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L-summands in their biduals have Pełczyński's property (V*)

by

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Abstract. Banach spaces which are L-summands in their biduals—for example l^1, the predual of any von Neumann algebra, or the dual of the disc algebra—have Pełczyński's property (V*), which means that, roughly speaking, the space in question is either reflexive or is weakly sequentially complete and contains many complemented copies of l^1.

During the last ten years every once in a while attention has been paid to Banach spaces which are L-summands in their biduals; occasionally we call such spaces for short L-embedded. The easiest nontrivial example is l^1, more generally a predual of a W*-algebra is an L-summand in its bidual [15, III.2.14]; other examples are the dual A' of the disc algebra, the Hardy space H_0^1 and the space L^1/H_0^1 [3]; concerning the latter there are certain conditions on a Banach space X such that $L^1(X)/H_0^1(X)$ is L-embedded (see [8, §3.11]).

For more details on L-summands in their biduals we refer to Chapter IV of [7].

As to our notation we denote the dual of a Banach space X by X' and recall that a series $\sum x_i$ in a Banach space X is called *weakly unconditionally Cauchy* (or wuC for short) if $\sum |x'(x_i)| < \infty$ for all $x' \in X'$; for a subset $M \subset X$ the annihilator of M in X' is denoted by $M^\perp$ and for $Y \subset X'$ we write $\overline{Y}^{w^*}$ for the closure of Y in the $\sigma(X', X)$-topology of X'. For further notations not explained here we refer to [12], [13] or [1].

Harmand [6] proved that L-embedded Banach spaces contain l^1-copies. Godefroy ([3], Lemme 4) improved this and showed w-sequential completeness for these spaces. Apart from a construction of uniformly complemented isomorphic copies of $l^1(n)$ in such spaces in [11], Li [10] also proved the following: If both a Banach space X and a subspace $Y \subset X$ are L-embedded, i.e. if $X'' = X \oplus_1 X_s$ and $Y'' = Y \oplus_1 Y_s$ with projections P onto X and π onto Y, respectively, then P and π are parallel, i.e. $P|_{Y^{\perp\perp}} = \pi$, where

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$Y^{\perp\perp} = \overline{Y}^{w^*} \subset X''$ is identified with Y''. Lemma 2 below provides a perturbation argument.

On the basis of the just mentioned results we prove (see Theorem 3 below) that L-embedded Banach spaces have Pełczyński's property (V*) (for the definition see [14] or Lemma 1 below), which means, roughly speaking, that they contain many complemented l^1-copies if they are not reflexive.

First we use a classical compactness argument in order to give a criterion for property (V*):

LEMMA 1. *For a Banach space X the following assertions are equivalent:*

(i) *X has property* (V*), *that is, by definition, for each set $\mathcal{K} \subset X$ which is not relatively w-compact there is a wuC-series $\sum x'_i$ in X' such that $\sup_{x\in\mathcal{K}} |x'_i(x)| \nrightarrow 0$ as $i \to \infty$.*

(ii) *For any (for some) number δ with $0 < \delta < 1$ the following holds: X is w-sequentially complete and if (y_k) is a $(1-\delta)$-copy of l^1 in X, that is, if $(1-\delta)\sum|\alpha_k| \le \|\sum \alpha_k y_k\| \le \sum|\alpha_k|$ for all scalar sequences (α_k), then there are a subsequence (y_{k_n}), positive numbers $\varepsilon = \varepsilon(\delta,(y_k))$, $M = M(\delta,(y_k)) < \infty$ and for each $n \in \mathbb{N}$ there is a finite sequence $(y_i'^{(n)})_{i=1}^n \subset X'$ such that*

$$|y_i'^{(n)}(y_{k_i})| > \varepsilon \quad \forall i \le n,$$
$$\Big\|\sum_{i=1}^n \alpha_i y_i'^{(n)}\Big\| \le M \max_{i\le n}|\alpha_i| \quad \forall (\alpha_i) \subset \mathbb{K}.$$

Proof. (i)⇒(ii). Let δ be any number with $0 < \delta < 1$ and (y_k) a $(1-\delta)$-copy of l^1 in X. Since $\mathcal{K} = \{y_k \mid k \in \mathbb{N}\}$ is not relatively w-compact, by (i) there are a wuC-series $\sum x'_i$, a subsequence (y_{k_n}) and a number $\varepsilon > 0$ such that $|x'_n(y_{k_n})| > \varepsilon$ for all $n \in \mathbb{N}$. Set $y_i'^{(n)} = x'_i$ for all $i, n \in \mathbb{N}$. That property (V*) implies w-sequential completeness is well-known (cf. [14] or [7]).

