and therefore

$$2^{n_0^2}T^{n_0}w = T^{n_0}\Big(\sum_{j=1}^{\nu} 2^{jn_0+n_0(n_0-1)/2}\beta_j x_{j+n_0}\Big)$$
$$= \sum_{j=1}^{\nu}\Big(\frac{1}{2}\Big)^{jn_0+n_0(n_0-1)/2} 2^{jn_0+n_0(n_0-1)/2}\beta_j x_j = v\,.$$

So $\|2^{n_0^2}T^{n_0}w - v\| = 0 \le \varepsilon$. ∎

4. Final remarks. 1) By analogy with the proof of Theorem 1, one can prove that for a complex separable Banach space there are operators with supercyclic vectors if and only if $\dim X \in \{0,1\}$ or $\dim X = \infty$.

2) In the infinite-dimensional case, the operator T in the proof of Theorem 1 is compact. An operator with hypercyclic vectors cannot be compact (see [6]).

3) In [2, p. 42] a supercyclic vector $x \in X$ for $T \in B(X)$ and X real is defined by $\overline{\{\lambda y : y \in \mathrm{Orb}(T,x),\ \lambda > 0\}} = X$. Also with this definition, Theorem 1 holds with exactly the same proof with the only difference that in the case $\dim X = 1$ we take minus identity.

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Functionals on transient stochastic processes with independent increments

by

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Abstract. The paper is devoted to the study of integral functionals $\int_0^\infty f(X(t,\omega))\,dt$ for a wide class of functions f and transient stochastic processes $X(t,\omega)$ with stationary and independent increments. In particular, for nonnegative processes a random analogue of the Tauberian theorem is obtained.

1. Notation and preliminaries. Let μ be a Borel measure on the real line $(-\infty,\infty)$. We denote by $L(\mu)$ the space of all complex-valued Borel functions f with the finite norm

$$\|f\|_\mu = \int_{-\infty}^{\infty} |f(x)|\,\mu(dx)\,.$$

The measure μ is said to be *shift-bounded* if

$$\sup\{\mu([a+x,b+x]) : x \in (-\infty,\infty)\} < \infty$$

for every bounded interval $[a,b]$. All measures under consideration in the sequel will tacitly be assumed to be shift-bounded and not identically equal to 0. The support of a function f is denoted by $\operatorname{supp} f$. The indicator of a set A is denoted by 1_A.

Put $\gamma(dx) = e^{-|x|}\,dx$. The space $L_\infty(\gamma)$ consists of all complex-valued Borel functions f with the finite norm

$$\|f\|_\infty = \operatorname{vrai\,sup}\{|f(x)| : x \in (-\infty,\infty)\}\,.$$

In the sequel we shall briefly say "almost everywhere" instead of "γ-almost everywhere".

If the integral $\int_{-\infty}^{\infty} |f(x+y)|\,\mu(dy)$ is finite for almost all x, then the convolution $f * \mu$ is defined by the formula

$$(f*\mu)(x) = \int_{-\infty}^{\infty} f(x+y)\,\mu(dy)\,.$$

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Observe that, by the shift-boundedness of the measure μ, the supremum

$$s(\mu) = \sup\left\{ \int_{-\infty}^{\infty} e^{-|x-y|}\, \mu(dy) : x \in (-\infty, \infty) \right\}$$

is finite. We define a mapping $\mu \to \mu^*$ by setting

$$\mu^*(dx) = \int_{-\infty}^{\infty} e^{-|x-y|}\, \mu(dy)\, dx\,.$$

It is clear that the measure μ^* is shift-bounded and equivalent to the measure γ. Moreover, we have the formula

$$\|f\|_{\mu^*} = \||f| * \mu\|_\gamma\,, \tag{1.1}$$

which shows that $f \in L(\mu^*)$ if and only if $|f| * \mu \in L(\gamma)$.

The following simple result will be needed below.

LEMMA 1.1. *Given a continuous mapping F from $L(\mu^*)$ into $L(\gamma)$ such that the norm $\|F(f)\|_\infty$ is locally bounded on $L(\mu^*)$, put $G(f) = fF(f)$. Then G is a continuous mapping from $L(\mu^*)$ into $L(\mu^*)$.*

Proof. Suppose that $f, f_n \in L(\mu^*)$ $(n = 1, 2, \ldots)$ and $\|f_n - f\|_{\mu^*} \to 0$ as $n \to \infty$. Since the norms $\|F(f_n)\|_\infty$ are uniformly bounded and the measures μ^* and γ are equivalent we have

$$\lim_{n\to\infty} \|(f - f_n)F(f_n)\|_{\mu^*} = 0\,. \tag{1.2}$$

Further, setting $A_k = \{x : |f(x)| > k\} \cup \{x : |x| > k\}$ we have

$$\lim_{k\to\infty} \|f 1_{A_k}\|_{\mu^*} = 0\,. \tag{1.3}$$

Denoting by B_k the complement of A_k we get

$$\|f 1_{B_k}(F(f) - F(f_n))\|_{\mu^*} \le k \int_{-k}^{k} |F(f) - F(f_n)|(x)\, \mu^*(dx)$$

$$\le s(\mu) k e^k \int_{-k}^{k} |F(f) - F(f_n)|(x) e^{-|x|}\, dx \le s(\mu) k e^k \|F(f) - F(f_n)\|_\gamma\,,$$

which yields

$$\lim_{n\to\infty} \|f 1_{B_k}(F(f) - F(f_n))\|_{\mu^*} = 0 \quad (k = 1, 2, \ldots)\,. \tag{1.4}$$

Since $\|f 1_{A_k}(F(f) - F(f_n))\|_{\mu^*} \le \|F(f) - F(f_n)\|_\infty \|f 1_{A_k}\|_{\mu^*}$ we have, by (1.3) and (1.4),

$$\lim_{n\to\infty} \|f(F(f) - F(f_n))\|_{\mu^*} = 0\,,$$

which together with (1.2) implies $\lim_{n\to\infty} \|G(f) - G(f_n)\|_{\mu^*} = 0$. The lemma is thus proved.

