An example of a subalgebra of $H^\infty$ on the unit disk whose stable rank is not finite

by

RAYMOND MORTINI (Karlsruhe)

Abstract. We present an example of a subalgebra with infinite stable rank in the algebra of all bounded analytic functions in the unit disk.

1. Introduction. Let $A$ be a commutative complex Banach algebra with identity. It is known that the notion of stable rank (introduced by H. Bass [1]) is closely related to the topology of the spectrum of $A$ (see, e.g., Corach–Suárez [4–6] and Vaserstein [15]). In order to get more insight into the structure of spectra of uniform algebras $A \subseteq C(X)$, it is therefore of interest to explicitly determine the stable rank of $A$. This was done for various algebras of analytic functions with smooth boundary values by Jones–Marshall–Wolff [9], Corach–Suárez [4–6] and Rupp [11, 12]. In particular, it was shown in [6] that the stable rank of the polydisk algebra $A(D^n)$ and the ball algebra $A(B^n)$ is $[n/2]+1$. Moreover, Rupp [11, 12] was able to show that the stable rank of many classes of subalgebras of $A(B^n)$ is less than $n$, in particular is finite. These algebras include, e.g., all subalgebras $A$ of $A^1(D^n) = \{ f \in A(D^n) : f' \in A(D^n) \}$ in which the weak Nullstellensatz holds, i.e., for which $(f_1, \ldots, f_n) = A$ if and only if the functions $f_j$ have no common zero in $D^n$. In particular, $\text{ber } A = 1$ whenever $A \subseteq A^1(D)$ and $A$ satisfies the weak Nullstellensatz (see [11]).

Only recently has the stable rank of the algebra $H^\infty$ of all bounded analytic functions in the unit disk $D$ been determined by Treil [14]: it is also one. This raises the following questions. What does the situation for subalgebras of $H^\infty$ look like? Are there any subalgebras $A$ of $H^\infty$ which do not have stable rank one? Can $\text{ber } A$ be infinite? It is the aim of this note to answer these questions. First we give some definitions.

Let $A$ be a commutative ring with identity element 1. An element $(a_1, \ldots, a_n) \in A^n$ is called unimodular if $\sum_{j=1}^n a_j A = A$. The set of

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all unimodular elements of $A^n$ is denoted by $U_n(A)$. We say that $a = (a_1, \ldots, a_{n+1}) \in U_{n+1}(A)$ is reducible if there exist $(x_1, \ldots, x_n) \in A^n$ such that $(a_1 + x_1 a_{n+1}, \ldots, a_n + x_n a_{n+1}) \in U_n(A)$. The (Base) stable rank of $A$, denoted by $\text{bsr } A$, is the least integer $n \in \mathbb{N}$ for which every $a \in U_{n+1}(A)$ is reducible. If there is no such integer $n$, we say that $A$ has infinite stable rank.

2. The infinite polydisk algebra. As mentioned in the introduction, the stable rank of the polydisk algebra

$$A(D^n) = \{ f \text{ continuous on the closed polydisk } D^n \text{ and analytic in its interior} \}$$

is $[n/2] + 1$.

We show next that the stable rank of the infinite polydisk algebra $[2]$ is not finite. This observation may be known, because, if we consider $A(D^n)$ as a quotient algebra of $A(D^\infty)$, we obtain

$$\text{bsr } A(D^\infty) = [n/2] + 1$$

for all $n$. We want, however, to give a self-contained proof along the lines of [12].

**Proposition 1.** Let $A = A(D^\infty)$ be the infinite polydisk algebra, i.e., the uniform closure of the algebra generated by the coordinate functions $z_1, z_2, \ldots$ on the countably infinite polydisk $D^\infty = \mathbb{D} \times \mathbb{D} \times \ldots$. Then the stable rank of $A$ is infinite.

**Proof.** Fix $n \in \mathbb{N}$. We claim that the element $(z_1, \ldots, z_n, g) \in A^{n+1}$, where $g(z) = \prod_{j=1}^{n} (1 - z_j z_{n+j})$, is not reducible.

First we note that $(z_1, \ldots, z_n, g)$ is unimodular. Assume that there exist $h_1, \ldots, h_n \in A$ such that

$$z_1 + g h_1, \ldots, z_n + g h_n$$

is unimodular in $A^n$.