(ii)⇒(i). Suppose (ii) holds for a fixed number δ with $0 < \delta < 1$. Let $\mathcal{K} \subset X$ be not relatively w-compact. If $\mathcal{K}$ is not bounded, there are $x_n \in \mathcal{K}$ and $x'_n \in X'$ such that $\|x'_n\| = 1$ and $2^{-n}x'_n(x_n) > 1$ for each $n \in \mathbb{N}$, and $\sum 2^{-n}x'_n$ is trivially a wuC-series. Therefore we assume $\mathcal{K}$ to be bounded. Since X is weakly sequentially complete, by Rosenthal's l^1-theorem $\mathcal{K}$ contains an l^1-basis (x_n) with basis constant $r > 0$, i.e. $r\sum|\alpha_n| \le \|\sum \alpha_n x_n\| \le \sum|\alpha_n|$. By James' distortion theorem [9] there are pairwise disjoint finite sets $A_k \subset \mathbb{N}$ and a sequence (λ_n) of scalars such that the sequence $y_k = \sum_{A_k} \lambda_n x_n$ satisfies

$$(1)\quad (1-\delta)\sum|\alpha_k| \le \Big\|\sum \alpha_k y_k\Big\| \le \sum|\alpha_k|, \qquad \sum_{n\in A_k} |\lambda_n| < \frac{1}{r} \quad \forall k \in \mathbb{N}.$$

The $(y_i'^{(n)})_{i=1}^n$ of (ii) give rise to operators

$$T_n : X \to l^1, \quad x \mapsto (y_1'^{(n)}(x), y_2'^{(n)}(x), \ldots, y_n'^{(n)}(x), 0, \ldots),$$

which are uniformly bounded, since there are scalars α_i of modulus one such that

$$\|T_n x\| = \sum_{i=1}^n |y_i'^{(n)}(x)| = \sum_{i=1}^n \alpha_i y_i'^{(n)}(x) = \Big(\sum_{i=1}^n \alpha_i y_i'^{(n)}\Big)(x)$$
$$\le M \max_{i\le n}|\alpha_i|\,\|x\| = M\|x\|.$$

Closed balls of $L(X, l^1)$ are compact in the w^*-operator topology. Therefore, denoting the usual bases of c_0 and l^1 by (e_n) and (e'_n) respectively, (T_n) has an accumulation point T in this topology with $\|T\| \le M$ satisfying $|(Ty_{k_i})(e_i)| \ge \varepsilon$ for all $i \in \mathbb{N}$ since $|(T_n y_{k_i})(e_i)| = |y_i'^{(n)}(y_{k_i})| > \varepsilon$ for all $n \ge i$. Put $x'_i = T'e_i$. Then $\sum x'_i$ is a wuC-series such that $|x'_i(y_{k_i})| = |(Ty_{k_i})(e_i)| > \varepsilon$. By (1) there is an x_{n_i} such that $|x'_i(x_{n_i})| > \varepsilon r$ for each $i \in \mathbb{N}$, because otherwise we would have

$$|x'_i(y_{k_i})| \le \sum_{n\in A_{k_i}} |\lambda_n||x'_i(x_n)| \overset{(1)}{<} \frac{1}{r}\varepsilon r = \varepsilon.$$

This proves (i). ■

LEMMA 2. *Let the Banach space X be an L-summand in its bidual, i.e. $X'' = X \oplus_1 X_s$ with projection P, and let the subspace $Y \subset X$ be an almost L-summand in its bidual in the sense that there is a number $0 < \varepsilon < 1/4$ such that $Y'' = Y \oplus Y_s$ and $\|y + y_s\| \ge (1-\varepsilon)(\|y\| + \|y_s\|)$ for all $y \in Y$, $y_s \in Y_s$. Then $\|P|_{Y^{\perp\perp}} - \pi\| \le 3\varepsilon^{1/2}$, where Y'' and $Y^{\perp\perp} = \overline{Y}^{w^*} \subset X''$ are identified and where π means the projection from Y'' onto Y.*