We denote by $L_+(\mu^*)$ the subset of $L(\mu^*)$ consisting of all functions f satisfying the condition

$$\lim_{h\to 0+} \int_{-\infty}^{\infty} |f(x + y + h) - f(x + y)|\, \mu(dy) = 0$$

for all $x \in (-\infty, \infty)$. The following statement is evident.

LEMMA 1.2. *The set $L_+(\mu^*)$ is invariant under multiplication by bounded functions continuous on the right. For any $f \in L_+(\mu^*)$ the function $f * \mu$ is continuous on the right and finite everywhere.*

We denote by $L(\mu, \infty)$ the space of all complex-valued Borel functions f defined on the real line for which $|f| * \mu \in L_\infty(\gamma)$. The norm in $L(\mu, \infty)$ is given by the formula

$$\|f\|_{\mu,\infty} = \||f| * \mu\|_\infty\,. \tag{1.5}$$

By (1.1) we have $\|f\|_{\mu,\infty} \ge \|f\|_{\mu^*}$, which yields $L(\mu, \infty) \subset L(\mu^*)$. It is clear that all bounded Borel functions with bounded support belong to $L(\mu, \infty)$. Moreover, by the shift-boundedness of μ, for any $a \in (0, \infty)$ the function $e^{-ax} 1_{[0,\infty)}(x)$ belongs to $L(\mu, \infty) \cap L_+(\mu^*)$.

Let B be a set of complex-valued functions. We shall denote by $\mathrm{Re}\, B$ the subset of B consisting of real-valued functions.

Let M be a bounded Borel measure on $(-\infty, \infty)$ and $a \in (-\infty, \infty)$. We shall denote by $e(a, M)$ the infinitely divisible probability measure with characteristic function

$$\widehat{e}(a, M)(s) = \exp\left(ias + \int_{-\infty}^{\infty} \left(e^{isx} - 1 - \frac{isx}{1 + x^2} \right) \frac{1 + x^2}{x^2}\, M(dx) \right).$$

Moreover, for a bounded Borel measure N concentrated on the half-line $[0, \infty)$ we shall denote by $e_+(N)$ the infinitely divisible probability measure concentrated on $[0, \infty)$ with Laplace transform

$$\widetilde{e}_+(N)(z) = \exp \int_0^{\infty} (e^{-zx} - 1) \frac{1 + x}{x}\, N(dx) \tag{1.6}$$

for $z \in [0, \infty)$. The function

$$B(N, z) = \int_0^{\infty} (1 - e^{-zx}) \frac{1 + x}{x}\, N(dx) \tag{1.7}$$

is called the *Bernstein function*. In the sequel we write briefly $B(z)$ instead of $B(N, z)$ if it causes no confusion.

Let $X(t,\omega)$, $t \in [0,\infty)$, be a real-valued stochastic process with stationary and independent increments, with sample functions continuous on the right and satisfying the initial condition $X(0,\omega) = 0$. Denote by τ_t the probability distribution of $X(t,\omega)$. It is well-known that

$$\tau_t = e(ta, tM)$$

for some a and M. Similarly, for nonnegative processes we have

$$\tau_t = e_+(tN)$$

for some measure N. In this case the Bernstein function $B(N,z)$ is said to be *associated* with the process $X(t,\omega)$.

The process $X(t,\omega)$ is said to be *transient* if the potential

$$\varrho(A) = \int_0^\infty \tau_t(A)\,dt$$

is finite for all bounded Borel sets A. By Proposition 13.10 in [4] the measure ϱ is always shift-bounded. It is known that $X(t,\omega)$ is transient if and only if the function

$$\operatorname{Re}(\log \widehat{\tau}_1(s))^{-1}$$

is integrable in a neighbourhood of the origin ([4], 13.17). Moreover, each nonnegative process which is not identically zero is transient and the Laplace transform of its potential is given by the formula

$$\widetilde{\varrho}(z) = B(z)^{-1} \tag{1.8}$$

for $z \in (0,\infty)$ where $B(z)$ is the associated Bernstein function ([4], 14.1).

Throughout this paper the processes $X(t,\omega)$ will tacitly be assumed to be transient.

2. Integral functionals. Let $X(t,\omega)$ be a transient process with potential ϱ. This section is devoted to the study of the probability distribution of the functionals $\int_0^\infty f(X(t,\omega))\,dt$ for $f \in \operatorname{Re} L(\varrho^*)$. We shall use the notation

$$I(f,x,\omega) = \int_0^\infty f(X(t,\omega)+x)\,dt \tag{2.1}$$

provided the right-hand side integral is well-defined.

Suppose that $(|f| * \varrho)(x_0)$ is finite. By standard calculations the expectation of $I(|f|,x_0,\omega)$ is

$$EI(|f|,x_0,\omega) = (|f| * \varrho)(x_0)\,.$$

It follows from (1.1) that for any $f \in L(\varrho^*)$ the convolution $|f| * \varrho$ is finite almost everywhere. This yields the following statement.

PROPOSITION 2.1. *Let $f \in L(\varrho^*)$. Then $I(f,x,\omega)$ is finite with probability 1 for almost every x and*

$$EI(f,x,\omega) = (f * \varrho)(x) \tag{2.2}$$

almost everywhere. If in addition $f \in L_+(\varrho^*)$, then $I(f,x,\omega)$ is finite with probability 1 for every x, formula (2.2) holds for every x and

$$\lim_{u\to 0+} E|I(f,x+u,\omega) - I(f,x,\omega)| = 0 \tag{2.3}$$

for every x.