Let $h = (h_1, \ldots, h_n)$. For $z = (z_1, \ldots, z_n) \in C^n$ we define

$$H(z) = \begin{cases} -h(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n, 0, \ldots) \prod_{j=1}^{n} (1 - |z_j|^2) & \text{for } |z_j| \leq 1 \ (j = 1, \ldots, n), \\ 0 & \text{otherwise}. \end{cases}$$

Then $H$ is a continuous map from $C^n$ into $C^n$. Because $\max_{z \in \overline{D^n}} |H(z)| = \sup_{z \in \overline{D^n}} |H(z)|$, it is easy to see that there exists a polydisk $\overline{D}^n \supset D^n$ such that $H$ maps $D^n$ into $\overline{D}^n$. Since $\overline{D}^n$ is compact and convex, by Brouwer's fixed point theorem there exists $\zeta \in \overline{D}^n$ such that $H(\zeta) = \zeta$. Since $H = 0$ outside $D^n$, we see that $\zeta \in \overline{D}^n$. Let $\zeta = (z_1, \ldots, z_n)$. Hence, for every $j \in \{1, \ldots, n\}$, we obtain

$$0 = z_j + h_j(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n, 0, \ldots) \prod_{j=1}^{n} (1 - |z_j|^2)$$

$$= z_j + (h_j(g))(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n, 0, \ldots),$$

which contradicts (1). Note that the spectrum of $A$ is the infinite polydisk $D^\infty$ itself [2].

3. The subalgebra $B_n$ of $H^\infty$. All proper subalgebras of $H^\infty$ for which the stable rank is presently known are subalgebras of the disk algebra $A(D)$. The results in this section will provide us with various classes of subalgebras $B_n$ of $H^\infty$ for which either $A(D) \subset B_n \subset H^\infty$ or $B_n \subset H^\infty$, $A(D) \not\subset B_n$, and for which the stable rank is $n + 1$ ($n \in \mathbb{N} \cup \{\infty\}$).

For our construction we need to work with interpolating Blaschke products. These are the Blaschke products

$$b(z) = \prod_{n=1}^{\infty} \frac{a_n}{|a_n|} \cdot \frac{a_n - z}{1 - a_n \bar{z}}$$

whose zero sequence $(a_n)$ is an interpolating sequence for $H^\infty$. Recall that $(a_n)$ is an interpolating sequence if for every bounded sequence $(w_n)$ of complex numbers there exists a function $f \in H^\infty$ such that $f(z_n) = w_n$ for every $n$. Reducibility of unimodular vectors whose last component is an interpolating Blaschke product is in fact rather easy to prove as the following lemma shows.

**Lemma 2.** Let $f_1, \ldots, f_n \in H^\infty$ and let $b$ be an interpolating Blaschke product. Suppose that $f = (f_1, \ldots, f_n, b)$ is a unimodular element in $(H^\infty)^{n+1}$. Then $f$ is reducible. Moreover, there exist $g_j, h_j \in H^\infty$ such that $\sum_{j=1}^{n} e^{h_j} f_j + g_j b = 1$.

**Proof.** By assumption we have $\sum_{j=1}^{n} |f_j(z_k)|^2 \geq \delta > 0$, where $\{z_k : k \in \mathbb{N}\}$ denotes the zero set of $b$ in $D$. Let $\alpha_j^{(k)} = -\arg f_j(z_k)$ whenever $f_j(z_k) \neq 0$ and $\alpha_j^{(k)} = 0$ otherwise (arg $f_j(z_k) \in (-\pi, \pi]$).

Because $\{z_k\}$ is an interpolating sequence, there exists $k_j \in H^\infty$ such that $k_j(z_k) = \alpha_j^{(k)}$ ($k = 1, 2, \ldots; j = 1, \ldots, n$). Hence $w_k = (\sum_{j=1}^{n} e^{h_j} f_j)(z_k) = \sum_{j=1}^{n} |f_j(z_k)|^2 \geq \delta > 0$ ($k = 1, 2, \ldots$). Let $F = \sum_{j=1}^{n} e^{h_j} f_j$. Since $w_k$ is a bounded sequence, again there exists a function $K \in H^\infty$ such that $K(z_k) = \log w_k$ ($k = 1, 2, \ldots$). Therefore $(F - e^{h_j} f_j)(z_k) = 0$, and hence $F = e^{h_j} - gb$ for some $g \in H^\infty$. This yields $\sum_{j=1}^{n} e^{h_j} f_j + g_j b = 1$, where $g_j = (b/g(n)) e^{-h_j}$. ■
Remark. The case \( n = 1 \) appears in [10], where it is also shown that, in general, the left factors of the summands above cannot be taken to be exponentials.