Proof. By assumption there is a subspace $Z \subset X''$ such that $Y'' \cong Y^{\perp\perp} = \overline{Y}^{w^*} = Y \oplus Z$ with $\|y + z\| \ge (1-\varepsilon)(\|y\| + \|z\|)$. Because of

$$\|Py^{\perp\perp} - \pi y^{\perp\perp}\| = \|P(y+z) - \pi(y+z)\| = \|Pz\|$$

(π denotes the projection from $Y^{\perp\perp}$ onto Y) and because of

$$(\varepsilon^{1/2} + 2\varepsilon)\|z\| \le \frac{\varepsilon^{1/2} + 2\varepsilon}{1-\varepsilon}\|y + z\| \le 3\varepsilon^{1/2}\|y + z\|$$

for any $y \in Y$, $z \in Z$, it is enough to show $\|Pz\| \le (\varepsilon^{1/2} + 2\varepsilon)\|z\|$ for each $z \in Z$. Decompose $z = x + x_s$ in $X'' = X \oplus_1 X_s$. Since we are done if $\|x\| = \|Pz\| \le \varepsilon^{1/2}\|z\|$, we assume $\|x\| > \varepsilon^{1/2}\|z\|$ from now on. We obtain

$$(2)\qquad \|y + x\| = \|(y + x) + x_s\| - \|x_s\| = \|y + z\| - \|x_s\|$$
$$\ge (1-\varepsilon)(\|y\| + \|z\|) - \|x_s\|$$

$$
\begin{aligned}
&= (1-\varepsilon)(\|y\| + \|x\| + \|x_s\|) - \|x_s\| \\
&= (1-\varepsilon)(\|y\| + \|x\|) - \varepsilon\|x_s\| \\
&\geq (1-\varepsilon)(\|y\| + \|x\|) - \varepsilon\|z\| \\
&\geq (1-\varepsilon)(\|y\| + \|x\|) - \varepsilon^{1/2}\|x\| \\
&\geq (1-2\varepsilon^{1/2})(\|y\| + \|x\|)
\end{aligned}
$$

for all $y \in Y$, which extends to all $y^{\perp\perp} \in Y^{\perp\perp}$:

$$\|y^{\perp\perp} + x\| \geq (1-2\varepsilon^{1/2})(\|y^{\perp\perp}\| + \|x\|)\,. \tag{3}$$

For the time being we take (3) for granted and have in particular for $z \in Y^{\perp\perp}$

$$
\begin{aligned}
\|x_s\| = \|-z+x\| &\geq (1-2\varepsilon^{1/2})(\|z\| + \|x\|) \\
&\geq (1-2\varepsilon^{1/2})(\|z\| + \varepsilon^{1/2}\|z\|)
\end{aligned}
$$

and finally

$$
\begin{aligned}
\|Pz\| &= \|x\| = \|z\| - \|x_s\| \\
&\leq \|z\| - (1-2\varepsilon^{1/2})(1+\varepsilon^{1/2})\|z\| = (\varepsilon^{1/2} + 2\varepsilon)\|z\|\,.
\end{aligned}
$$

Now we prove (3). We note that $x \notin Y$, because otherwise we would have $0 = \|-x+x\| \geq (1-2\varepsilon^{1/2})(\|-x\| + \|x\|)$, hence $x = 0$, which contradicts $\|x\| > \varepsilon^{1/2}\|z\|$. Thus $G = Y \oplus \mathbb{K}x \subset X$ is well-defined and we can define ι to be the identity from $G = Y \oplus \mathbb{K}x \subset X$ onto $\widetilde{G} = Y \oplus_1 \mathbb{K}x$, thus $\widetilde{G}'' \cong Y^{\perp\perp} \oplus_1 \mathbb{K}x$ and $\|\iota\| \leq 1/(1-2\varepsilon^{1/2})$ by (2). Inequality (3) now follows with $y^{\perp\perp} + x \in G^{\perp\perp}$ from

$$\|y^{\perp\perp}\| + \|x\| = \|\iota''(y^{\perp\perp} + x)\| \leq \frac{1}{1-2\varepsilon^{1/2}}\|y^{\perp\perp} + x\|\,. \ \blacksquare$$

THEOREM 3. *If a Banach space X is an L-summand in its bidual then it has Pełczyński's property* (V*).

