and therefore
\[ 2n^2 T^{n_0 w} = T^{n_0} \left( \sum_{j=1}^{\nu} 2^{j n_0 + n_0 (n_0 - 1)/2} \beta_j x_j + n_0 \right) \]
\[ = \sum_{j=1}^{\nu} \left( \frac{1}{2} \right)^{j n_0 + n_0 (n_0 - 1)/2} 2^{j n_0 + n_0 (n_0 - 1)/2} \beta_j x_j = v. \]
So \( \|2n^2 T^{n_0 w} - v\| = 0 \leq \epsilon. \)

4. Final remarks. 1) By analogy with the proof of Theorem 1, one can prove that for a complex separable Banach space there are operators with supercyclic vectors if and only if \( \dim X \in \{0, 1, 2\} \) or \( \dim X = \infty \).

2) In the infinite-dimensional case, the operator \( T \) in the proof of Theorem 1 is compact. An operator with hypercyclic vectors cannot be compact (see [6]).

3) In [2, p. 42] a supercyclic vector \( x \in X \) for \( T \in B(X) \) and \( X \) real is defined by \( \{ \lambda y : y \in \text{Orb}(T, x), \lambda > 0 \} = X \). Also with this definition, Theorem 1 holds with exactly the same proof with the only difference that in the case \( \dim X = 1 \) we take minus identity.

References


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We denote by \( L_+(\mu^*) \) the subset of \( L(\mu^*) \) consisting of all functions \( f \) satisfying the condition
\[
\lim_{h \to 0^+} \int_{-\infty}^{\infty} |f(x+y+h) - f(x+y)| \mu(dy) = 0
\]
for all \( x \in (-\infty, \infty) \). The following statement is evident.

**Lemma 1.2.** The set \( L_+(\mu^*) \) is invariant under multiplication by bounded functions continuous on the right. For any \( f \in L_+(\mu^*) \), the function \( f \ast \mu \) is continuous on the right and finite everywhere.

We denote by \( L(\mu, \infty) \) the space of all complex-valued Borel functions \( f \) defined on the real line for which \( |f| \ast \mu \in L_\infty(\gamma) \). The norm in \( L(\mu, \infty) \) is given by the formula
\[
\|f\|_{L(\mu, \infty)} = \|f \ast \mu\|_\gamma.
\]
By (1.1) we have \( \|f\|_{L(\mu, \infty)} \geq \|f\|_{\mu^*} \), which yields \( L(\mu, \infty) \subset L(\mu^*) \). It is clear that all bounded Borel functions with bounded support belong to \( L(\mu, \infty) \). Moreover, by the shift-boundedness of \( \mu_1 \), for any \( a \in (0, \infty) \) the function \( e^{-ax}1_{[0, \infty)}(x) \) belongs to \( L(\mu, \infty) = L_+(\mu^*) \).

Let \( B \) be a set of complex-valued functions. We shall denote by \( \operatorname{Re} B \) the subset of \( B \) consisting of real-valued functions.

Let \( M \) be a bounded Borel measure on \((-\infty, \infty)\), and \( a \in (-\infty, \infty) \). We shall denote by \( e(a, M) \) the infinitely divisible probability measure with characteristic function
\[
\tilde{e}(a, M)(s) = \exp \left( ias + \int_{-\infty}^{\infty} \left( e^{i \xi x} - 1 - \frac{i \xi x - 1/2}{1 + x^2} \right) M(dx) \right).
\]
Moreover, for a bounded Borel measure \( N \) concentrated on the half-line \([0, \infty)\), we shall denote by \( \tilde{e}_+(N) \) the infinitely divisible probability measure concentrated on \([0, \infty)\) with Laplace transform
\[
\tilde{e}_+(N)(z) = \exp \int_0^\infty (e^{-xz} - 1) \frac{1 + x^2}{x^2} N(dx)
\]
for \( z \in [0, \infty) \). The function
\[
B(N, z) = \int_0^\infty (1 - e^{-zx}) \frac{1 + x^2}{x^2} N(dx)
\]
is called the Bernstein function. In the sequel we write briefly \( B(z) \) instead of \( B(N, z) \) if it causes no confusion.
Let \( X(t, \omega), t \in [0, \infty), \) be a real-valued stochastic process with stationary and independent increments, with sample functions continuous on the right and satisfying the initial condition \( X(0, \omega) = 0. \) Denote by \( \tau_t \) the probability distribution of \( X(t, \omega). \) It is well-known that
\[
\tau_t = e(\alpha t, tM)
\]
for some \( \alpha \) and \( M. \) Similarly, for nonnegative processes we have
\[
\tau_t = e_+(\alpha tN)
\]
for some measure \( N. \) In this case the Bernstein function \( B(N, z) \) is said to be associated with the process \( X(t, \omega). \)

The process \( X(t, \omega) \) is said to be transient if the potential
\[
\varrho(A) = \int_0^\infty \tau_t(A) \, dt
\]
is finite for all bounded Borel sets \( A. \) By Proposition 13.10 in [4] the measure \( \varrho \) is always shift-bounded. It is known that \( X(t, \omega) \) is transient if and only if the function
\[
\text{Re}(\log \varrho(s))^{-1}
\]
is integrable in a neighbourhood of the origin ([4], 13.17). Moreover, each nonnegative process which is not identically zero is transient and the Laplace transform of its potential is given by the formula
\[
\varrho(z) = B(z)^{-1}
\]
for \( z \in (0, \infty) \) where \( B(z) \) is the associated Bernstein function ([4], 14.1).

