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An example of a subalgebra of H^∞ on the unit disk whose stable rank is not finite

by

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Abstract. We present an example of a subalgebra with infinite stable rank in the algebra of all bounded analytic functions in the unit disk.

1. Introduction. Let A be a commutative complex Banach algebra with identity. It is known that the notion of stable rank (introduced by H. Bass [1]) is closely related to the topology of the spectrum of A (see, e.g., Corach–Suárez [4–6] and Vaserstein [15]). In order to get more insight into the structure of spectra of uniform algebras $A \subseteq C(X)$, it is therefore of interest to explicitly determine the stable rank of A. This was done for various algebras of analytic functions with smooth boundary values by Jones–Marshall–Wolff [9], Corach–Suárez [4–6] and Rupp [11, 12]. In particular, it was shown in [6] that the stable rank of the polydisk algebra $A(\mathbb{D}^n)$ and the ball algebra $A(\mathbb{B}^n)$ is $[n/2]+1$. Moreover, Rupp [11, 12] was able to show that the stable rank of many classes of subalgebras of $A(\mathbb{B}^n)$ is less than n, in particular is finite. These algebras include, e.g., all subalgebras A of $A^1(\mathbb{D}^n) = \{f \in A(\mathbb{D}^n) : f' \in A(\mathbb{D}^n)\}$ in which the weak Nullstellensatz holds, i.e., for which $(f_1, \ldots, f_n) = A$ if and only if the functions f_j have no common zero in $\overline{\mathbb{D}}^n$. In particular, $\operatorname{bsr} A = 1$ whenever $A \subseteq A^1(\mathbb{D})$ and A satisfies the weak Nullstellensatz (see [11]).

Only recently has the stable rank of the algebra H^∞ of all bounded analytic functions in the unit disk $\mathbb{D}$ been determined by Treil [14]: it is also one. This raises the following questions. What does the situation for subalgebras of H^∞ look like? Are there any subalgebras A of H^∞ which do not have stable rank one? Can $\operatorname{bsr} A$ be infinite? It is the aim of this note to answer these questions. First we give some definitions.

Let A be a commutative ring with identity element 1. An element $(a_1, \ldots, a_n) \in A^n$ is called *unimodular* if $\sum_{j=1}^n a_j A = A$. The set of

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all unimodular elements of A^n is denoted by $U_n(A)$. We say that $a = (a_1, \ldots, a_{n+1}) \in U_{n+1}(A)$ is *reducible* if there exist $(x_1, \ldots, x_n) \in A^n$ such that $(a_1 + x_1 a_{n+1}, \ldots, a_n + x_n a_{n+1}) \in U_n(A)$. The *(Bass) stable rank* of A, denoted by $\operatorname{bsr} A$, is the least integer $n \in \mathbb{N}$ for which every $a \in U_{n+1}(A)$ is reducible. If there is no such integer n, we say that A has *infinite* stable rank.

2. The infinite polydisk algebra.

As mentioned in the introduction, the stable rank of the polydisk algebra

$$A(\mathbb{D}^n) = \{f \text{ continuous on the closed polydisk } \overline{\mathbb{D}}^n \text{ and analytic in its interior}\}$$

is $[n/2] + 1$.

We show next that the stable rank of the infinite polydisk algebra [2] is not finite. This observation may be known, because, if we consider $A(\mathbb{D}^n)$ as a quotient algebra of $A(\mathbb{D}^\infty)$, we obtain

$$(*) \qquad \operatorname{bsr} A(\mathbb{D}^\infty) \ge \operatorname{bsr} A(\mathbb{D}^n) = [n/2] + 1$$

for all n. We want, however, to give a self-contained proof along the lines of [12].

