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Weak uniform normal structure in direct sum spaces

by

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Abstract. The weak normal structure coefficient $WCS(X)$ is computed or bounded when X is a finite or infinite direct sum of reflexive Banach spaces with a monotone norm.

A Banach space X is said to have *normal structure* (resp. *weak normal structure*) if every closed convex (resp. weakly compact convex) subset of X contains a nondiametral point. It is well known that if X has (weak) normal structure, C is a weakly compact convex subset of X and $T : C \to C$ is a nonexpansive mapping, then T has a fixed point.

In [B] the following coefficients related to the normal structure are defined:

$$N(X) = \inf\{\operatorname{diam} A/r(A) : A \text{ a bounded subset of } X\}$$

where $\operatorname{diam} A$ is the diameter of A and $r(A)$ is the Chebyshev radius of A, and

$$WCS(X) = \inf\{\operatorname{diam}_a(x_n)/r_a(x_n) : (x_n) \text{ is a weakly convergent sequence which is not norm convergent}\}$$

where $r_a(x_n)$ is the *asymptotic radius* of (x_n), i.e. $r_a(x_n) = \inf\{\limsup \|x_n - y\| : y \in \operatorname{co} x_n\}$, and $\operatorname{diam}_a(x_n)$ is the *asymptotic diameter* of (x_n), i.e. $\operatorname{diam}_a(x_n) = \lim_k \sup_{n,m\ge k} \|x_n - x_m\|$. The definition of $WCS(X)$ makes sense when X does not have the Schur property and we can say, by convention, that $WCS(X) = 2$ if X has this property. It is well known that $1 \le N(X) \le WCS(X) \le 2$, X has normal structure if $N(X) > 1$, and weak normal structure if $WCS(X) > 1$. Since the converse results are not true and X is said to have *uniform normal structure* if $N(X) > 1$, we can say that X has *weak uniform normal structure* if $WCS(X) > 1$.

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Another normal structure coefficient, denoted by $D(X)$, was defined in [M]. In [P] it is proved that this coefficient is actually $1/WCS(X)$ for reflexive spaces.

A recently studied problem is the permanence property of normal structure and weak normal structure under the (finite or infinite) direct sum operation (see [L1, L2, C]). In this paper we evaluate the coefficient $WCS(X)$ when X is a direct sum of reflexive spaces, with a monotone norm in the finite case. In the infinite case we only obtain a lower bound for the weak normal structure coefficient either using the Clarkson modulus of convexity or directly, if the substitution space has an Orlicz norm. This bound is the actual value of $WCS(X)$ when the Orlicz sequence space is ℓ^p. In particular, we prove that the direct sum $X_1 \oplus \ldots \oplus X_n$ with a p-norm $(1 \le p \le \infty)$ has weak uniform normal structure whenever all X_i are reflexive spaces with this property. It is noteworthy that, for $p = 1$, weak normal structure is not preserved under finite direct sum operation (see [L2]) and it is unknown (see [C]) if uniform normal structure is preserved under the same operation.

We recall that a norm in $\mathbb{R}^k$ is called *monotone* if $\|(a_1, \ldots, a_k)\| \le \|(b_1, \ldots, b_k)\|$ when $0 \le a_i \le b_i$ for every $i = 1, \ldots, k$. This condition is satisfied if, for instance, the norm is *symmetric*, i.e. $\|(a_1, \ldots, a_k)\| = \|(\varepsilon_1 a_1, \ldots, \varepsilon_k a_k)\|$ for any $\varepsilon_i = \pm 1$. This is the case of the p-norms or Orlicz norms. Assume that X is $\mathbb{R}^k$ with a monotone norm. We denote by $(X_1 \oplus \ldots \oplus X_k)_X$ the product space $X_1 \times \ldots \times X_k$ with the norm $\|(x_1, \ldots, x_k)\| = \|(\|x_1\|, \ldots, \|x_k\|)\|$. When a p-norm in $\mathbb{R}^k$ is considered we use the notation $(X_1 \oplus \ldots \oplus X_k)_p$.