Throughout this paper the processes \( X(t, \omega) \) will tacitly be assumed to be transient.

2. Integral functionals. Let \( X(t, \omega) \) be a transient process with potential \( \varrho. \) This section is devoted to the study of the probability distribution of the functionals \( \int_0^\infty f(X(t, \omega)) \, dt \) for \( f \in \text{Re} L(q^*). \) We shall use the notation
\[
I(f, x, \omega) = \int_0^\infty f(X(t, \omega) + x) \, dt
\]
provided the right-hand side integral is well-defined.

Suppose that \( \langle f, x_0, \omega \rangle \) is finite. By standard calculations the expectation of \( I(f, x_0, \omega) \) is
\[
EI(f, x_0, \omega) = \langle f, x_0, \omega \rangle.
\]
It follows from (1.1) that for any \( f \in L(q^*) \) the convolution \( f \ast \varrho \) is finite almost everywhere. This yields the following statement.

**Proposition 2.1.** Let \( f \in L(q^*). \) Then \( I(f, x, \omega) \) is finite with probability 1 for almost every \( x \) and
\[
EI(f, x, \omega) = \langle f, x, \omega \rangle
\]
amost everywhere. If in addition \( f \in L_+(q^*), \) then \( I(f, x, \omega) \) is finite with probability 1 for every \( x, \) formula (2.2) holds for every \( x \) and
\[
\lim_{\theta \to 0^+} E[I(f, x + u, \omega) - I(f, x, \omega)] = 0
\]
for every \( x. \)

Consider the characteristic function \( Q(f, x, s) = e^{isI(f, x, \omega)} \) for \( f \in \text{Re} L(q^*). \)

**Theorem 2.1.** Let \( f \in \text{Re} L(q^*). \) Then for every \( s \in (-\infty, \infty) \) the function \( Q(f, \cdot, s) \) satisfies the equation
\[
Q(f, x, s) = 1 + is \int_{-\infty}^{\infty} f(x + y)Q(f, x + y, s) \varrho(dy)
\]
for almost every \( x. \) If in addition \( f \in \text{Re} L_+(q^*), \) then the above equation is satisfied for every \( x. \)

**Proof.** Introduce the notation
\[
F_s(f)(x) = Q(f, x, s), \quad G_s(f)(x) = f(x)Q(f, x, x).
\]
First we shall prove that for any \( s \in (-\infty, \infty), \) \( F_s(\cdot) \) and \( G_s(\cdot) \ast \varrho \) are continuous mappings from \( \text{Re} L(q^*) \) into \( L(\gamma). \) Taking into account the obvious inequality
\[
\|F_s(f)\|_\infty \leq 1
\]
and (1.1) we infer that \( F_s(f) \in L(\gamma) \) and \( G_s(f) \ast \varrho \in L(\gamma) \) for every \( f \in \text{Re} L(q^*). \) Further, for any \( f, g \in \text{Re} L(q^*) \) we have, by (2.2),
\[
|F_s(f)(x) - F_s(g)(x)| \leq |s|E[I(f, x, \omega) - I(g, x, \omega)] \leq |s|\|f - g\|_\gamma.
\]
almost everywhere, which yields, by (1.1),
\[
\|F_s(f) - F_s(g)\|_\gamma \leq |s|\|f - g\|_\gamma.
\]
Consequently, the mapping \( F_s(\cdot) \) from \( \text{Re} L(q^*) \) into \( L(\gamma) \) is continuous. Now from (2.5) and Lemma 1.1 it follows that \( G_s(\cdot) \ast \varrho \) is continuous from \( \text{Re} L(q^*) \) into \( L(q^*). \) Consequently, by (1.1), \( G_s(\cdot) \ast \varrho \) is continuous from \( \text{Re} L(q^*) \) into \( L(\gamma). \)

Equation (2.4) can be written in the form
\[
F_s(f) = 1 + is(G_s(f) \ast \varrho).
\]
Since both sides of the above equation are continuous mappings from \( \text{Re} L(q^*) \) into \( L(\gamma) \) it suffices to prove it for real-valued continuous functions \( f \) with a bounded support, which form a dense subset of \( \text{Re} L(q^*). \)
For such \( f \) and \( u \in (0, \infty) \) we denote by \( Q_u(f, x, s) \) the characteristic function of the functional \( \int_0^u \int f(x, \omega + z) \, dz \). By the Skorokhod Theorem ([8], Chapter 4.1, 6),
\[
Q_u(f, x, s) = 1 + i u \int_0^u f(x + y) \, Q_{u-1}(f, x + y, s) \, \tau_y(dy) \, dt.
\]
It is clear that \( Q_u(f, x, s) \to Q(f, x, s) \) as \( u \to \infty \). Moreover, the right-hand side of the above equation tends to \( 1 + i u \tau(G_2(\gamma) + q(x)) \) as \( u \to \infty \), which yields (2.6) for real-valued continuous functions \( f \) with a bounded support. This completes the proof of (2.4).