PROPOSITION 1. *Let $A = A(\mathbb{D}^\infty)$ be the infinite polydisk algebra, i.e., the uniform closure of the algebra generated by the coordinate functions $z_1, z_2, \ldots$ on the countably infinite polydisk $\overline{\mathbb{D}}^\infty = \overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \ldots$ Then the stable rank of A is infinite.*

Proof. Fix $n \in \mathbb{N}$. We claim that the element $(z_1, \ldots, z_n, g)$ of A^{n+1}, where $g(z) = \prod_{j=1}^n (1 - z_j z_{n+j})$, $z = (z_1, z_2, \ldots) \in \overline{\mathbb{D}}^\infty$, is not reducible.

First we note that $(z_1, \ldots, z_n, g)$ is unimodular. Assume that there exist $h_1, \ldots, h_n \in A$ such that

$$(1) \qquad (z_1 + gh_1, \ldots, z_n + gh_n) \text{ is unimodular in } A^n.$$

Let $h = (h_1, \ldots, h_n)$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we define

$$H(z) = \begin{cases} -h(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n, 0, \ldots) \prod_{j=1}^n (1 - |z_j|^2) \\ \qquad\qquad \text{for } |z_j| \le 1 \ (j = 1, \ldots, n), \\ 0 \quad \text{otherwise.} \end{cases}$$

Then H is a continuous map from $\mathbb{C}^n$ into $\mathbb{C}^n$. Because $\max_{z \in \overline{\mathbb{D}}^n} |H(z)| = \sup_{z \in \mathbb{C}^n} |H(z)|$, it is easy to see that there exists a polydisk $\overline{D}^n \supseteq \overline{\mathbb{D}}^n$ such that H maps $\overline{D}^n$ into $\overline{D}^n$. Since $\overline{D}^n$ is compact and convex, by Brouwer's fixed point theorem there exists $\zeta \in \overline{D}^n$ such that $H(\zeta) = \zeta$. Since $H = 0$ outside $\overline{\mathbb{D}}^n$, we see that $\zeta \in \overline{\mathbb{D}}^n$. Let $\zeta = (z_1, \ldots, z_n)$. Hence, for every $j \in \{1, \ldots, n\}$, we obtain

$$\begin{aligned} 0 &= z_j + h_j(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n, 0, \ldots) \prod_{j=1}^n (1 - |z_j|)^2 \\ &= z_j + (h_j g)(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n, 0, \ldots), \end{aligned}$$

which contradicts (1). Note that the spectrum of A is the infinite polydisk $\overline{\mathbb{D}}^\infty$ itself [2]. ■

3. The subalgebra B_n of H^∞.

All proper subalgebras of H^∞ for which the stable rank is presently known are subalgebras of the disk algebra $A(\mathbb{D})$. The results in this section will provide us with various classes of subalgebras B_n of H^∞ for which either $A(\mathbb{D}) \subset B_n \subset H^\infty$ or $B_n \subset H^\infty$, $A(\mathbb{D}) \not\subset B_n$, and for which the stable rank is n ($n \in \mathbb{N} \cup \{\infty\}$).

For our construction we need to work with *interpolating Blaschke products*. These are the Blaschke products

$$b(z) = \prod_{n=1}^\infty \frac{\overline{a}_n}{|a_n|} \cdot \frac{a_n - z}{1 - \overline{a}_n z}$$

whose zero sequence (a_n) is an interpolating sequence for H^∞. Recall that (z_n) is an *interpolating sequence* if for every bounded sequence (w_n) of complex numbers there exists a function $f \in H^\infty$ such that $f(z_n) = w_n$ for every n. Reducibility of unimodular vectors whose last component is an interpolating Blaschke product is in fact rather easy to prove as the following lemma shows.

LEMMA 2. *Let $f_1, \ldots, f_n \in H^\infty$ and let b be an interpolating Blaschke product. Suppose that $f = (f_1, \ldots, f_n, b)$ is a unimodular element in $(H^\infty)^{n+1}$. Then f is reducible. Moreover, there exist $g_j, h_j \in H^\infty$ such that $\sum_{j=1}^n e^{h_j}(f_j + g_j b) = 1$.*

Proof. By assumption we have $\sum_{j=1}^n |f_j(z_k)| \ge \delta > 0$, where $\{z_k : k \in \mathbb{N}\}$ denotes the zero set of b in $\mathbb{D}$. Let $\alpha_j^{(k)} = -i \arg f_j(z_k)$ whenever $f_j(z_k) \ne 0$ and $\alpha_j^{(k)} = 0$ otherwise ($\arg f_j(z_k) \in (-\pi, \pi]$).