Proof. Let $X'' = X \oplus_1 X_s$ and let P be the corresponding L-projection onto X. Denote the usual basis of l^1 by (e'_n), and denote by ϱ the canonical projection from $(l^1)'' = l^1 \oplus_1 c_0^\perp$ onto l^1. The w^*-closure of the set $\{e'_n \mid n \in \mathbb{N}\} \subset l^1$ in the bidual of l^1 contains an accumulation point $\mu \in \ker \varrho$ of norm $\|\mu\| = 1$.

Let ε, δ be numbers such that $0 < \varepsilon < 1/4$, $0 < \delta < \varepsilon^2/9^2$, and choose a sequence (ε_n) of positive numbers such that $\prod_{n\geq 1}(1-\varepsilon_n) \geq 1-\varepsilon$ and $\prod_{n\geq 1}(1+\varepsilon_n) \leq 1+\varepsilon$.

We will show (ii) of Lemma 1 in order to show property (V*). As mentioned above, we know by a result of Godefroy that X is w-sequentially complete.

Let (y_k) be an l^1-copy as in (ii) of Lemma 1. Put $Y = \overline{\mathrm{lin}}\{y_k \mid k \in \mathbb{N}\}$. The canonical isomorphism $S : Y \to l^1$ satisfies $\|y''\| \leq \|S''y''\| \leq \frac{1}{1-\delta}\|y''\|$

for all $y'' \in Y''$. In particular, $1-\delta \leq \|z_s\| \leq 1$ for $z_s = (S'')^{-1}(\mu)$. Consider $z_s \in X''$ via the identification of Y'' and $Y^{\perp\perp} \subset X''$. Denote by π the canonical projection from $Y^{\perp\perp}$ onto Y (i.e. $\pi = (S'')^{-1}\varrho S''$). Then $z_s \in Y_s = \ker \pi$ follows from $\mu \in \ker \varrho$, and z_s is a $\sigma(X'', X')$-accumulation point of the set $\{y_k \mid k \in \mathbb{N}\}$ in Y_s.

For the decomposition $y^{\perp\perp} = y + y_s$ in $Y^{\perp\perp} = Y \oplus Y_s$ of any element $y^{\perp\perp} \in Y^{\perp\perp}$ we have

$$
\begin{aligned}
\|y^{\perp\perp}\| &\geq (1-\delta)\|S''y^{\perp\perp}\| = (1-\delta)\|S''y + S''y_s\| \\
&= (1-\delta)(\|S''y\| + \|S''y_s\|) \geq (1-\delta)(\|y\| + \|y_s\|)\,.
\end{aligned}
$$

Put $x_s = (\mathrm{Id}_{X''} - P)(z_s) \in \ker P = X_s$. Lemma 2 gives

$$\|x_s - z_s\| = \|Pz_s\| = \|Pz_s - \pi z_s\| \leq 3\delta^{1/2}\|z_s\|\,, \tag{4}$$

$$\|x_s\| = \|z_s - Pz_s\| = \|z_s\| - \|Pz_s\| \geq (1-3\delta^{1/2})\|z_s\| \geq 1 - 4\delta^{1/2}\,. \tag{5}$$

Choose $t \in \ker P' \subset X'''$ such that $\|t\| = 1$ and $t(x_s) = \|x_s\|$.

Before starting the construction of sequences $(y_i'^{(n)})$ as desired in (ii) of Lemma 1 we finish these preparations with the remark that by (4), x_s is near to an accumulation point of the y_k: For any number $\eta > 0$ and any $x' \in X'$ there is an index $k = k(\eta, x')$ such that

$$
\begin{aligned}
|x_s(x') - x'(y_k)| &\leq |x_s(x') - z_s(x')| + |z_s(x') - x'(y_k)| \\
&\leq 3\delta^{1/2}\|x'\| + \eta\,.
\end{aligned} \tag{6}
$$