Suppose now that \( f \in R\ell E_L(\gamma) \). Then, by Lemma 1.2 and Proposition 2.1, both sides of (2.4) are continuous on the right in \( x \), which yields the assertion of the theorem.

**Theorem 2.2.** Let \( f \in R\ell E_L(\gamma) \), \( \|f\|_{0, \infty} < 1 \) and \( A \supset \text{supp } f \). Suppose that a Borel function \( H_s \) satisfies \( H_s 1_A \in L_{0, \gamma} \) and
\[
H_s(x) = 1 + i u \int f(x + y) \, H_s(x + y) \, \rho(dy)
\]
for almost every \( x \in A \). Then \( H_s(x) = Q(f, x, s) \) for almost every \( x \in A \).

**Proof.** Setting \( U_s(x) = Q(f, x, s) 1_A(x) - H_s(x) 1_A(x) \), we get a function belonging to \( L_{0, \gamma} \) and, by the obvious equality \( f(x) = f(x) 1_A(x) \), and by Theorem 2.1.
\[
U_s(x) = i u 1_A(x) \int f(x + y) \, U_s(x + y) \, \rho(dy)
\]
for almost every \( x \in (-\infty, \infty) \). Consequently, by (1.5),
\[
\|U_s\|_{0, \infty} \leq \|s\| \|U_s\|_{0, \gamma} \|f\|_{0} \gamma = \|s\| \|U_s\|_{0, \gamma} \|f\|_{0, \infty},
\]
which yields \( \|U_s\|_{0, \infty} = 0 \). This implies the assertion of the theorem.

Given \( f \in R\ell E_L(\gamma) \) we define a sequence of functions \( q_n(f, x) \) \( (n = 0, 1, \ldots) \) on the real line by setting
\[
q_0(f, x) = 1,
\]
\[
q_{n+1}(f, x) = \int f(x + y) \, q_n(f, x + y) \, \rho(dy) \quad (n = 0, 1, \ldots).
\]
By induction we get
\[
\|q_n(f, x)\|_{0, \infty} \leq \|f\|_{0, \infty} \quad (n = 0, 1, \ldots),
\]
which shows that \( q_n(f, \cdot) \in L_{0, \gamma} \). If in addition \( f \in L_{+}(\gamma) \), then, by Lemma 1.2, the functions \( q_n(f, \cdot) \) are continuous on the right.

**Theorem 2.3.** Let \( f \in \text{Re } L(\gamma, \infty) \). For almost every \( x \) the characteristic function \( Q(f, x, \cdot) \) can be extended to an analytic function in the strip \( \|x\|_{0, \infty} < 1 \), and in the circle \( \|x\|_{r, \infty} < 1 \) it has the power series representation
\[
Q(f, x, z) = \sum_{n=0}^{\infty} (iz)^n q_n(f, x)
\]
for almost every \( x \). In addition if \( f \in L_{+}(\gamma) \), then the above formula is true for every \( x \).

**Proof.** By (2.9) the function \( H_s(x) = \sum_{n=0}^{\infty} (iz)^n q_n(f, x) \) is analytic in the circle \( \|x\|_{r, \infty} < 1 \) for almost every \( x \). Further, for any fixed real \( x \) the function \( H_s(x) \) belongs to \( L_{0, \gamma} \) and, by (2.8), satisfies equation (2.7) for almost every \( x \). Applying Theorem 2.2 we get \( Q(f, x, s) = H_s(x) \) for almost every \( x \). Hence \( Q(f, x, \cdot) \) can be extended to an analytic function in the circle \( \|x\|_{r, \infty} < 1 \) and, consequently, in the strip \( \|x\|_{r, \infty} < 1 \) ([6], p. 212). The last assertion is an immediate consequence of the continuity on the right of the functions \( Q(f, \cdot, s) \) and \( q_n(f, \cdot) \) \( (n = 0, 1, \ldots) \) for \( f \in L_{+}(\gamma) \).

As a consequence of the above theorem we get a formula for the moments of the functional \( I(f, x, \omega) \).

**Corollary 2.1.** For any \( f \in R\ell E_L(\gamma) \) and almost every \( x \)
\[
E I^n(f, x, \omega) = n! q_n(f, x) \quad (n = 0, 1, \ldots).
\]
If in addition \( f \in L_{+}(\gamma) \), then the above formula is true for every \( x \).

For nonnegative functions \( f \) from \( L_{+}(\gamma) \) we denote by \( H(f, x, z) \) the Laplace transform of the probability distribution of \( I(f, x, \omega) \), i.e.
\[
H(f, x, z) = E e^{-zf(f, x, \omega)}
\]
for \( z \in [0, \infty) \). As an immediate consequence of Theorems 2.1–2.3 we get the following statement.

**Theorem 2.4.** Let \( f \) be a nonnegative function from \( L(\gamma, \infty) \). Then for any \( x \) satisfying \( \|x\|_{0, \infty} < 1 \) the function \( H(f, x, z) \) is the only solution of the equation
\[
H(f, x, z) = 1 - z \int f(x + y) H(f, x, y, z) \, \rho(dy)
\]
for almost every \( x \in \text{supp } f \). Moreover,
\[
H(f, x, z) = \sum_{n=0}^{\infty} (-z)^n q_n(f, x)
\]
almost everywhere. In addition if \( f \in L_{+}(\gamma) \), then (2.11) and (2.12) hold for every \( x \).
3. Examples. The results of the preceding section may serve for the determining of the probability distribution of functionals $I(f, z, \omega)$. We shall illustrate this by some examples.