Because $\{z_k\}$ is an interpolating sequence, there exists $k_j \in H^\infty$ such that $k_j(z_k) = \alpha_j^{(k)}$ $(k = 1, 2, \ldots; j = 1, \ldots, n)$. Hence $w_k = (\sum_{j=1}^n e^{k_j} f_j)(z_k) = \sum_{j=1}^n |f_j(z_k)| \ge \delta > 0$ $(k = 1, 2, \ldots)$. Let $F = \sum_{j=1}^n e^{k_j} f_j$. Since $\log w_k$ is a bounded sequence, again there exists a function $K \in H^\infty$ such that $K(z_k) = \log w_k$ $(k = 1, 2, \ldots)$. Therefore $(F - e^K)(z_k) = 0$, and hence $F = e^K - gb$ for some $g \in H^\infty$. This yields $\sum_{j=1}^n e^{k_j - K}(f_j + g_j b) = 1$, where $g_j = (g/n)e^{-k_j}$. ■

Remark. The case $n = 1$ appears in [10], where it is also shown that, in general, the left factors of the summands above cannot be taken to be exponentials.

We are now able to construct for every $n \in \mathbb{N} \cup \{\infty\}$ subalgebras B_n of H^∞ whose stable rank is n.

To this end let $M(H^\infty)$ denote the spectrum of H^∞, that is, the space of nonzero multiplicative linear functionals on H^∞ endowed with the weak-$*$ topology. Because H^∞ is a uniform algebra, we can identify functions f in H^∞ with their Gelfand transforms $\widehat{f} : M(H^\infty) \to \mathbb{C}$ defined by $\widehat{f}(m) = m(f)$ (see [8, §186). Let b be an interpolating Blaschke product and let $Z(b) = \{m \in M(H^\infty) : m(b) = 0\}$. By [8, p. 379] we know that $Z(b)$ equals the (weak-$*$) closure $\overline{\{z_k\}}$ of the zero set of b in $\mathbb{D}$. Hence $Y = Z(b)$ is homeomorphic to the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Moreover, we see that the restriction of H^∞ to Y, denoted by $H^\infty|_Y$, equals $C(Y)$, the space of all complex-valued continuous functions on Y.

We now have the following result.

THEOREM 3. *Let $Y \subseteq M(H^\infty)$ be as above and let $A \subseteq C(Y)$ be a uniform algebra whose stable rank is n ($n \in \mathbb{N} \cup \{\infty\}$). Then $B = \{f \in H^\infty : f|_Y \in A\}$ is a uniform algebra with the same stable rank.*

Proof. Step 1. The first assertion clearly follows from the fact that B inherits the norm of H^∞ and $H^\infty|_Y = C(Y)$. Let ϱ denote the restriction mapping from H^∞ onto $C(Y)$. Suppose that $n \in \mathbb{N}$ and let $(f_1, \ldots, f_n, f) \in U_{n+1}(B)$. Obviously, we have $(\varrho(f_1), \ldots, \varrho(f_n), \varrho(f)) \in U_{n+1}(A)$. Because $\operatorname{bsr} A = n$, there exist $a_j \in A$ ($j = 1, \ldots, n$) such that $(\varrho(f_1) + a_1\varrho(f), \ldots \ldots, \varrho(f_n) + a_n\varrho(f)) \in U_n(A)$, i.e., $\sum_{j=1}^n A_j(\varrho(f_j) + a_j\varrho(f)) = 1$ for some $A_j \in A$. Let $g_j, G_j \in H^\infty$ be so that $\varrho(g_j) = g_j|_Y = a_j$ and $\varrho(G_j) = G_j|_Y = A_j$ ($j = 1, \ldots, n$). Hence there exists $h \in H^\infty$ such that