Construct by induction on $n = 1, 2, \ldots$ finite sequences $(y_i'^{(n)})_{i=1}^n \subset X'$ and a subsequence (y_{k_n}) of (y_k) such that

$$|y_i'^{(n)}(y_{k_i})| > 1 - 9\delta^{1/2} \quad \forall i \leq n\,, \tag{7}$$

$$
\begin{aligned}
\Big(\prod_{i=1}^n (1-\varepsilon_i)\Big) \max_{i\leq n} |\alpha_i| &\leq \Big\|\sum_{i=1}^n \alpha_i y_i'^{(n)}\Big\| \\
&\leq \Big(\prod_{i=1}^n (1+\varepsilon_i)\Big) \max_{i\leq n} |\alpha_i| \quad \forall (\alpha_i) \subset \mathbb{K}\,.
\end{aligned} \tag{8}
$$

For $n = 1$ we set $k_1 = 1$ and choose $y_1'^{(1)}$ so as to have $\|y_1'^{(1)}\| = 1$ and $y_1'^{(1)}(y_{k_1}) = \|y_{k_1}\| \geq 1-\delta$; then $y_1'^{(1)}$ also satisfies (8).

For the induction step $n \mapsto n+1$ we observe that $P'|_{X'}$ is an isometric isomorphism from X' onto $X_s^\perp$, that $X''' = X^\perp \oplus_\infty X_s^\perp$ and that $(P'x')|_X = x'|_X$ for all $x' \in X'$. Take

$$
\begin{aligned}
E &= \mathrm{lin}(\{P'y_i'^{(n)} \mid i \leq n\} \cup \{t\}) \subset X'''\,, \\
F &= \mathrm{lin}(\{y_{k_i} \mid i \leq n\} \cup \{x_s\}) \subset X''\,.
\end{aligned}
$$

Local reflexivity gives an operator $R: E \to X'$ and $y_i'^{(n+1)} = R(P'y_i'^{(n)})$ for $i \le n$ and $y_{n+1}'^{(n+1)} = Rt$ such that E, a good copy of $l^\infty(n+1)$ (note that $P'y_i'^{(n)} \in \operatorname{ran} P' \perp \ker P' \ni t$) in X''', becomes a good copy of $l^\infty(n+1)$ in X', more precisely the $(y_i'^{(n+1)})_{i=1}^{n+1}$ fulfill $(8, n+1)$ by

$$\begin{aligned}
\Big(\prod_{i=1}^{n+1}(1-\varepsilon_i)\Big)\max_{i\le n+1}|\alpha_i| &\le (1-\varepsilon_{n+1})\max\Big(\Big(\prod_{i=1}^{n}(1-\varepsilon_i)\Big)\max_{i\le n}|\alpha_i|, |\alpha_{n+1}|\Big)\\
&\overset{(7,n)}{\le} (1-\varepsilon_{n+1})\max\Big(\Big\|\sum_{i=1}^{n}\alpha_i y_i'^{(n)}\Big\|, \|\alpha_{n+1}t\|\Big)\\
&= (1-\varepsilon_{n+1})\max\Big(\Big\|\sum_{i=1}^{n}\alpha_i P'y_i'^{(n)}\Big\|, \|\alpha_{n+1}t\|\Big)\\
&= (1-\varepsilon_{n+1})\Big\|\Big(\sum_{i=1}^{n}\alpha_i P'y_i'^{(n)}\Big) + \alpha_{n+1}t\Big\|\\
&\overset{\text{loc. refl.}}{\le} \Big\|\sum_{i=1}^{n+1}\alpha_i y_i'^{(n+1)}\Big\|\\
&\overset{\text{loc. refl.}}{\le} (1+\varepsilon_{n+1})\Big\|\Big(\sum_{i=1}^{n}\alpha_i P'y_i'^{(n)}\Big) + \alpha_{n+1}t\Big\|\\
&= (1+\varepsilon_{n+1})\max\Big(\Big\|\sum_{i=1}^{n}\alpha_i P'y_i'^{(n)}\Big\|, \|\alpha_{n+1}t\|\Big)\\
&= (1+\varepsilon_{n+1})\max\Big(\Big\|\sum_{i=1}^{n}\alpha_i y_i'^{(n)}\Big\|, \|\alpha_{n+1}t\|\Big)\\
&\overset{(7,n)}{\le} (1+\varepsilon_{n+1})\max\Big(\Big(\prod_{i=1}^{n}(1+\varepsilon_i)\Big)\max_{i\le n}|\alpha_i|, |\alpha_{n+1}|\Big)\\
&\le \Big(\prod_{i=1}^{n+1}(1+\varepsilon_i)\Big)\max_{i\le n+1}|\alpha_i|.
\end{aligned}$$