Example 3.1 (A compound Poisson process). Let $X(t, \omega)$ be a process with probability distribution $\tau_1 = c_\pm(N_1)$ where $N_1(dx) = \frac{1}{1 + x} e^{-x} dx$. The associated Bernstein function is $B(x) = \frac{1}{1 + x}$, which, by (1.8), yields $q(dx) = \delta_0(dx) + 1_{[0, \infty)}(x) dx$ where $\delta_0$ is the probability measure concentrated at the origin. For any $f \in L(\rho, \infty)$ and almost every $x$ we have

$$Q(f, x, s) = (1 - isf(x))^{-1} \exp \left( is \int_0^\infty \frac{f(y)}{1 - isf(y)} dy \right).$$

(3.1)

In fact, by Theorem 2.1,

$$Q(f, x, s) = 1 + isf(x)Q(f, x, s) + is \int_0^\infty f(y)Q(f, y, s) dy$$

for almost every $x$. It is easy to verify that the right-hand side of (3.1) also satisfies the above equation, which, by Theorem 2.2, yields formula (3.1). Observe that for every $f \in L(\rho, \infty)$ and almost every $x$ the random variable $I(f, x, \omega)$ is infinitely divisible.

Example 3.2 (Nonnegative stable processes). Let $X(t, \omega)$ be a nonnegative stable process with associated Bernstein function $B(x) = x^\beta$ where $p \in (0, 1)$. From (1.8) we get

$$\vartheta(dx) = \frac{x^{p-1}}{\Gamma(p)} 1_{[0, \infty)}(x) dx.$$  

(3.2)

Given $b \in (0, \infty)$ we put

$$M_b(dx) = \frac{1}{\Gamma(p)} \frac{1}{1 - \frac{b}{p + b}} 1_{(\infty, 0)}(x) dx,$$

$$M_b(dx) = \frac{|x| \left( \exp \frac{px}{p + b} - \exp \frac{x}{p + b} \right)}{(1 + x^2)(1 - \exp x) \left(1 - \exp \frac{x}{p + b} \right)} 1_{(-\infty, 0)}(x) dx,$$

$$M_b(dx) = \int_{-\infty}^\infty \left( \frac{x^{-1} + (1 + x^{-2})}{1 + x^{-2}} \right) \frac{1}{\Gamma(p)} \frac{1 + x^{-2}}{1 - \frac{x^{-1}}{p + b}} 1_{(-\infty, 0)}(x) dx.$$  

(3.3)

We shall show that $e(a_b, M_b)$ is the probability distribution of the random variable

$$\log \int_0^\infty X_b(t, \omega) 1_{[0, 1]}(X(t, \omega)) dt.$$  

First observe that the functions $f_b(x) = x^\beta 1_{[0, 1]}(x)$ belong to $L(\rho, \infty) \cap L_+(\rho^\alpha)$ and, by (2.8) and (3.2),

$$q_n(f_b, 0) = \frac{1}{\Gamma(p)} \int_{b_n}^\infty \cdots \int_{b_n}^\infty \frac{y_1^b y_2^b \cdots y_n^b y_{1-p}^b \cdots y_{n-p}^b dy_1 \cdots dy_n}{\Gamma(p)}$$

where $b_n = \{(y_1, \ldots, y_n) : \sum_{i=1}^n y_i < 1, y_i > 0, j = 1, \ldots, n\}$. Taking the one-to-one mapping $(y_1, \ldots, y_n) \rightarrow (y_1, \ldots, y_n)$ from $B_n$ onto the $n$th Cartesian power of the interval $(0, 1)$ defined by

$$y_1 = u_1 \cdots u_{k-1}, \quad y_k = u_{k+1} \cdots u_{n+1} (1 - u_{k-1}) (k = 2, \ldots, n)$$

we get, by Corollary 2.1, the formula

$$E I^n(f_b, 0, \omega) = \frac{n!}{\Gamma(p)} \frac{1}{\Gamma(p)} \int_0^1 \cdots \int_0^1 u_n^{(p+b)-1} \prod_{k=1}^{n-1} u_k^{(p+b)-1} (1 - u_k)^p du_1 \cdots du_n.$$  

Hence, by simple calculation we get

$$E I^n(f_b, 0, \omega) = (p + b)^{-n} \prod_{k=0}^{n-1} \frac{\Gamma(k(p + b) + 1)}{\Gamma(k(p + b) + p)} (n = 1, 2, \ldots).$$  

(3.4)

Let $Y_0(\omega)$ be a random variable with probability distribution $e(a_0, M_0)$. It is clear that the characteristic function $e(a_0, M_0)$ can be extended to an analytic function in the half-plane $\text{Im} z < 0$. Thus

$$E \exp(nY_0(\omega)) = e(a_0, M_0)(-in) \quad (n = 1, 2, \ldots).$$  

(3.5)

Applying Malmsten's formula (1.1, 1.9),

$$\log \Gamma(z) = \int_0^\infty \left( z - 1 - \frac{e^{-x} - 1}{1 - e^{-x}} \right) e^{-x} dx \quad (\text{Re} z > 0).$$  

(3.6)

to the right-hand side of (3.3) we get, by simple calculation,

$$E I^n(f_b, 0, \omega) = e(a_0, M_0)(-in) \quad (n = 1, 2, \ldots).$$  

(3.7)

By Theorem 2.3 the probability distribution of $I(f_b, 0, \omega)$ is uniquely determined by its moments. Comparing (3.4) and (3.6) we conclude that $e(a_0, M_0)$ is the probability distribution of $\log I(f_b, 0, \omega)$.