Consider the characteristic function $Q(f,x,s) = Ee^{isI(f,x,\omega)}$ for $f \in \operatorname{Re} L(\varrho^*)$.

THEOREM 2.1. *Let $f \in \operatorname{Re} L(\varrho^*)$. Then for every $s \in (-\infty,\infty)$ the function $Q(f,\cdot,s)$ satisfies the equation*

$$Q(f,x,s) = 1 + is\int_{-\infty}^{\infty} f(x+y)Q(f,x+y,s)\,\varrho(dy) \tag{2.4}$$

for almost every x. If in addition $f \in \operatorname{Re} L_+(\varrho^*)$, then the above equation is satisfied for every x.

Proof. Introduce the notation

$$F_s(f)(x) = Q(f,x,s), \qquad G_s(f)(x) = f(x)Q(f,x,s)\,.$$

First we shall prove that for any $s \in (-\infty,\infty)$, $F_s(\cdot)$ and $G_s(\cdot) * \varrho$ are continuous mappings from $\operatorname{Re} L(\varrho^*)$ into $L(\gamma)$. Taking into account the obvious inequality

$$\|F_s(f)\|_\infty \le 1 \tag{2.5}$$

and (1.1) we infer that $F_s(f) \in L(\gamma)$ and $G_s(f) * \varrho \in L(\gamma)$ for every $f \in \operatorname{Re} L(\varrho^*)$. Further, for any $f, g \in \operatorname{Re} L(\varrho^*)$ we have, by (2.2),

$$|F_s(f)(x) - F_s(g)(x)| \le |s|E|I(f,x,\omega) - I(g,x,\omega)| \le |s|(|f-g| * \varrho)(x)$$

almost everywhere, which yields, by (1.1),

$$\|F_s(f) - F_s(g)\|_\gamma \le |s|\,\|f-g\|_*\,.$$

Consequently, the mapping $F_s(\cdot)$ from $\operatorname{Re} L(\varrho^*)$ into $L(\gamma)$ is continuous. Now from (2.5) and Lemma 1.1 it follows that $G_s(\cdot)$ is continuous from $\operatorname{Re} L(\varrho^*)$ into $L(\varrho^*)$. Consequently, by (1.1), $G_s(\cdot) * \varrho$ is continuous from $\operatorname{Re} L(\varrho^*)$ into $L(\gamma)$.

Equation (2.4) can be written in the form

$$F_s(f) = 1 + is(G_s(f) * \varrho)\,. \tag{2.6}$$

Since both sides of the above equation are continuous mappings from $\operatorname{Re} L(\varrho^*)$ into $L(\gamma)$ it suffices to prove it for real-valued continuous functions f with a bounded support, which form a dense subset of $\operatorname{Re} L(\varrho^*)$.

For such f and $u \in (0,\infty)$ we denote by $Q_u(f,x,s)$ the characteristic function of the functional $\int_0^u f(X(t,\omega)+x)\,dt$. By the Skorokhod Theorem ([8], Chapter 4.1, 6),

$$Q_u(f,x,s) = 1 + is \int_0^u \int_{-\infty}^{\infty} f(x+y)Q_{u-t}(f,x+y,s)\,\tau_t(dy)\,dt\,.$$

It is clear that $Q_u(f,x,s) \to Q(f,x,s)$ as $u \to \infty$. Moreover, the right-hand side of the above equation tends to $1 + is(G_s(f) * \varrho)(x)$ as $u \to \infty$, which yields (2.6) for real-valued continuous functions f with a bounded support. This completes the proof of (2.4).

Suppose now that $f \in \operatorname{Re} L_+(\varrho^*)$. Then, by Lemma 1.2 and Proposition 2.1, both sides of (2.4) are continuous on the right in x, which yields the assertion of the theorem.

THEOREM 2.2. *Let $f \in \operatorname{Re} L(\varrho,\infty)$, $|s|\|f\|_{\varrho,\infty} < 1$ and $A \supset \operatorname{supp} f$. Suppose that a Borel function H_s satisfies $H_s 1_A \in L_\infty(\gamma)$ and*

$$H_s(x) = 1 + is \int_{-\infty}^{\infty} f(x+y)H_s(x+y)\,\varrho(dy) \tag{2.7}$$

for almost every $x \in A$. Then $H_s(x) = Q(f,x,s)$ for almost every $x \in A$.

Proof. Setting $U_s(x) = Q(f,x,s)1_A(x) - H_s(x)1_A(x)$ we get a function belonging to $L_\infty(\gamma)$ and, by the obvious equality $f(x) = f(x)1_A(x)$ and Theorem 2.1,

$$U_s(x) = is1_A(x) \int_{-\infty}^{\infty} f(x+y)U_s(x+y)\,\varrho(dy)$$

for almost every $x \in (-\infty,\infty)$. Consequently, by (1.5),

$$\|U_s\|_\infty \le |s|\|U_s\|_\infty \||f| * \varrho\|_\infty = |s|\|U_s\|_\infty \|f\|_{\varrho,\infty}\,,$$

which yields $\|U_s\|_\infty = 0$. This implies the assertion of the theorem.