THEOREM 1. *Let $X_1, \ldots, X_k$ be reflexive spaces and let X be $\mathbb{R}^k$ with a monotone norm. Then*

$$WCS((X_1 \oplus \ldots \oplus X_k)_X) = \min\{WCS(X_i) : i = 1, \ldots, k\}.$$

Proof. Using some results of [P] and [D1] it is proved in [D2] that for any reflexive space X we have

$$WCS(X) = \inf\{\lim_{n,m;n\neq m} \|x_n - x_m\| : (x_n) \text{ is a weakly null sequence}, \lim \|x_n\| = 1 \text{ and } \lim_{n,m;n\neq m} \|x_n - x_m\| \text{ exists}\}.$$

Let (x_n) be a weakly null sequence in $(X_1 \oplus \ldots \oplus X_k)_X$ such that $\lim_{n,m;n\neq m} \|x_n - x_m\| = l$ and $\lim \|x_n\| = 1$. We write $x_n = (x_n(1), \ldots \ldots, x_n(k))$. We can assume that $(X_1 \oplus \ldots \oplus X_k)_X$ is infinite-dimensional because otherwise all X_i are finite-dimensional and Theorem 1 is obvious. From Lemma 3.4 of [D1] and a diagonal argument we know that for every bounded sequence (a_n) in a metric space S there exists a subsequence (b_n) of (a_n) such that $\lim_{n,m;n\neq m} d(b_n, b_m)$ exists. Applying this k times we can assume that $\lim_n \|x_n(i)\| = a(i)$, and $\lim_{n,m;n\neq m} \|x_n(i) - x_m(i)\| = l(i)$ for $i = 1, \ldots, k$. Since the topology in $(X_1 \oplus \ldots \oplus X_k)_X$ is the product topology of $X_1 \times \ldots \times X_k$ it is clear that $1 = \lim \|x_n\| = \|(a(i))\|$ and $l = \lim_{n,m;n\neq m} \|x_n - x_m\| = \|(l(i))\|$.

Define $w = \min\{WCS(X_i) : i = 1, \ldots, k\}$. Since $(x_n(i))_n$ is a weakly null sequence in X_i we have $l(i) \ge w a(i)$ for every $i = 1, \ldots, k$ (even if X_i is finite-dimensional). Thus $l = \|(l(i))\| \ge \|(wa(i))\| = w\|(a(i))\| = w$. Hence $WCS((X_1 \oplus \ldots \oplus X_k)_X) \ge w$. On the other hand, it is clear that the converse inequality holds because each X_i equipped with the norm $|x| = \|x\|\|e_i\|$ (e_i is the ith canonical vector of $\mathbb{R}^k$) is a subspace of $(X_1 \oplus \ldots \oplus X_k)_X$. ∎

COROLLARY 1. *Let $X_1, \ldots, X_k$ be reflexive spaces. Then*

$$WCS((X_1 \oplus \ldots \oplus X_k)_p) = \min\{WCS(X_i) : i = 1, \ldots, k\}$$

for every $p \in [1, \infty]$.

Remark 1. The statement of Theorem 1 is not true if the norm in $\mathbb{R}^k$ is arbitrary. Indeed, let $\|\cdot\|$ be any norm in $\mathbb{R}^2$ such that $\|(1,1)\| = \|(2^{1/2}, 2^{1/3})\| = 1$. Then $\ell^2 \oplus \ell^3$ does not have weak normal structure (or normal structure) because the weakly null sequence $((e_n, e_n))_n$ (e_n the basic vectors in ℓ_2 and ℓ_3) satisfies $\|(e_n, e_n) - (e_m, e_m)\| = 1$, $n \neq m$, and $\|(e_n, e_n)\| = 1$.