Example 3.3 (Brownian motion with drift). Let $X(t, \omega) = \text{W}(t, \omega) + ct$ where $\text{W}(t, \omega)$ is the standard Brownian motion and the drift coefficient $c$ is positive. The potential of this process is of the form

$$\vartheta(dx) = c^{-1} 1_{[0, \infty)}(x) dx + c^{-1} e^{2cx} 1_{(-\infty, 0)}(x) dx.$$  

(3.8)

Denote by $a_1(r) < a_2(r) < \cdots$ the sequence of all positive zeros of the Bessel function $J_\nu (r > -1)$. Define a finite Borel measure $N_0$ on the half-line $[0, \infty)$
by setting
\[ N_0(dx) = (1 + x)^{-1} \left( \sum_{n=1}^{\infty} \exp(-8^{-1}a_n^2(2c - 1)x) \right) dx. \]

We shall prove that \( e_+(N_0) \) is the probability distribution of the functional
\[ \int_0^\infty e^{-X(t,\omega)} 1_{[0,\infty)}(X(t,\omega)) \, dt. \]

Observe that the Laplace transform of \( e_+(N_0) \) is
\[ E_+(N_0)(x) = \prod_{n=1}^\infty \left( 1 + 8\varepsilon a_n^{-2}(2c - 1) \right)^{-1} = \frac{\langle 2x \rangle}{\sqrt{2\pi} I_1(2c) F_{2c-1}(\sqrt{2x})} \]

where \( I_r \) denotes the modified Bessel function of the first kind. As we mentioned in Section 1 the function \( f_0(x) = e^{-x} 1_{[0,\infty)}(x) \) belongs to \( L(\varrho, \infty) \cap L_1(\varrho^+) \). Taking into account (3.7) we infer, by Theorem 2.4, that the Laplace transform \( H(f_0, x, z) \) of the probability distribution of the functional \( I(f_0, x, \omega) \) is the only solution of the equation
\[ H(f_0, x, z) = 1 - c^{-1} z \int_x^\infty e^{-y} H(f_0, y, z) \, dy 
- c^{-1} e^{-2cZ} \int_0^z e^{(2c-1)y} H(f_0, y, z) \, dy \]

for \( x \in \text{supp} f_0 = [0, \infty) \) and \( |z|/\|f_0\|_{\varrho, \infty} < 1 \).

In order to solve the above equation we introduce auxiliary functions \( F_r(u, z) \) for \( r \in (0, \infty) \), \( u \in [0, 1] \) and \( z \in [0, \infty) \) by setting
\[ F_r(u, z) = \frac{(uz)^r}{2^{r-1} F(r) I_{r-1}(x)} (I_r(uz) K_{r-1}(z) + K_r(uz) I_{r-1}(z)) \]

where \( K_r \) denotes the modified Bessel function of the third kind. Using the well-known properties of the modified Bessel functions \( I_r \) and \( K_r \) (\cite{2}, 7.11, formulae (19)–(22) and (39)) we can easily check that
\[ F_r(u, z) = 1 - (2r^{-1} z)^2 \int_0^1 y F_r(y, z) \, dy 
- (2r^{-1} z)^2 \int_1^\infty y^{-2r} F_r(y, z) \, dy \]

and
\[ F_r(1, z) = \frac{z^{r-1}}{2^{r-1} F(r) I_{r-1}(z)}. \]

Moreover, taking into account the asymptotic behaviour of \( I_r \) and \( K_r \) as \( x \to \infty \) (\cite{2}, 7.4, formula (1)) we have
\[ \lim_{u \to 0^+} F_r(u, u^{-1} z) = \frac{z^r K_r(z)}{2^{r-1} F(r)}. \]

Starting from equation (3.11) one can easily check that the function \( F_{2c}(e^{-x/2}, \sqrt{2z}) \) satisfies equation (3.9), which yields
\[ H(f_0, x, z) = F_{2c}(e^{-x/2}, \sqrt{2z}) \]

for \( x \in [0, \infty) \) and \( z \in [0, \infty) \). In particular, \( H(f_0, 0, z) = F_{2c}(1, \sqrt{2z}) \), which, by (3.8) and (3.12), shows that \( H(f_0, 0, z) \) is the Laplace transform of \( e_+(N_0) \). This completes the proof of our assertion.