Given $f \in \operatorname{Re} L(\varrho,\infty)$ we define a sequence of functions $q_n(f,x)$ $(n = 0,1,\ldots)$ on the real line by setting

$$\begin{aligned} &q_0(f,x) = 1\,,\\ &q_{n+1}(f,x) = \int_{-\infty}^{\infty} f(x+y)q_n(f,x+y)\,\varrho(dy) \quad (n=0,1,\ldots). \end{aligned} \tag{2.8}$$

By induction we get

$$\|q_n(f,x)\|_\infty \le \|f\|^n_{\varrho,\infty} \quad (n=0,1,\ldots)\,, \tag{2.9}$$

which shows that $q_n(f,\cdot) \in L_\infty(\gamma)$. If in addition $f \in L_+(\varrho^*)$, then, by Lemma 1.2, the functions $q_n(f,\cdot)$ are continuous on the right.

THEOREM 2.3. *Let $f \in \operatorname{Re} L(\varrho,\infty)$. For almost every x the characteristic function $Q(f,x,\cdot)$ can be extended to an analytic function in the strip $|\operatorname{Im} z|\|f\|_{\varrho,\infty} < 1$, and in the circle $|z|\|f\|_{\varrho,\infty} < 1$ it has the power series representation*

$$Q(f,x,z) = \sum_{n=0}^{\infty} (iz)^n q_n(f,x) \tag{2.10}$$

for almost every x. If in addition $f \in L_+(\varrho^*)$, then the above formula is true for every x.

Proof. By (2.9) the function $H_z(x) = \sum_{n=0}^{\infty}(iz)^n q_n(f,x)$ is analytic in the circle $|z|\|f\|_{\varrho,\infty} < 1$ for almost every x. Further, for any fixed real s the function $H_s(x)$ belongs to $L_\infty(\gamma)$ and, by (2.8), satisfies equation (2.7) for almost every x. Applying Theorem 2.2 we get $Q(f,x,s) = H_s(x)$ for almost every x. Hence $Q(f,x,\cdot)$ can be extended to an analytic function in the circle $|z|\|f\|_{\varrho,\infty} < 1$ and, consequently, in the strip $|\operatorname{Im} z|\|f\|_{\varrho,\infty} < 1$ ([6], p. 212). The last assertion is an immediate consequence of the continuity on the right of the functions $Q(f,\cdot,s)$ and $q_n(f,\cdot)$ $(n = 0,1,\ldots)$ for $f \in L_+(\varrho^*)$.

As a consequence of the above theorem we get a formula for the moments of the functional $I(f,x,\omega)$.

COROLLARY 2.1. *For any $f \in \operatorname{Re} L(\varrho,\infty)$ and almost every x*

$$EI^n(f,x,\omega) = n!q_n(f,x) \quad (n=0,1,\ldots)\,.$$

If in addition $f \in L_+(\varrho^*)$, then the above formula is true for every x.

For nonnegative functions f from $L(\varrho^*)$ we denote by $H(f,x,z)$ the Laplace transform of the probability distribution of $I(f,x,\omega)$, i.e.

$$H(f,x,z) = Ee^{-zI(f,x,\omega)}$$

for $z \in [0,\infty)$. As an immediate consequence of Theorems 2.1–2.3 we get the following statement.

THEOREM 2.4. *Let f be a nonnegative function from $L(\varrho,\infty)$. Then for any z satisfying $|z|\|f\|_{\varrho,\infty} < 1$ the function $H(f,x,z)$ is the only solution of the equation*

$$H(f,x,z) = 1 - z \int_{-\infty}^{\infty} f(x+y)H(f,x+y,z)\,\varrho(dy) \tag{2.11}$$

for almost every $x \in \operatorname{supp} f$. Moreover,

$$H(f,x,z) = \sum_{n=0}^{\infty} (-z)^n q_n(f,x) \tag{2.12}$$

almost everywhere. If in addition $f \in L_+(\varrho^*)$, then (2.11) and (2.12) hold for every x.

3. Examples. The results of the preceding section may serve for the determining of the probability distribution of functionals $I(f,x,\omega)$. We shall illustrate this by some examples.

EXAMPLE 3.1 (A compound Poisson process). Let $X(t,\omega)$ be a process with probability distribution $\tau_1 = e_+(N_1)$ where $N_1(dx) = \frac{x}{1+x}e^{-x}\,dx$. The associated Bernstein function is $B(z) = \frac{z}{1+z}$, which, by (1.8), yields $\varrho(dx) = \delta_0(dx) + 1_{[0,\infty)}(x)\,dx$ where δ_0 is the probability measure concentrated at the origin. For any $f \in \operatorname{Re} L(\varrho,\infty)$ and almost every x we have

$$Q(f,x,s) = (1-isf(x))^{-1}\exp\left(is\int_x^\infty \frac{f(y)}{1-isf(y)}\,dy\right). \tag{3.1}$$

In fact, by Theorem 2.1,

$$Q(f,x,s) = 1 + isf(x)Q(f,x,s) + is\int_x^\infty f(y)Q(f,y,s)\,dy$$

for almost every x. It is easy to verify that the right-hand side of (3.1) also satisfies the above equation, which, by Theorem 2.2, yields formula (3.1). Observe that for every $f \in \operatorname{Re} L(\varrho,\infty)$ and almost every x the random variable $I(f,x,\omega)$ is infinitely divisible.