We now study a similar problem for infinite direct sums of reflexive spaces. Let X be a Banach space with Schauder basis $\{e_i\}$ and monotone norm (i.e. $\|\sum_{i=1}^\infty a_i e_i\| \le \|\sum_{i=1}^\infty b_i e_i\|$ if $0 \le a_i \le b_i$ for every $i \in \mathbb{N}$). If (X_i) is a sequence of Banach spaces we denote by $\bigoplus_X X_i$ the substitution space formed by all sequences (x_i) such that $x_i \in X_i$ and $\sum_{i=1}^\infty \|x_i\| e_i \in X$, with the norm $\|(x_i)\| = \|\sum_{i=1}^\infty \|x_i\| e_i\|$. If M is an Orlicz function and X is the Orlicz space h_M (for standard facts about Orlicz sequence spaces we refer to [L1]) we denote the substitution space by $\bigoplus_M X_i$, and by $\bigoplus_p X_i$ if $M(s) = s^p$ (i.e. $X = \ell^p$). We shall assume throughout this paper that M is an Orlicz nondegenerate function satisfying the Δ_2-condition at zero and without loss of generality that $M(1) = 1$.

We start with the weak normal structure coefficient for an infinite direct sum with Orlicz norm. The following lemma for Orlicz sequence spaces is proved in [D3]:

LEMMA 1. *Let M be an Orlicz nondegenerate function satisfying the Δ_2-condition at zero, φ the inverse function of M and $\alpha = \inf\{\varphi(t)/\varphi(t/2) : t \in (0,1]\}$. Then $\alpha > 1$ and $M(t/\alpha) \ge \frac{1}{2}M(t)$ for every $t \in [0,1]$. Furthermore, $WCS(h_M) \ge \alpha$ and $\alpha = 2^{1/p}$ if $M(t) = t^p$.*

The proof of the following theorem is inspired by the arguments in [Pa, Theorem 1], [K, Lemma 8] and [D3, Lemma 3].

THEOREM 2. *Let M be an Orlicz nondegenerate function satisfying the Δ_2-condition at zero and (X_i) a sequence of reflexive Banach spaces. Then*

$$\inf\{\alpha, WCS(X_i) : i \in \mathbb{N}\} \le WCS\Big(\bigoplus_M X_i\Big) \le \inf\{WCS(X_i) : i \in \mathbb{N}\}.$$

If, in addition, h_M is reflexive then $WCS(\bigoplus_M X_i) \le WCS(h_M)$.

Proof. Since every space X_i with the norm $|x| = \|x\|\|e_i\|$ (e_i is the ith basic vector of h_M) is a subspace of $\bigoplus_M X_i$ it is clear that $WCS(\bigoplus_M X_i) \le \inf\{WCS(X_i) : i \in \mathbb{N}\}$. If h_M is reflexive, so is $\bigoplus_M X_i$. Let $\varepsilon > 0$. There exists a weakly null sequence (ξ_n) such that $\|(\xi_n)\| \to 1$ and

$$\lim_{n,m;n\ne m} \|\xi_n - \xi_m\| < WCS(h_M) + \varepsilon.$$

Since each coordinate sequence $(\xi_n(i))_n$ converges to zero we can assume without loss of generality that $\operatorname{supp}\xi_n \cap \operatorname{supp}\xi_m = \emptyset$ if $n \ne m$ where $\operatorname{supp}\xi = \{i \in \mathbb{N} : \xi(i) \ne 0\}$. Choose for every $n \in \mathbb{N}$ and $i \in \mathbb{N}$ a vector $x_n(i) \in X_i$ such that $\|x_n(i)\| = |\xi_n(i)|$. Then the sequence (x_n) is weakly null and satisfies $\|x_n\| = \|\xi_n\|$ and $\|x_n - x_m\| = \|\xi_n - \xi_m\|$ for every $n, m \in \mathbb{N}$. Since ε is arbitrary we obtain $WCS(\bigoplus_M X_i) \le WCS(h_M)$.