Observe that, by the strong law of large numbers for the Brownian motion,
\[ \lim_{t \to \infty} \frac{X(t, \omega)}{t} = c \]

with probability 1. Consequently, the integral \( \int_0^{\infty} e^{-X(t, \omega)} \, dt \) is finite with probability 1. Denote by \( H(x) \) the Laplace transform of its probability distribution. Since \( \lim_{x \to \infty} e^{-X(x, \omega)} = \int_0^{\infty} e^{-X(t, \omega)} \, dt \) with probability 1, we have \( \lim_{x \to \infty} H(f_0, x, e^{-x/2} z) = H(x) \), which, by (3.13) and (3.14), yields
\[ H(z) = F_{2c}(e^{-z/2}, \sqrt{2z}) \]

for \( z \in [0, \infty) \). Since \( K_{2c}(\sqrt{2z}) \) is a bounded function, the random variable \( \int_0^{\infty} e^{-X(t, \omega)} \, dt \) has probability density function \( F_{2c}(1, e^{-x/2}, \sqrt{2z}) \) when \( x \) and \( \omega \) are independent.

**Example 3.4 (Nonnegative processes).** Let \( X(t, \omega) \) be a nonnegative process with Bernstein function \( B(z) \). We shall prove that for any \( u \in (0, \infty) \)

\[ E \left( \int_0^{\infty} e^{-uX(t, \omega)} \, dt \right) = \mathfrak{n}! \prod_{k=1}^{\mathfrak{n}} B(ku)^{-1} \quad (n = 1, 2, \ldots). \]

As we mentioned in Section 1 the functions \( g_u(x) = e^{-ux} 1_{[0, \infty)}(x) \) belong to \( L(\varrho, \infty) \cap L_1(\varrho^+) \) for every \( u \in (0, \infty) \); it is clear that the potential \( \varrho \) of the process in question is concentrated on the half-line \([0, \infty)\). Moreover, \( I(g_u, x, \omega) = e^{-ux} I(g_u, 0, \omega) \) for \( x \in [0, \infty) \) and
\[ I(g_u, 0, \omega) = \int_0^{\infty} e^{-uX(t, \omega)} \, dt. \]

Consequently, by Corollary 2.1, \( g_u(x, \omega) = e^{-ux} g_u(x, \omega) \) for \( x \in (0, \infty) \), which, by (2.8), yields
\[ g_{n+1}(x, 0) = g_n(x, 0) \int_0^{\infty} e^{-(n+1)y} g(dy) \quad \text{for} \quad g_n(x, 0) = g_n((n+1)u). \]

Applying formula (1.8) and Corollary 2.1 we get our assertion.
4. A family of probability measures. Define a bounded Borel measure $M_0$ on the real line and a constant $a_0$ by setting

$$M_0(dx) = \frac{|x|e^x}{(1 - e^x)(1 + x^2)}1_{(-\infty, 0)}(x) \, dx,$$

$$a_0 = \int_{-\infty}^{0} \left( \frac{e^y}{(e^y - 1)(1 + y^2)} - \frac{e^y}{y} \right) dy.$$

Given $p \in [0, \infty)$ we denote by $Z_p(\omega)$ a random variable with probability distribution $\varepsilon_p N(0, pM_0)$. Since the measure $M_0$ is absolutely continuous with respect to the Lebesgue measure and the function $\frac{\varepsilon^2}{x} \frac{dM_0}{dx}$ does not increase on $(-\infty, 0)$ and on $(0, \infty)$, we infer, by criterion B in [6], p. 324, that the random variables $Z_p(\omega)$ are self-decomposable. Using Mal'msteyn's formula (3.3) we get, by standard calculation, the formula

$$E \exp(isZ_p(\omega)) = \Gamma(1 + is)^p$$

where the principal branch of the $p$th power is taken. From the asymptotic behaviour

$$\lim_{|s| \to \infty} \frac{\Gamma(1 + is)||s|^{-1/2}}{\exp(\pi|s|/2)} = \sqrt{2\pi}$$

([1], 1.18, formula 6) we conclude that for $p \in (0, \infty)$ the characteristic function (4.1) is integrable on $(-\infty, \infty)$. Consequently, the probability distribution of $Z_p(\omega)$ is absolutely continuous for $p \in (0, \infty)$.

Denote by $\lambda_p$ the probability distribution of the random variable $Z_p(\omega)$. For $p \in (0, \infty)$ the measure $\lambda_p$ is also absolutely continuous. Since the characteristic function (4.1) has an analytic extension to the half-plane $\text{Im} \, z < 0$ we have

$$\int_{0}^{\infty} z^n \lambda_p(dx) = \Gamma(1 + n)^p = (n!)^p \quad (n = 1, 2, \ldots),$$

which for $p \in [0, 1)$ yields the expansion

$$\tilde{\lambda}_p(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{(n!)^{1-p}}$$

in the whole complex plane.

Given two independent random variables $X(\omega)$ and $Y(\omega)$ with probability distributions $\mu$ and $\nu$ respectively we shall denote by $\mu\nu$ the probability distribution of the random variable $X(\omega)Y(\omega)$. It is clear that

$$\lambda_p \lambda_q = \lambda_{p+q}$$

for all $p, q \in [0, \infty)$.

**Proposition 4.1.** The probability distribution $\lambda_p$ is infinitely divisible if and only if $p \in \{0\} \cup [1, \infty)$.

