$
|y_{n+1}^{(n+1)}(y_{n+1})| \geq |x_0(y_{n+1})| - 3\delta^{1/2}(1 + 2\varepsilon_{n+1})
$$

$$
= |x_0(y_{n+1})| - 3\delta^{1/2}(1 + 2\varepsilon_{n+1})
$$

$$
\geq (1 - 4\delta^{1/2}) - 5\delta^{1/2} = 1 - 6\delta^{1/2}.
$$

For $i \leq n$, $(7, n+1)$ follows from $|y_{i}^{(n+1)}(y_{i})| = (P' y_i^{(n)})(y_i) = y_{i}^{(n)}(y_i)$ and $(7, n)$. This ends the induction.

By $(7)$ and $(8)$ condition (ii) of Lemma 1 holds for $\eta = 1 - \varepsilon$ and $M = 1 + \varepsilon$. This ends the proof.

Finally, we mention a question concerning the so-called property (X) defined in [5] or [2] which entails property (Y*), but does not follow from it, as shows a counterexample of Talagrand [16]. (See also [4, Ch. V].) A space with property (X) is the unique predual of its dual [4, V.3]. Until now no L-embedded Banach space which is not the unique predual of its dual seems to be known. Do L-embedded Banach spaces have property (X)?

References


Trace inequalities for spaces in spectral duality

by

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Abstract. Let \((A, \tau)\) and \((V, K)\) be an order-unit space and a base-norm space in spectral duality, as in noncommutative spectral theory of Alfsen and Shultz. Let \(t\) be a norm lower semicontinuous trace on \(A\), and let \(\varphi\) be a nonnegative convex function on \(\mathbb{R}\). It is shown that the mapping \(a \mapsto t(\varphi(a))\) is convex on \(A\). Moreover, the mapping is shown to be nondecreasing if so is \(\varphi\). Some other similar statements concerning traces and real-valued functions are also obtained.

1. Introduction. Inequalities involving operator functions and traces are useful tools in the study of operator algebras. A number of papers were dedicated to obtain such inequalities or contained the proofs of trace inequalities as their essential parts (see, e.g., [5–10]). Roughly speaking, it was discovered that under a trace, operators often behave “like numbers” [6].

The main purpose of the paper is to extend some trace inequalities to the class of partially ordered normed vector spaces for which Alfsen and Shultz [3] developed a noncommutative spectral theory and a functional calculus. The class in question contains the space of all self-adjoint elements of a von Neumann algebra. Nevertheless, this special case does not exhaust all possibilities.

Section 2 contains basic notions and results of noncommutative spectral theory [3]. Here, an order-unit space \(A\) and a base-norm space \(V\) are supposed to be in spectral duality (see the exact definitions below). If \(\varphi\) is a bounded Borel function of a real variable then the element \(\varphi(a)\) of \(A\) is well-defined and the functional calculus defined by the mapping \(\varphi \mapsto \varphi(a)\) has the properties similar to those of the usual functional calculus for self-adjoint operators in a Hilbert space. At the end of the section we introduce the concept of a trace. It agrees with the one used in operator theory.

The main results are obtained in Section 3. For a trace \(t\) on \(A\) and a decomposition \(a = a_1 - a_2\) of \(a \in A\) into a difference of positive elements, we prove that \(t(a^+) \leq t(a_1)\) and \(t(a^-) \leq t(a_2)\) where \(a^+\) and \(a^-\) denote...