To prove the inequality $\inf\{\alpha, WCS(X_i) : i \in \mathbb{N}\} \le WCS(\bigoplus_M X_i)$ note that if (x_n) is a bounded sequence in a Banach space we can find a subsequence (y_n) such that $\limsup\|x_n\| = \lim\|y_n\|$ and $\lim_{n,m;n\ne m}\|y_n - y_m\|$ exists. Hence

$$\frac{\operatorname{diam_a}(x_n)}{r_a(x_n)} \ge \frac{\operatorname{diam_a}(x_n)}{\limsup\|x_n\|} \ge \frac{\lim_{n,m;n\ne m}\|y_n - y_m\|}{\lim\|y_n\|}.$$

Thus we only need to prove that $l \ge \inf\{\alpha, WCS(X_i) : i \in \mathbb{N}\}$ for every normalized weakly null sequence (x_n) in $\bigoplus_M X_i$ with $\lim_{n,m;n\ne m}\|x_n - x_m\| = l$. Write $w = \inf\{\alpha, WCS(X_i) : i \in \mathbb{N}\}$ and let (x_n) be a normalized weakly null sequence in $\bigoplus_M X_i$ such that $\lim_{n,m;n\ne m}\|x_n - x_m\| = l$. Taking subsequences and using a diagonal argument we can assume that $\lim\|x_n(i)\| = a(i)$ and $\lim_{n,m;n\ne m}\|x_n(i) - x_m(i)\| = l(i)$ for any $i \in \mathbb{N}$. Since $\sum_{i=1}^{\infty} M(\|x_n(i)\|) = 1$ and using the continuity of M it is easy to check that $(a(i))$ is in h_M and $\|(a(i))\| \le 1$.

Assume, by contradiction, that $l < w$ and choose $\varepsilon > 0$ with $w(1-2\varepsilon) > l + 4\varepsilon$. Since (e_i) is a Schauder basis of h_M there exists $i_1 \in \mathbb{N}$ such that $\|\sum_{i>i_1} a(i)e_i\| < \varepsilon$. Choose $n_1 \in \mathbb{N}$ large enough such that

$$\Big|\Big\|\sum_{i\le i_1}\|x_m(i)\|e_i\Big\| - \Big\|\sum_{i\le i_1} a(i)e_i\Big\|\Big| < \varepsilon,$$

$\|x_{n_1} - x_m\| < l + \varepsilon$ and

$$\Big|\Big\|\sum_{i\le i_1}\|x_{n_1}(i) - x_m(i)\|e_i\Big\| - \Big\|\sum_{i\le i_1} l(i)e_i\Big\|\Big| < \varepsilon$$

for any $m > n_1$. There exists $i_2 \in \mathbb{N}$ such that $\|\sum_{i>i_2}\|x_{n_1}(i)\|e_i\| < \varepsilon$. Choose $n_2 \in \mathbb{N}$ large enough such that $\|\sum_{i=i_1+1}^{i_2}\|x_{n_2}(i)\|e_i\| < \varepsilon$. (Recall that $(x_n(i))_n$ converges to $(a(i))$ and $\|\sum_{i\ge i_1} a(i)e_i\| < \varepsilon$.) Then

$$\begin{aligned} w(1-2\varepsilon) &> l + 4\varepsilon > \|x_{n_1} - x_{n_2}\| + 3\varepsilon \\ &> \Big\|\sum_{i\le i_1} l(i)e_i + \sum_{i=i_1+1}^{i_2}\|x_{n_1}(i)\|e_i + \sum_{i>i_2}\|x_{n_2}(i)\|e_i\Big\|. \end{aligned}$$