We are now able to construct for every \( n \in \mathbb{N} \cup \{ \infty \} \) subalgebras \( B_n \) of \( H^\infty \) whose stable rank is \( n \).

To this end let \( M(H^\infty) \) denote the spectrum of \( H^\infty \), that is, the space of nonzero multiplicative linear functionals on \( H^\infty \) endowed with the weak-* topology. Because \( H^\infty \) is a uniform algebra, we can identify functions \( f \) in \( H^\infty \) with their Gelfand transforms \( f : M(H^\infty) \to \mathbb{C} \) defined by \( f(m) = m(f) \) (see [8, §186]). Let \( b \) be an interpolating Blaschke product and let \( Z(b) = \{ m \in M(H^\infty) : m(b) = 0 \} \). By [8, p. 379] we know that \( Z(b) \) equals the (weak-*) closure \( \overline{\{b_k\}} \) of the zero set of \( b \) in \( \mathbb{D} \). Hence \( Y = Z(b) \) is homeomorphic to the Stone–Čech compactification \( \beta \mathbb{N} \) of \( \mathbb{N} \). Moreover, we see that the restriction of \( H^\infty \) to \( Y \), denoted by \( H^\infty|Y \), equals \( C(Y) \), the space of all complex-valued continuous functions on \( Y \).

We now have the following result.

**Theorem 3.** Let \( Y \subseteq M(H^\infty) \) be as above and let \( A \subseteq C(Y) \) be a uniform algebra whose stable rank is \( n \) \((n \in \mathbb{N} \cup \{ \infty \})\). Then \( B = \{ f \in H^\infty : f|Y \in A \} \) is a uniform algebra with the same stable rank.

**Proof.** Step 1. The first assertion clearly follows from the fact that \( B \) inherits the norm of \( H^\infty \) and \( H^\infty|Y = C(Y) \). Let \( \varphi \) denote the restriction mapping from \( H^\infty \) onto \( C(Y) \). Suppose that \( n \in \mathbb{N} \) and let \( (f_1, \ldots, f_n, f) \in U_{n+1}(B) \). Obviously, we have that \( (\varphi(f_1), \ldots, \varphi(f_n), \varphi(f)) \in U_{n+1}(A) \). Because \( A = n \), there exist \( a_j \in A \) \((j = 1, \ldots, n)\) such that \( \varphi(f_j) = a_j \varphi(f) \). Hence there exists \( \lambda \in H^\infty \) such that \( G_j(f) = \lambda^j g_j(f) \).

\[
\sum_{j=1}^{n} G_j(f_j + g_j f) = 1 + nh.
\]

In general, \( h \) is not identically zero, and we cannot conclude that \( (f_1 + g_1 f, \ldots, f_n + g_n f) \in U_n(B) \). However, as we shall show, there do exist functions \( y_j \in H^\infty \) such that

\[
(f_1 + g_1 y, \ldots, f_n + g_n y) \in U_n(B).
\]

Note that \( \varphi(g_j + y) = \varphi(g_j) = a_j \) and that \( (f_1 + g_1 y, \ldots, f_n + g_n y) \in U_{n+1}(H^\infty) \).

To this end we first use Treil's result [14] that \( \text{bsr } H^\infty = 1 \leq n \) in order to conclude that there exist \( y_j \in H^\infty \) such that

\[
(f_1 + g_1 y, \ldots, f_n + g_n y) \in U_n(H^\infty).
\]

Let \( K = h + \sum_{j=1}^{n} y_j G_j f \). By (3) there exist \( x_j \in H^\infty \) such that

\[
\sum_{j=1}^{n} x_j [(f_j + g_j f) + y_j f] = -K.
\]

A simple calculation finally yields

\[
\sum_{j=1}^{n} (G_j + x_j y)(f_j + (g_j + y_j f)b) = 1.
\]

This gives assertion (2). Hence \( \text{bsr } B \leq n \).