EXAMPLE 3.2 (Nonnegative stable processes). Let $X(t,\omega)$ be a nonnegative stable process with associated Bernstein function $B(z) = z^p$ where $p \in (0,1)$. From (1.8) we get

$$\varrho(dx) = \frac{x^{p-1}}{\Gamma(p)}1_{[0,\infty)}(x)\,dx. \tag{3.2}$$

Given $b \in (0,\infty)$ we put

$$M_b(dx) = \frac{|x|\left(\exp\dfrac{px}{p+b} - \exp\dfrac{x}{p+b}\right)}{(1+x^2)(1-\exp x)\left(1-\exp\dfrac{x}{p+b}\right)}1_{(-\infty,0)}(x)\,dx,$$

$$a_b = \int_{-\infty}^0 (x^{-1} + (1-e^x)(1+x^{-2}))\,M_b(dx) - \log(p+b) - \log\Gamma(p).$$

We shall show that $e(a_b,M_b)$ is the probability distribution of the random variable

$$\log\int_0^\infty X^b(t,\omega)1_{[0,1)}(X(t,\omega))\,dt.$$

First observe that the functions $f_b(x) = x^b 1_{[0,1)}(x)$ belong to $L(\varrho,\infty)\cap L_+(\varrho^*)$ and, by (2.8) and (3.2),

$$q_n(f_b,0) = \frac{1}{\Gamma(p)^n}\int\limits_{B_n}\cdots\int y_1^b(y_1+y_2)^b\ldots(y_1+\ldots+y_n)^b y_1^{p-1}\ldots y_n^{p-1}\,dy_1\ldots dy_n$$

where $B_n = \{(y_1,\ldots,y_n) : \sum_{k=1}^n y_k < 1,\ y_j > 0,\ j=1,\ldots,n\}$. Taking the one-to-one mapping $(y_1,\ldots,y_n)\to(u_1,\ldots,u_n)$ from B_n onto the nth Cartesian power of the interval $(0,1)$ defined by

$$y_1 = u_1\ldots u_n,\quad y_k = u_k u_{k+1}\ldots u_n(1-u_{k-1})\quad (k=2,\ldots,n)$$

we get, by Corollary 2.1, the formula

$$EI^n(f_b,0,\omega) = \frac{n!}{\Gamma(p)}\int_0^1\cdots\int_0^1 u_n^{n(p+b)-1}\prod_{k=1}^{n-1}u_k^{k(p+b)-1}(1-u_k)^{p-1}du_1\ldots du_n.$$

Hence, by simple calculation we get

$$EI^n(f_b,0,\omega) = (p+b)^{-n}\prod_{k=0}^{n-1}\frac{\Gamma(k(p+b)+1)}{\Gamma(k(p+b)+p)}\quad (n=1,2,\ldots). \tag{3.3}$$

Let $Y_b(\omega)$ be a random variable with probability distribution $e(a_b,M_b)$. It is clear that the characteristic function $\widehat{e}(a_b,M_b)$ can be extended to an analytic function in the half-plane $\operatorname{Im} z < 0$. Thus

$$E\exp(nY_b(\omega)) = \widehat{e}(a_b,M_b)(-in)\quad (n=1,2,\ldots). \tag{3.4}$$

Applying Malmsten's formula ([1], 1.9)

$$\log\Gamma(z) = \int_0^\infty\left(z-1-\frac{1-e^{-(z-1)x}}{1-e^{-x}}\right)\frac{e^{-x}}{x}\,dx\quad (\operatorname{Re} z > 0) \tag{3.5}$$

to the right-hand side of (3.3) we get, by simple calculation,

$$EI^n(f_b,0,\omega) = \widehat{e}(a_b,M_b)(-in)\quad (n=1,2,\ldots). \tag{3.6}$$

By Theorem 2.3 the probability distribution of $I(f_b,0,\omega)$ is uniquely determined by its moments. Comparing (3.4) and (3.6) we conclude that $e(a_b,M_b)$ is the probability distribution of $\log I(f_b,0,\omega)$.

EXAMPLE 3.3 (Brownian motion with drift). Let $X(t,\omega) = W(t,\omega)+ct$ where $W(t,\omega)$ is the standard Brownian motion and the drift coefficient c is positive. The potential of this process is of the form

$$\varrho(dx) = c^{-1}1_{[0,\infty)}(x)\,dx + c^{-1}e^{2cx}1_{(-\infty,0)}(x)\,dx. \tag{3.7}$$

Denote by $a_1(r) < a_2(r) < \ldots$ the sequence of all positive zeros of the Bessel function J_r $(r > -1)$. Define a finite Borel measure N_0 on the half-line $[0,\infty)$

by setting

$$N_0(dx) = (1+x)^{-1}\Big(\sum_{n=1}^{\infty}\exp(-8^{-1}a_n^2(2c-1)x)\Big)\,dx\,.$$

We shall prove that $e_+(N_0)$ is the probability distribution of the functional

$$\int_0^\infty e^{-X(t,\omega)}1_{[0,\infty)}(X(t,\omega))\,dt\,.$$

Observe that the Laplace transform of $e_+(N_0)$ is

(3.8) $$\widetilde{e}_+(N_0)(z) = \prod_{n=1}^{\infty}(1+8za_n^{-2}(2c-1))^{-1} = \frac{(2z)^c}{\sqrt{2z}\,\Gamma(2c)I_{2c-1}(\sqrt{8z})}$$

where I_r denotes the modified Bessel function of the first kind. As we mentioned in Section 1 the function $f_0(x) = e^{-x}1_{[0,\infty)}(x)$ belongs to $L(\varrho,\infty)\cap L_+(\varrho^*)$. Taking into account (3.7) we infer, by Theorem 2.4, that the Laplace transform $H(f_0,x,z)$ of the probability distribution of the functional $I(f_0,x,\omega)$ is the only solution of the equation

(3.9) $$H(f_0,x,z) = 1 - c^{-1}z\int_x^\infty e^{-y}H(f_0,y,z)\,dy - c^{-1}ze^{-2cx}\int_0^x e^{(2c-1)y}H(f_0,y,z)\,dy$$

for $x\in \operatorname{supp} f_0 = [0,\infty)$ and $|z|\|f_0\|_{\varrho,\infty} < 1$.