Thus

$$\begin{aligned} 1 &> \sum_{i\le i_1} M\Big(\frac{l(i)}{w(1-2\varepsilon)}\Big) + \sum_{i=i_1+1}^{i_2} M\Big(\frac{\|x_{n_1}(i)\|}{w(1-2\varepsilon)}\Big) + \sum_{i>i_2} M\Big(\frac{\|x_{n_2}(i)\|}{w(1-2\varepsilon)}\Big) \\ &\ge \frac12\Big[\sum_{i\le i_1} M\Big(\frac{a(i)}{1-2\varepsilon}\Big) + \sum_{i=i_1+1}^{i_2} M\Big(\frac{\|x_{n_1}(i)\|}{1-2\varepsilon}\Big)\Big] \\ &\quad + \frac12\Big[\sum_{i\le i_1} M\Big(\frac{a(i)}{1-2\varepsilon}\Big) + \sum_{i>i_2} M\Big(\frac{\|x_{n_2}(i)\|}{1-2\varepsilon}\Big)\Big] > 1 \end{aligned}$$

because both norms $\|\sum_{i\le i_1} a(i)e_i + \sum_{i=i_1+1}^{i_2}\|x_{n_1}(i)\|e_i\|$ and $\|\sum_{i\le i_1} a(i)e_i + \sum_{i>i_2}\|x_{n_2}(i)\|e_i\|$ are greater than $1-2\varepsilon$. This contradiction proves that $l \ge w$. ■

If $M(s) = s^p$ the substitution space $\bigoplus_M X_i$ will be denoted by $\bigoplus_p X_i$, which is obviously the set of all sequences $(x(i))$ where $x(i)$ belongs to X_i and $\sum_{i=1}^{\infty}\|x(i)\|^p < \infty$ with the norm $\|x(i)\| = (\sum_{i=1}^{\infty}\|x(i)\|^p)^{1/p}$. Since in this case $\alpha = WCS(\ell_p) = 2^{1/p}$ the following corollary is clear:

COROLLARY 2. *Let (X_i) be a sequence of reflexive Banach spaces. Then for any $p \in [1,\infty)$ we have*

$$WCS\Big(\bigoplus_p X_i\Big) = \inf\{WCS(X_i), 2^{1/p} : i \in \mathbb{N}\}.$$

Remark 2. Having in mind Theorem 1 and Corollary 2 one can expect $WCS(\bigoplus_X X_i) = \inf\{WCS(X_i), WCS(X) : i \in \mathbb{N}\}$ for any substitution space with monotone norm (it is clear from Remark 1 that this result is not true for arbitrary norms). If this result were true, the following conjecture of Kottmann (see [K]) would also be true: $K(\bigoplus_X X_i) = \sup\{K(X_i), K(X) : i \in \mathbb{N}\}$ where $K(X)$ is the maximal separation for a sequence in the unit ball of X, i.e. $K(X) = \sup\{\varepsilon > 0 :$ there exists a sequence (x_n) in X satisfying $\|x_n\| \le 1$ and $\|x_n - x_m\| \ge \varepsilon$ for every $n, m \in \mathbb{N}$, $n \ne m\}$. Unfortunately, we do not know how to prove the above equality. However, we have the following weaker result:

THEOREM 3. *Let (X_i) be a sequence of reflexive Banach spaces and X a reflexive Banach space with Schauder basis $\{e_i\}$ and monotone norm. Define $w = \inf\{WCS(X_i) : i \in \mathbb{N}\}$ and let $\delta(\cdot)$ be the Clarkson modulus of convexity of X, i.e.*

$$\delta(\varepsilon) = 1 - \sup\left\{\frac{\|x+y\|}{2} : \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \varepsilon\right\}.$$

Then

$$WCS\Big(\bigoplus_X X_i\Big) \ge \left(1 - \delta\left(\frac{w-1}{2}\right)\right)^{-1}.$$