$$\sum_{j=1}^n G_j(f_j + g_j f) = 1 + hb. \tag{1}$$

In general, h is not identically zero, and we cannot conclude that $(f_1 + g_1 f, \ldots, f_n + g_n f) \in U_n(B)$. However, as we shall show, there do exist functions $y_j \in H^\infty$ such that

$$(f_1 + (g_1 + y_1 b)f, \ldots, f_n + (g_n + y_n b)f) \in U_n(B). \tag{2}$$

Note that $\varrho(g_j + y_j b) = \varrho(g_j) = a_j$ and that $(f_1 + g_1 f, \ldots, f_n + g_n f, fb) \in U_{n+1}(H^\infty)$.

To this end we first use Treil's result [14] that $\operatorname{bsr} H^\infty = 1 \le n$ in order to conclude that there exist $y_j \in H^\infty$ such that

$$((f_1 + g_1 f) + y_1 fb, \ldots, (f_n + g_n f) + y_n fb) \in U_n(H^\infty). \tag{3}$$

Let $K = h + \sum_{j=1}^n y_j G_j f$. By (3) there exist $x_j \in H^\infty$ such that

$$\sum_{j=1}^n x_j[(f_j + g_j f) + y_j fb] = -K.$$

A simple calculation finally yields

$$\sum_{j=1}^n (G_j + x_j b)(f_j + (g_j + y_j b)f) = 1.$$

This gives assertion (2). Hence $\operatorname{bsr} B \le n$.

Step 2. Let $a = (a_1, \ldots, a_n)$ be a unimodular element in A^n which is not reducible. Choose $f_1, \ldots, f_n \in B$ such that $\varrho(f_j) = f_j|_Y = a_j$ ($j = 1, \ldots, n$). Hence there exist $h_j \in B$, $h \in H^\infty$ such that

$$\sum_{j=1}^n h_j f_j = 1 + hb. \tag{4}$$

In general, $(f_1, \ldots, f_n) \notin U_n(B)$. However, by the same reasoning as before, there exist $y_j \in H^\infty$ such that

$$(f_1 + y_1 b, \ldots, f_n + y_n b) \in U_n(B) \tag{5}$$

(just put $f \equiv 1$, $g_j \equiv 0$) ([1]).

Let $F_j = f_j + y_j b$. Assuming that $\operatorname{bsr} B < n$, there exist $k_j \in B$ ($j = 1, \ldots, n-1$) such that $(F_1 + k_1 F_n, \ldots, F_{n-1} + k_{n-1} F_n) \in U_{n-1}(B)$. Because $\varrho(F_j) = a_j$, we obtain

$$(a_1 + \varrho(k_1)a_n, \ldots, a_{n-1} + \varrho(k_{n-1})a_n) \in U_{n-1}(A).$$

But this contradicts the choice of the vector a. Thus we have proven that $\operatorname{bsr} B = n$.

If $\operatorname{bsr} A = \infty$, then the preceding arguments applied to an arbitrary natural number n show that $\operatorname{bsr} B = \infty$. ∎

The algebra $A \subseteq C(Y)$ in the previous theorem will now be realized as an isometric, isomorphic image of some polydisk algebra $A(\mathbb{D}^n)$ ($n = 1, 2, \ldots, \infty$). The idea of this construction appears in [13]. For the reader's convenience we shall reproduce it here.

In fact, let $X = T^n$, where T is the torus $\{z \in \mathbb{C} : |z| = 1\}$ ($n = 1, 2, \ldots, \infty$). Enumerate a dense subset $x_1, x_2, \ldots$ of X and fix a bijection α between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. Let π_1 denote the projection of $\mathbb{N} \times \mathbb{N}$ onto the first coordinate. Put $\tau(n) = x_{(\pi_1 \circ \alpha)(n)}$. Then τ extends to a continuous map, called again τ, of $\beta\mathbb{N}$ onto X [8, p. 186]. Note that τ actually maps $\beta\mathbb{N} \setminus \mathbb{N}$

([1]) The careful reader may have noticed that in this case it is sufficient to use Lemma 2 instead of Treil's result in order to prove relation (5).

onto X. Identifying Y with $\beta\mathbb{N}$, it is now easy to see that $\tau^* f = f \circ \tau$ defines an isometric algebra isomorphism of $C(X)$ onto $C(Y)$.