In order to solve the above equation we introduce auxiliary functions $F_r(u,z)$ for $r\in(0,\infty)$, $u\in[0,1]$ and $z\in[0,\infty)$ by setting

(3.10) $$F_r(u,z) = \frac{(uz)^r}{2^{r-1}\Gamma(r)I_{r-1}(z)}(I_r(uz)K_{r-1}(z) + K_r(uz)I_{r-1}(z))$$

where K_r denotes the modified Bessel function of the third kind. Using the well-known properties of the modified Bessel functions I_r and K_r ([2], 7.11, formulae (19)–(22) and (39)) we can easily check that

(3.11) $$F_r(u,z) = 1 - (2r)^{-1}z^2\int_0^u yF_r(y,z)\,dy - (2r)^{-1}u^{2r}z^2\int_u^1 y^{1-2r}F_r(y,z)\,dy$$

and

(3.12) $$F_r(1,z) = \frac{z^{r-1}}{2^{r-1}\Gamma(r)I_{r-1}(z)}\,.$$

Moreover, taking into account the asymptotic behaviour of I_r and K_r as $z\to\infty$ ([2], 7.4, formula (1)) we have

(3.13) $$\lim_{u\to 0+}F_r(u,u^{-1}z) = \frac{z^rK_r(z)}{2^{r-1}\Gamma(r)}\,.$$

Starting from equation (3.11) one can easily check that the function $F_{2c}(e^{-x/2},\sqrt{8z})$ satisfies equation (3.9), which yields

(3.14) $$H(f_0,x,z) = F_{2c}(e^{-x/2},\sqrt{8z})$$

for $x\in[0,\infty)$ and $z\in[0,\infty)$. In particular, $H(f_0,0,z) = F_{2c}(1,\sqrt{8z})$, which, by (3.8) and (3.12), shows that $H(f_0,0,z)$ is the Laplace transform of $e_+(N_0)$. This completes the proof of our assertion.

Observe that, by the strong law of large numbers for the Brownian motion,

$$\lim_{t\to\infty}\frac{X(t,\omega)}{t} = c$$

with probability 1. Consequently, the integral $\int_0^\infty e^{-X(t,\omega)}\,dt$ is finite with probability 1. Denote by $H(z)$ the Laplace transform of its probability distribution. Since $\lim_{x\to\infty} e^xI(f_0,x,\omega) = \int_0^\infty e^{-X(t,\omega)}\,dt$ with probability 1, we have $\lim_{x\to\infty}H(f_0,x,e^xz) = H(z)$, which, by (3.13) and (3.14), yields $H(z) = \Gamma(2c)^{-1}2^{c+1}z^cK_{2c}(\sqrt{8z})$. Hence ([3], p. 283, formula 40) the random variable $\int_0^\infty e^{-X(t,\omega)}\,dt$ has probability density function $\Gamma(2c)^{-1}4^c\times x^{-2c-1}e^{-2/x}1_{[0,\infty)}(x)$.

EXAMPLE 3.4 (Nonnegative processes). Let $X(t,\omega)$ be a nonnegative process with Bernstein function $B(z)$. We shall prove that for any $u\in(0,\infty)$

(3.15) $$E\Big(\int_0^\infty e^{-uX(t,\omega)}\,dt\Big)^n = n!\prod_{k=1}^{n}B(ku)^{-1}\qquad (n=1,2,\ldots)\,.$$

As we mentioned in Section 1 the functions $g_u(x) = e^{-ux}1_{[0,\infty)}(x)$ belong to $L(\varrho,\infty)\cap L_+(\varrho^*)$ for every $u\in(0,\infty)$. It is clear that the potential ϱ of the process in question is concentrated on the half-line $[0,\infty)$. Moreover, $I(g_u,x,\omega) = e^{-ux}I(g_u,0,\omega)$ for $x\in[0,\infty)$ and

$$I(g_u,0,\omega) = \int_0^\infty e^{-uX(t,\omega)}\,dt\,.$$

Consequently, by Corollary 2.1, $q_n(g_u,x) = e^{-nux}q_n(g_u,0)$ for $x\in[0,\infty)$, which, by (2.8), yields

$$q_{n+1}(g_u,0) = q_n(g_u,0)\int_0^\infty e^{-(n+1)uy}\,\varrho(dy) = q_n(g_u,0)\widetilde{\varrho}((n+1)u)\,.$$

Applying formula (1.8) and Corollary 2.1 we get our assertion.

4. A family of probability measures. Define a bounded Borel measure M_0 on the real line and a constant a_0 by setting

$$M_0(dx) = \frac{|x|e^x}{(1-e^x)(1+x^2)} 1_{(-\infty,0)}(x)\,dx\,,$$

$$a_0 = \int_{-\infty}^{0} \left(\frac{e^y}{(e^y-1)(1+y^2)} - \frac{e^y}{y} \right) dy\,.$$

Given $p \in [0,\infty)$ we denote by $Z_p(\omega)$ a random variable with probability distribution $e(pa_0, pM_0)$. Since the measure M_0 is absolutely continuous with respect to the Lebesgue measure and the function $\frac{1+x^2}{x}\frac{dM_0}{dx}$ does not increase on $(-\infty,0)$ and on $(0,\infty)$, we infer, by criterion B in [6], p. 324, that the random variables $Z_p(\omega)$ are self-decomposable. Using Malmsten's formula (3.5) we get, by standard calculation, the formula

$$E\exp(isZ_p(\omega)) = \Gamma(1+is)^p \tag{4.1}$$

where the principal branch of the pth power is taken. From the asymptotic behaviour

$$\lim_{|s|\to\infty} |\Gamma(1+is)||s|^{-1/2}\exp(\pi|s|/2) = \sqrt{2\pi}$$

([1], 1.18, formula 6) we conclude that for $p \in (0,\infty)$ the characteristic function (4.1) is integrable on $(-\infty,\infty)$. Consequently, the probability distribution of $Z_p(\omega)$ is absolutely continuous for $p \in (0,\infty)$.