Proof. Let (x_n) be a sequence in $\bigoplus_X X_i$ such that $\|x_n\| = 1$ and $\lim_{n,m;n\ne m} \|x_n - x_m\| = l$. We shall prove

$$l \ge \left[1 - \delta\left(\frac{w-1}{2}\right)\right]^{-1}.$$

Indeed, otherwise choose $\varepsilon > 0$ such that

$$l + \varepsilon < (1-\varepsilon)\left[1 - \delta\left(\frac{(w-1)(1-\varepsilon)}{2+\varepsilon}\right)\right]^{-1}.$$

(Recall that the Clarkson modulus of convexity is a continuous function.) Following an argument as in the proof of Theorem 2 we can assume that $\lim \|x_n(i)\| = a(i)$ and $\lim_{n,m;n\ne m} \|x_n(i) - x_m(i)\| = l(i)$ for any $i \in \mathbb{N}$.

Note that $\sup_k \|\sum_{i=1}^k a(i)e_i\| \le 1$. Indeed, if for some $k \in \mathbb{N}$ we have $\|\sum_{i=1}^k a(i)e_i\| > 1$ we can choose n large enough such that $\|\sum_{i=1}^k \|x_n(i)\| e_i\| > 1$, which is a contradiction because $\|\sum_{i=1}^\infty \|x_n(i)\| e_i\| = 1$.

Since X is reflexive, the basis $\{e_n\}$ is boundedly complete (see, for instance, [Be, Th. 2.1.5]). Thus there exists $x \in X$ such that $x = \sum_{i=1}^\infty a(i)e_i$. In particular, there exists $i_1 \in \mathbb{N}$ such that $\|\sum_{i>i_1} a(i)e_i\| > \varepsilon$. In a similar way to the proof of Theorem 2 we can find $k_1 \in \mathbb{N}$, $i_2 \in \mathbb{N}$, $n_1 \in \mathbb{N}$, $n_2 \in \mathbb{N}$ such that

$$|\|a+y\| - 1| < \varepsilon, \quad |\|a+z\| - 1| < \varepsilon, \quad |\|b+y+z\| - l| < \varepsilon$$

where

$$a = \sum_{i \le i_1} a(i)e_i,\ y = \sum_{i=i_1+1}^{i_2} \|x_{n_1}(i)\| e_i,\ z = \sum_{i>i_2} \|x_{n_2}(i)\| e_i,\ b = \sum_{i \le i_1} l(i)e_i.$$

Since $l(i) \ge wa(i)$, the monotonicity of the norm implies $l+\varepsilon \ge \|wa+y+z\|$. Since $\|(wa+y+z)-(a+z)\| = \|(w-1)a+z\| \ge (w-1)\|a+z\| \ge (w-1)(1-\varepsilon)$ and

$$\frac{1}{l+\varepsilon}\|wa+y+z\| \le 1, \quad \frac{1}{l+\varepsilon}\|a+z\| \le 1$$

we have from the definition of the Clarkson modulus of convexity

$$\frac{1}{l+\varepsilon}\left\|\frac{w+1}{2}a + \frac{y}{2} + z\right\| \le 1 - \delta\left((w-1)\frac{1-\varepsilon}{2+\varepsilon}\right).$$

Using again the monotonicity of the norm we obtain

$$\frac{1}{l+\varepsilon}\|a+z\| \le 1 - \delta\left((w-1)\frac{1-\varepsilon}{2+\varepsilon}\right),$$

which implies

$$\frac{1-\varepsilon}{l-\varepsilon} \le 1 - \delta\left((w-1)\frac{1-\varepsilon}{2+\varepsilon}\right),$$

that is,

$$l - \varepsilon \ge (1-\varepsilon)\left[1 - \delta\left((w-1)\frac{1-\varepsilon}{2+\varepsilon}\right)\right]^{-1},$$

contradicting the choice of ε. ∎

Since a Banach space is uniformly convex if and only if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$, the following corollary is clear:

COROLLARY 3. *Let X be a uniformly convex Banach space and (X_i) a sequence of reflexive Banach spaces such that $\inf\{WCS(X_i) : i \in \mathbb{N}\} > 1$. Then $\bigoplus_X X_i$ has uniform weak normal structure.*

References

[Be] B. Beauzamy, *Introduction to Banach Spaces and their Geometry*, North-Holland, Amsterdam 1982.