**Proof.** It follows from (4.2) that $\lambda_0 = \delta_1$ and $\lambda_1$ is the exponential distribution $\lambda_1(dx) = e^{-x}1_{[0, \infty)}(x) \, dx$. Thus $\lambda_0$ and $\lambda_1$ are both infinitely divisible. For $p \in (1, \infty)$ we have, by (4.4), $\lambda_p = \lambda_1\lambda_{p-1}$, which, by the Goldie Theorem ([5]), shows that $\lambda_p$ is infinitely divisible.

Suppose now that $p \in [0, 1)$ and $\lambda_p$ is infinitely divisible. Then $\lambda_p = e^{\mu}(N)$ where the measure $\mu$ is not concentrated at the origin because, by (4.2), $\lambda_p$ is not concentrated at a single point. Consequently, for some $\alpha \in (0, \infty)$ we have $\mu = N(\alpha, \infty)$, which means that $\lambda_p$ is not concentrated at a single point. Hence, for each $y \in [0, \infty)$ we have $\lambda_p(dx) = \exp(-b) \frac{e^{b^2x}}{x}$ for $x \geq y$. Let $y = \exp(b)$ and $x = \exp(b^2)$. Then

$$\int_{0}^{\infty} e^{b^2x} \lambda_p(dx) = \tilde{\lambda}_p(-y)$$

$$\geq \exp(b^2) \int_{0}^{\infty} (e^{b^2x} - 1) \frac{1}{x} \, N(dx) \geq \exp(b(e^{2b} - 1))$$

for every $y \in [0, \infty)$. Let $k$ be a positive integer such that $p + k^2 \leq 1$. Put $q = k^{-1}$. Taking independent random variables $X_1(\omega), \ldots, X_k(\omega)$ with the same probability distribution $\lambda_q$ we have for $x \in [0, \infty)$

$$P\left( \sum_{j=1}^{k} X_j(\omega) \geq x \right) \leq \sum_{j=1}^{k} P(X_j(\omega) \geq x) = k\lambda_q([x, \infty)).$$

By (4.4) the left-hand side of the above inequality is equal to $\lambda_1([x^k, \infty))$. Since $\lambda_1$ is the exponential distribution we finally get

$$\exp(-x^k) \leq k\lambda_q([x, \infty))$$

for every $x \in [0, \infty)$.

Observe that, by (4.3), the Laplace transform $\tilde{\lambda}_{p+q}(z)$ is an entire function and, consequently, the integral

$$c = \int_{0}^{\infty} e^{z} \lambda_{p+q}(dx)$$

is finite. Taking into account (4.4) and (4.5) we have for any $x \in [0, \infty)$

$$c = \int_{0}^{\infty} \int_{0}^{\infty} e^{\mu} \lambda_p(d\mu) \lambda_q(dy) \geq \int_{0}^{\infty} \exp(b(e^{\mu^2} - 1)) \lambda_q(dy)$$

$$\geq \lambda_q([x, \infty)) \exp(b(e^{\mu^2} - 1)),$$

which yields

$$\lambda_q([x, \infty)) \leq c \exp(b(1 - e^{b^2})).$$

Comparing the above inequality with (4.6) we get a contradiction, which shows that $\lambda_p$, for $p \in (0, 1)$ is not infinitely divisible. The proposition is thus proved.
Let $Y(\omega)$ be a random variable with probability distribution $\nu$ and $c \in (0, \infty)$. We shall denote by $T_2\nu$ the probability distribution of the random variable $cY(\omega)$.

We conclude this section with the following example.

**Example 4.1.** Let $X(t, \omega)$ be a nonnegative stable process with Bernstein function $B(z) = z^p$ where $p \in (0, 1]$. We shall show that for any $u \in (0, \infty)$, $T_u\lambda_{1-\gamma}$ with $\gamma = u^{-p}$ is the probability distribution of the functional

$$
\int_0^\infty e^{-uX(t, \omega)} \, dt.
$$

In fact, by (3.15), the $n$th moment of the above random variable is equal to $u^{-np}(n!)^{1-p}$. Comparing this with (4.2) we get our assertion.

### 5. A limit problem.

Throughout this section $X(t, \omega)$ will denote a nonnegative process. It was shown in Example 3.4 that for any $u \in (0, \infty)$ the integral $\int_0^\infty e^{-uX(t, \omega)} \, dt$ is finite with probability 1. For simplicity of notation $\Delta$ stands for either 0 or $\infty$. The problem we study can be formulated as follows: suppose that for a given process $X(t, \omega)$ there exist sequences $u_n$ and $v_n$ of positive numbers satisfying

$$
\lim_{n \to \infty} u_n = \Delta, \quad \lim_{n \to \infty} v_{n+1}/v_n = 1
$$

such that the sequence of the probability distributions of the normalized integrals

$$
v_n \int_0^\infty e^{-u_n X(t, \omega)} \, dt
$$

converges to a probability distribution other than $\delta_0$. What can be said about the limit distribution and the process $X(t, \omega)$? The results presented in this section can be regarded as a random analogue of Tauberian theorems.

We note that by the convergence of types theorem ([6], p. 203), if for a given process and a sequence $u_n$ we have two normalizing sequences $v_n$ and $v'_n$ then $\lim_{n \to \infty} v'_n/v_n$ exists and is positive.