Denote by λ_p the probability distribution of the random variable $\exp Z_p(\omega)$. For $p \in (0,\infty)$ the measure λ_p is also absolutely continuous. Since the characteristic function (4.1) has an analytic extension to the half-plane $\operatorname{Im} z < 0$ we have

$$\int_0^\infty x^n\,\lambda_p(dx) = \Gamma(1+n)^p = (n!)^p \quad (n = 1,2,\ldots)\,, \tag{4.2}$$

which for $p \in [0,1)$ yields the expansion

$$\widetilde{\lambda}_p(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{(n!)^{1-p}} \tag{4.3}$$

in the whole complex plane.

Given two independent random variables $X(\omega)$ and $Y(\omega)$ with probability distributions μ and ν respectively we shall denote by $\mu\nu$ the probability distribution of the random variable $X(\omega)Y(\omega)$. It is clear that

$$\lambda_p\lambda_q = \lambda_{p+q} \tag{4.4}$$

for all $p, q \in [0,\infty)$.

PROPOSITION 4.1. *The probability distribution λ_p is infinitely divisible if and only if $p \in \{0\} \cup [1,\infty)$.*

Proof. It follows from (4.2) that $\lambda_0 = \delta_1$ and λ_1 is the exponential distribution $\lambda_1(dx) = e^{-x}1_{[0,\infty)}(x)\,dx$. Thus λ_0 and λ_1 are both infinitely divisible. For $p \in (1,\infty)$ we have, by (4.4), $\lambda_p = \lambda_1\lambda_{p-1}$, which, by the Goldie Theorem ([5]), shows that λ_p is infinitely divisible.

Suppose now that $p \in (0,1)$ and λ_p is infinitely divisible. Then $\lambda_p = e_+(N)$ where the measure N is not concentrated at the origin because, by (4.2), λ_p is not concentrated at a single point. Consequently, for some $a \in (0,\infty)$ we have $b = N([a,\infty)) > 0$. Moreover, by (4.3), the Laplace transform $\widetilde{\lambda}_p(z)$ is an entire function, which, by (1.6), yields

$$\int_0^\infty e^{yx}\,\lambda_p(dx) = \widetilde{\lambda}_p(-y) \tag{4.5}$$

$$\geq \exp\int_a^\infty (e^{yx}-1)\frac{1+x}{x}\,N(dx) \geq \exp(b(e^{ya}-1))$$

for every $y \in [0,\infty)$. Let k be a positive integer such that $p + k^{-1} < 1$. Put $q = k^{-1}$. Taking independent random variables $X_1(\omega),\ldots,X_k(\omega)$ with the same probability distribution λ_q we have for $x \in [0,\infty)$

$$P\Big(\prod_{j=1}^{k} X_j(\omega) \geq x^k\Big) \leq \sum_{j=1}^{k} P(X_j(\omega) \geq x) = k\lambda_q([x,\infty))\,.$$

By (4.4) the left-hand side of the above inequality is equal to $\lambda_1([x^k,\infty))$. Since λ_1 is the exponential distribution we finally get

$$\exp(-x^k) \leq k\lambda_q([x,\infty)) \tag{4.6}$$

for every $x \in [0,\infty)$.

Observe that, by (4.3), the Laplace transform $\widetilde{\lambda}_{p+q}(z)$ is an entire function and, consequently, the integral

$$c = \int_0^\infty e^x\,\lambda_{p+q}(dx)$$

is finite. Taking into account (4.4) and (4.5) we have for any $x \in [0,\infty)$

$$c = \int_0^\infty\int_0^\infty e^{uy}\,\lambda_p(du)\,\lambda_q(dy) \geq \int_x^\infty \exp(b(e^{ya}-1))\,\lambda_q(dy)$$

$$\geq \lambda_q([x,\infty))\exp(b(e^{xa}-1))\,,$$

which yields

$$\lambda_q([x,\infty)) \leq c\exp(b(1-e^{xa}))\,.$$

Comparing the above inequality with (4.6) we get a contradiction, which shows that for $p \in (0,1)$ the measures λ_p are not infinitely divisible. The proposition is thus proved.

Let $Y(\omega)$ be a random variable with probability distribution ν and $c \in (0,\infty)$. We shall denote by $T_c\nu$ the probability distribution of the random variable $cY(\omega)$.

We conclude this section with the following example.

EXAMPLE 4.1. Let $X(t,\omega)$ be a nonnegative stable process with Bernstein function $B(z) = z^p$ where $p \in (0,1]$. We shall show that for any $u \in (0,\infty)$, $T_v\lambda_{1-p}$ with $v = u^{-p}$ is the probability distribution of the functional

$$\int_0^\infty e^{-uX(t,\omega)}\,dt\,.$$

In fact, by (3.15), the nth moment of the above random variable is equal to $u^{-np}(n!)^{1-p}$. Comparing this with (4.2) we get our assertion.

5. A limit problem. Throughout this section $X(t,\omega)$ will denote a nonnegative process. It was shown in Example 3.4 that for any $u \in (0,\infty)$ the integral $\int_0^\infty e^{-uX(t,\omega)}\,dt$ is finite with probability 1. For simplicity of notation Δ stands for either 0 or ∞. The problem we study can be formulated as follows: suppose that for a given process $X(t,\omega)$ there exist sequences u_n and v_n of positive numbers satisfying

$$(5.1)\qquad \lim_{n\to\infty} u_n = \Delta, \qquad \lim_{n\to\infty} u_{n+1}/u_n = 1$$

such that the sequence of the probability distributions of the normalized integrals

$$(5.2)\qquad v_n \int_0^\infty e^{-u_nX(t,\omega)}\,dt$$

converges to a probability distribution other than δ_0. What can be said about the limit distribution and the process $X(t,\omega)$? The results presented in this section can be regarded as a random analogue of Tauberian theorems.