Finally, we note that $A(\mathbb{D}^n)$ is isometrically isomorphic to $A(\mathbb{D}^n)|_X$, because T^n is the Shilov boundary of these algebras (see [7] and [2]).

Now let $A_n = \tau^* A(\mathbb{D}^n)$. Then A_n is a uniform subalgebra of $C(Y)$. Let

$$B_n = \{f \in H^\infty : f|_Y \in A_n\} \quad (n = 1, 2, \ldots, \infty).$$

THEOREM 4. *The stable rank of the algebra B_n is $[n/2] + 1$ for $n \in \mathbb{N}$ and infinite for B_∞.*

Proof. Recall that $B_n = \{f \in H^\infty : f|_Y \in \tau^* A(\mathbb{D}^n)\}$. Because the stable rank is invariant under algebra isomorphisms, Corach–Suárez's result yields that bsr $\tau^* A(\mathbb{D}^n) = [n/2]+1$ ($n \in \mathbb{N}$). Hence, by Theorem 3, bsr $B_n = [n/2] + 1$.

Now we consider the algebra

$$B_\infty = \{f \in H^\infty : f|_Y \in \tau^* A(\mathbb{D}^\infty)\}.$$

Fix $n \in \mathbb{N}$. Choose, according to Proposition 1, a vector $(g_1, \ldots, g_n) \in U_n(A(\mathbb{D}^\infty))$ which is not reducible. Let $a = (\tau^* g_1, \ldots, \tau^* g_n)$. Then $a \in U_n(\tau^* A(\mathbb{D}^\infty))$, and a is not reducible.

By the proof of Step 2 in Theorem 3 we obtain a vector $f = (F_1, \ldots, F_n) \in U_n(B_\infty)$ which is not reducible in B_∞. This shows that bsr $B_\infty \geq n$. Since n can be chosen arbitrarily, bsr $B_\infty = \infty$. ∎

Remarks. 1. Using the footnote, we see that our proof for $n = \infty$ is independent of Treil's result.

2. In general, the algebras B_n do not contain the disk algebra. In fact, let Y be the weak-$*$ closure of the interpolating sequence $z_n = 1 - 2^{-n}$, $n \in \mathbb{N}$, in $M(H^\infty)$. Then the Gelfand transform $\widehat{z}$ of z is constantly one on $\overline{\{z_n\}} \setminus \{z_n\}$, which is homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$. Now assume that $z \in B_n$. Because τ maps $\beta\mathbb{N} \setminus \mathbb{N}$ onto $X = T^n$ and X is the Shilov boundary of $A(\mathbb{D}^n)$, we see that $\widehat{z}(z_n) \equiv 1$, which is of course absurd.

However, if we modify a bit the definition of the algebras B_n by setting

$$\widetilde{B}_n = \{f \in H^\infty : f|_{Y^*} \in \tau^* A(\mathbb{D}^n)\},$$

where $Y^* = \overline{\{z_n\}} \setminus \{z_n\}$ ($z_n = 1 - 2^{-n}$), then $A(\mathbb{D}) \subseteq \widetilde{B}_n$ ($n = 1, 2, \ldots, \infty$). Also in this case, bsr $\widetilde{B}_n = [n/2] + 1$ (just replace the interpolating Blaschke product b by a suitable function $\widetilde{b} \in H^\infty$ vanishing on Y^*).

S. Scheinberg [13] showed that for none of these algebras $\widetilde{B}_n$ ($n \geq 2$) does the corona theorem hold; i.e., the unit disk is not dense in the spectrum of $\widetilde{B}_n$. It is at present an open problem whether there exist natural subalgebras A of H^∞ whose stable rank is not one, but for which $\mathbb{D}$ is dense in $M(A)$.

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