Denote by $D_\Delta$ the family of all processes admitting sequences $u_n$ and $v_n$ with the properties described above and by $K_\Delta$ the set of all possible limit probability distributions.

We recall that a positive-valued measurable function $F$ defined on the half-line $(0, \infty)$ is said to be regularly varying of order $p$ at $\Delta$ if for every $x \in (0, \infty)$

$$
\lim_{y \to \Delta} F(xy)/F(y) = x^p.
$$

Observing that each Bernstein function $B(z)$ is increasing and, by (1.7), the function $B(z)/z$ is decreasing we infer that the order $p$ of a Bernstein function regularly varying at $\Delta$ belongs to the interval $[0, 1]$. Moreover, for every $p \in [0, 1]$ there exists a Bernstein function regularly varying of order $p$ at $\Delta$. Indeed, for $p \in [0, 1]$ it suffices to take $B(z) = z^p$. For $p = 0$ the functions $B(z) = 1/\log(1 + z^{-1})$ [4], p. 129) and $B(z) = z/(1 + z)$ have this property at 0 and $\infty$ respectively.

**Lemma 5.1.** The Bernstein functions associated with processes from $D_\Delta$ are regularly varying at $\Delta$.

**Proof.** Let $B(z)$ be the Bernstein function associated with a process from $D_\Delta$. Suppose that $u_n$ and $v_n$ have the required properties and denote by $H_n(z)$ the Laplace transform of the probability distribution of (5.2). By assumption $H_n(z)$ tends to the Laplace transform $H(z)$ of the limit distribution. By Theorem 2.4,

$$
H_n(z) = 1 - v_n z \int_0^\infty e^{-u_n y} H_n(ze^{-u_n y}) \, d\varrho(y)
$$

where $\varrho$ is the potential of the process in question and $z \in [0, \infty)$. Since $H_n(z)$ is decreasing the above equation yields

$$
H_n(z) \leq 1 - v_n z H(z) \varrho(v_n).
$$

Now, by virtue of (1.8), we get

$$
\frac{v_n}{B(u_n)} \leq \frac{1 - H(1)}{H(1)}.
$$

Since $H(1) > 0$ we conclude that $v_n \leq cB(u_n)$ $(n = 1, 2, \ldots)$ for a positive constant $c$. By (3.15) we have for sufficiently small $|z|

$$
H_n(z) = \sum_{k=0}^\infty (-v_n z)^k \prod_{j=1}^k B(ju_n)^{-1}.
$$

Since $B(z)$ is nondecreasing the absolute value of the $k$th coefficient in the above power series is not greater than $c^k$. Hence the limit function $H(z)$ is analytic in a neighbourhood of the origin. Setting

$$
H(z) = \sum_{k=0}^\infty a_k (-z)^k
$$

we have

$$
\lim_{n \to \infty} u_n \prod_{j=1}^k B(ju_n)^{-1} = a_k \quad (k = 1, 2, \ldots).
$$
Observe that all \(a_k\) are positive because the limit measure is different from \(\delta_0\). Consequently, for any positive integer \(k\)

\[
\lim_{n \to \infty} \frac{B(ku_n)}{B(u_n)} = \frac{a_1a_k-1}{a_k}.
\]

Further, from (5.1) it follows that the sequence of functions

\[
(e^{-x} - e^{-u_n+1/x/u_n})(1 - e^{-x})^{-1}
\]

tends uniformly to 0 on the half-line \([0, \infty)\). Hence, in view of (1.7), we get

\[
\lim_{n \to \infty} B(u_{n+1})/B(u_n) = 1.
\]

Now our assertion follows from Theorem 1.9 in [7].

**Lemma 5.2.** Suppose that the Bernstein function \(B(x)\) associated with a process \(X(t, \omega)\) is regularly varying of order \(p \in [0, 1]\). Then the limit distribution of

\[
B(u) \int_0^\infty e^{-ux}X(t, \omega)\, dt
\]

exists as \(u \to \Delta\) and is equal to \(\lambda_{1-p}\).

**Proof.** Denote by \(G_u(x)\) the Laplace transform of the probability distribution of (5.3). By (3.15), \(G_u(x)\) can be expanded into a power series in the unit circle:

\[
G_u(x) = \sum_{n=0}^{\infty} a_n(u)(-x)^n
\]

where \(a_0(u) = 1\) and \(a_n(u) = B(u)^n \prod_{k=1}^{n} B(ku)^{-1} (n = 1, 2, \ldots)\). Since \(B(x)\) is increasing we have \(0 < a_n(u) \leq 1 (n = 0, 1, \ldots)\). Moreover, \(\lim_{u \to \Delta} G_u(x) = \lambda_{1-p}(x)\), which completes the proof.

As an immediate consequence of Lemmas 5.1 and 5.2 we get the following theorems.

**Theorem 5.1.** A process belongs to \(D_\Delta\) if and only if its Bernstein function is regularly varying at \(\Delta\).

**Theorem 5.2.** \(K_0 = K_\infty = \{T_{u, \lambda p} : p \in [0, 1], c \in (0, \infty)\}\).

For \(\Delta = 0\) the assertion of Theorem 5.1 can be written in an equivalent form.

**Corollary 5.1.** A process belongs to \(D_0\) if and only if the probability distribution \(T_1\) belongs to the domain of attraction of a stable law.