We note that by the convergence of types theorem ([6], p. 203), if for a given process and a sequence u_n we have two normalizing sequences v_n and v'_n, then $\lim_{n\to\infty} v'_n/v_n$ exists and is positive.

Denote by D_Δ the family of all processes admitting sequences u_n and v_n with the properties described above and by K_Δ the set of all possible limit probability distributions.

We recall that a positive-valued measurable function F defined on the half-line $(0,\infty)$ is said to be *regularly varying* of order p at Δ if for every $x \in (0,\infty)$

$$\lim_{y\to\Delta} F(xy)/F(y) = x^p\,.$$

Observing that each Bernstein function $B(z)$ is increasing and, by (1.7), the function $B(z)/z$ is decreasing we infer that the order p of a Bernstein function regularly varying at Δ belongs to the interval $[0,1]$. Moreover, for every $p \in [0,1]$ there exists a Bernstein function regularly varying of order p at Δ. Indeed, for $p \in (0,1]$ it suffices to take $B(z) = z^p$. For $p = 0$ the functions $B(z) = 1/\log(1+z^{-1})$ ([4], p. 129) and $B(z) = z/(1+z)$ have this property at 0 and ∞ respectively.

LEMMA 5.1. *The Bernstein functions associated with processes from D_Δ are regularly varying at Δ.*

Proof. Let $B(z)$ be the Bernstein function associated with a process from D_Δ. Suppose that u_n and v_n have the required properties and denote by $H_n(z)$ the Laplace transform of the probability distribution of (5.2). By assumption $H_n(z)$ tends to the Laplace transform $H(z)$ of the limit distribution. By Theorem 2.4,

$$H_n(z) = 1 - v_nz\int_0^\infty e^{-u_ny}H_n(ze^{-u_ny})\,\varrho(dy)$$

where ϱ is the potential of the process in question and $z \in [0,\infty)$. Since $H_n(z)$ is decreasing the above equation yields

$$H_n(z) \le 1 - v_nzH_n(z)\widetilde{\varrho}(u_n)\,.$$

Now, by virtue of (1.8), we get

$$\frac{v_n}{B(u_n)} \le \frac{1-H_n(1)}{H_n(1)}\,.$$

Since $H(1) > 0$ we conclude that $v_n \le cB(u_n)$ $(n = 1,2,\ldots)$ for a positive constant c. By (3.15) we have for sufficiently small $|z|$

$$H_n(z) = \sum_{k=0}^\infty (-v_nz)^k \prod_{j=1}^k B(ju_n)^{-1}\,.$$

Since $B(z)$ is nondecreasing the absolute value of the kth coefficient in the above power series is not greater than c^k. Hence the limit function $H(z)$ is analytic in a neighbourhood of the origin. Setting

$$H(z) = \sum_{k=0}^\infty a_k(-z)^k$$

we have

$$\lim_{n\to\infty} v_n^k \prod_{j=1}^k B(ju_n)^{-1} = a_k \qquad (k = 1,2,\ldots)\,.$$

Observe that all a_k are positive because the limit measure is different from δ_0. Consequently, for any positive integer k

$$\lim_{n\to\infty} \frac{B(ku_n)}{B(u_n)} = \frac{a_1 a_{k-1}}{a_k}.$$

Further, from (5.1) it follows that the sequence of functions

$$(e^{-x} - e^{-u_{n+1}x/u_n})(1-e^{-x})^{-1}$$

tends uniformly to 0 on the half-line $[0,\infty)$. Hence, in view of (1.7), we get

$$\lim_{n\to\infty} B(u_{n+1})/B(u_n) = 1.$$

Now our assertion follows from Theorem 1.9 in [7].

LEMMA 5.2. *Suppose that the Bernstein function $B(z)$ associated with a process $X(t,\omega)$ is regularly varying of order p at Δ ($p \in [0,1]$). Then the limit distribution of*

$$B(u) \int_0^\infty e^{-uX(t,\omega)}\,dt \tag{5.3}$$

exists as $u \to \Delta$ and is equal to λ_{1-p}.

Proof. Denote by $G_u(z)$ the Laplace transform of the probability distribution of (5.3). By (3.15), $G_u(z)$ can be expanded into a power series in the unit circle:

$$G_u(z) = \sum_{n=0}^{\infty} a_n(u)(-z)^n$$

where $a_0(u) = 1$ and $a_n(u) = B(u)^n \prod_{k=1}^n B(ku)^{-1}$ ($n = 1,2,\ldots$). Since $B(z)$ is increasing we have $0 < a_n(u) \le 1$ ($n = 0,1,\ldots$). Moreover, $\lim_{u\to\Delta} G_u(z) = \widetilde{\lambda}_{1-p}(z)$, which completes the proof.

As an immediate consequence of Lemmas 5.1 and 5.2 we get the following theorems.

THEOREM 5.1. *A process belongs to D_Δ if and only if its Bernstein function is regularly varying at Δ.*

THEOREM 5.2. $K_0 = K_\infty = \{T_c\lambda_p : p \in [0,1],\ c \in (0,\infty)\}$.

For $\Delta = 0$ the assertion of Theorem 5.1 can be written in an equivalent form.

COROLLARY 5.1. *A process belongs to D_0 if and only if the probability distribution τ_1 belongs to the domain of attraction of a stable law.*

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