Before we show $(7, n+1)$ let us use (6) with $\eta \le 3\delta^{1/2}\varepsilon_{n+1}$ in order to find $y_{k_{n+1}}$ so as to have

$$|x_s(y_{n+1}'^{(n+1)}) - y_{n+1}'^{(n+1)}(y_{k_{n+1}})| \le 3\delta^{1/2}\|y_{n+1}'^{(n+1)}\| + \eta \overset{\text{loc. refl.}}{\le} 3\delta^{1/2}(1+2\varepsilon_{n+1}).$$

Thus $(7, n+1)$ holds true for $i = n+1$ by

$$\begin{aligned}
|y_{n+1}'^{(n+1)}(y_{k_{n+1}})| &\ge |x_s(y_{n+1}'^{(n+1)})| - 3\delta^{1/2}(1+2\varepsilon_{n+1})\\
&\overset{\text{loc. refl.}}{=} |t(x_s)| - 3\delta^{1/2}(1+2\varepsilon_{n+1})\\
&\overset{(5)}{\ge} (1-4\delta^{1/2}) - 5\delta^{1/2} = 1 - 9\delta^{1/2}.
\end{aligned}$$

For $i \le n$, $(7, n+1)$ follows from $|y_i'^{(n+1)}(y_{k_i})| = (P'y_i'^{(n)})(y_{k_i}) = y_i'^{(n)}(y_{k_i})$ and $(7, n)$. This ends the induction.

By (7) and (8) condition (ii) of Lemma 1 holds for $\eta = 1-\varepsilon$ and $M = 1+\varepsilon$. This ends the proof. ∎

Finally, we mention a question concerning the so-called property (X) defined in [5] or [2] which entails property (V*), but does not follow from it, as shows a counterexample of Talagrand [16]. (See also [4, Ch. V].) A space with property (X) is the unique predual of its dual [4, V.3]. Until now no L-embedded Banach space which is not the unique predual of its dual seems to be known. Do L-embedded Banach spaces have property (X)?

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Trace inequalities for spaces in spectral duality

by

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Abstract. Let (A, e) and (V, K) be an order-unit space and a base-norm space in spectral duality, as in noncommutative spectral theory of Alfsen and Shultz. Let t be a norm lower semicontinuous trace on A, and let φ be a nonnegative convex function on $\mathbb{R}$. It is shown that the mapping $a \mapsto t(\varphi(a))$ is convex on A. Moreover, the mapping is shown to be nondecreasing if so is φ. Some other similar statements concerning traces and real-valued functions are also obtained.

1. Introduction. Inequalities involving operator functions and traces are useful tools in the study of operator algebras. A number of papers were dedicated to obtain such inequalities or contained the proofs of trace inequalities as their essential parts (see, e.g., [5–10]). Roughly speaking, it was discovered that under a trace, operators often behave "like numbers" [6].

The main purpose of the paper is to extend some trace inequalities to the class of partially ordered normed vector spaces for which Alfsen and Shultz [3] developed a noncommutative spectral theory and a functional calculus. The class in question contains the space of all self-adjoint elements of a von Neumann algebra. Nevertheless, this special case does not exhaust all possibilities.

Section 2 contains basic notions and results of noncommutative spectral theory [3]. Here, an order-unit space A and a base-norm space V are supposed to be in spectral duality (see the exact definitions below). If φ is a bounded Borel function of a real variable then the element $\varphi(a)$ of A is well-defined and the functional calculus defined by the mapping $\varphi \mapsto \varphi(a)$ has the properties similar to those of the usual functional calculus for self-adjoint operators in a Hilbert space. At the end of the section we introduce the concept of a trace. It agrees with the one used in operator theory.

The main results are obtained in Section 3. For a trace t on A and a decomposition $a = a_1 - a_2$ of $a \in A$ into a difference of positive elements, we prove that $t(a^+) \le t(a_1)$ and $t(a^-) \le t(a_2)$ where a^+ and a^- denote

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