[B] W. L. Bynum, *Normal structure coefficients for Banach spaces*, Pacific J. Math. 86 (1980), 427–435.

[C] E. Casini, *Degree of convexity and product spaces*, Comment. Math. Univ. Carolinae 31 (4) (1990), 637–641.

[D1] T. Domínguez Benavides, *Some properties of the set and ball measures of non-compactness and applications*, J. London Math. Soc. 34 (2) (1986), 120–128.

[D2] T. Domínguez Benavides and G. Lopez Acedo, *Lower bounds for normal structure coefficients*, Proc. Roy. Soc. Edinburgh Sect. A, to appear.

[D3] T. Domínguez Benavides and R. J. Rodriguez, *Some geometrical constants in Orlicz sequence spaces*, Nonlinear Anal., to appear.

[K] C. A. Kottmann, *Subsets of the unit ball that are separated by more than one*, Studia Math. 53 (1975), 15–25.

[L1] T. Landes, *Permanence property of normal structure*, Pacific J. Math. 111 (1) (1984), 125–143.

[L2] —, *Normal structure and the sum property*, ibid. 123 (1) (1986), 127–147.

[Li] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer, Berlin 1977.

[M] E. Maluta, *Uniform normal structure and related coefficients*, Pacific J. Math. 111 (1984), 357–367.

[Pa] J. P. Partington, *On nearly uniformly convex Banach spaces*, Math. Proc. Cambridge Philos. Soc. 93 (1983), 127–129.

[P] S. Prus, *On Bynum's fixed point theorem*, Atti Sem. Mat. Fis. Univ. Modena 38 (1990), 535–545.

[S] M. A. Smith, *Rotundity and extremity in* $\ell^p(X_i)$ *and* $L^p(\mu, X)$, in: Contemp. Math. 52, Amer. Math. Soc., 1986, 143–162.

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Banach spaces and bilipschitz maps

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Abstract. We show that a normed space E is a Banach space if and only if there is no bilipschitz map of E onto $E \setminus \{0\}$.

1. Introduction. A map $f : X \to Y$ between metric spaces X and Y is *bilipschitz* if there is a number $M \ge 1$ such that

$$|x - y|/M \le |f(x) - f(y)| \le M|x - y|$$

for all $x, y \in X$. We also say that f is *M-bilipschitz*. The inverse $f^{-1} : fX \to X$ of an M-bilipschitz map is also M-bilipschitz. A bilipschitz map preserves Cauchy sequences and maps complete sets onto complete sets. In particular, if E and E' are Banach spaces and if $f : E \to E'$ is bilipschitz, then fE is closed in E'. Hence f cannot map a Banach space E onto an open proper subset of E. The purpose of this note is to show that this property characterizes the Banach spaces in the class of all normed vector spaces. We formulate the result below; for some variations see Remark 6.

2. THEOREM. *A normed space E (real or complex) is a Banach space if and only if there is no bilipschitz map of E onto $E \setminus \{0\}$.*

3. *Notation.* The norm of a vector $x \in E$ is written as $|x|$. We let $B(x, r)$ and $\overline{B}(x, r)$ denote the open and the closed ball in E, respectively, with center x and radius r. The boundary sphere $\partial B(x, r)$ is written as $S(x, r)$.

The proof of Theorem 2 will be based on the following elementary construction:

4. LEMMA. *Let $a, b \in E$, and let $r \ge 2|a - b|$. Then there is a homeomorphism $h : E \to E$ such that*

(1) $h(a) = b$,
(2) $h(x) = x$ *if* $|x - a| \ge r$,
(3) h *is M-bilipschitz with* $M = 1 + 2|a - b|/r$.

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