**Step 2.** Let \( a = (a_1, \ldots, a_n) \) be a unimodular element in \( A^n \) which is not reducible. Choose \( f_1, \ldots, f_n \in B \) such that \( \varphi(f_j) = a_j \varphi(f) \). Hence there exist \( h_j \in B, h \in H^\infty \) such that

\[
\sum_{j=1}^{n} h_j f_j = 1 + nh.
\]

In general, \( (f_1, \ldots, f_n) \in U_n(B) \). However, by the same reasoning as before, there exist \( y_j \in H^\infty \) such that

\[
(f_1 + y_1 b, \ldots, f_n + y_n b) \in U_n(B)
\]

(1).

Let \( B = \{ f \in H^\infty : f|Y \in A \} \).

This contradicts the choice of the vector \( a \). Thus we have proven that there exist \( h_j \in B, h \in H^\infty \) such that

(3) \(
(f_1 + g_1 y, \ldots, f_n + g_n y) \in U_n(H^\infty)
\).

The algebra \( A \subseteq C(Y) \) in the previous theorem will now be realized as an isometric, isomorphic image of some polydisk algebra \( A(D^n) \) \((n = 1, 2, \ldots, \infty)\). The idea of this construction appears in [13]. For the reader's convenience we shall reproduce it here.

In fact, let \( X = \mathbb{T}^n \), where \( \mathbb{T} \) is the torus \( \{ x \in \mathbb{C} : |x| = 1 \} \). Enumerate a dense subset \( \tau_1, \tau_2, \ldots \) of \( X \) and fix a bijection \( \alpha \) between \( N \) and \( N \times N \). Let \( \tau_i \) denote the projection of \( N \times N \) onto the first coordinate. Put \( \tau(n) = x_{(\tau_1(n), \alpha(n))} \). Then \( \tau \) extends to a continuous map, called again \( \tau \), of \( \beta \mathbb{N} \) onto \( X \) [8, p. 186]. Note that \( \tau \) actually maps \( \beta \mathbb{N} \setminus \mathbb{N} \)

(1) The careful reader may have noticed that in this case it is sufficient to use Lemma 2 instead of Treil's result in order to prove relation (5).
onto $X$. Identifying $Y$ with $\beta N$, it is now easy to see that $\tau^* f = f \circ \tau$ defines an isometric algebra isomorphism of $C(X)$ onto $C(Y)$.

Finally, we note that $A(\mathbb{D}^n)$ is isometrically isomorphic to $A(\mathbb{D}^n)|_X$, because $T^n$ is the Shilov boundary of these algebras (see [7] and [2]).

Now let $A_n = \tau^* A(\mathbb{D}^n)$. Then $A_n$ is a uniform subalgebra of $C(Y)$. Let

$$B_n = \{ f \in H^\infty : f|_Y \in A_n \} \quad (n = 1, 2, \ldots, \infty).$$

**Theorem 4.** The stable rank of the algebra $B_n$ is $[n/2] + 1$ for $n \in \mathbb{N}$ and infinite for $B_\infty$.

Proof. Recall that $B_n = \{ f \in H^\infty : f|_Y \in \tau^* A(\mathbb{D}^n) \}$. Because the stable rank is invariant under algebra isomorphisms, Corach–Suárez’s result yields that $\text{sr} \tau^* A(\mathbb{D}^n) = [n/2] + 1$ ($n \in \mathbb{N}$). Hence, by Theorem 3, $\text{sr} B_n = [n/2] + 1$.

Now we consider the algebra

$$B_\infty = \{ f \in H^\infty : f|_Y \in \tau^* A(\mathbb{D}^\mathbb{N}) \}.$$

Fix $n \in \mathbb{N}$. Choose, according to Proposition 1, a vector $(g_1, \ldots, g_n) \in U_n(A(\mathbb{D}^n))$ which is not reducible. Let $a = (\tau^* g_1, \ldots, \tau^* g_n)$. Then $a \in U_n(\tau^* A(\mathbb{D}^\mathbb{N}))$, and $a$ is not reducible.

By the proof of Step 2 in Theorem 3 we obtain a vector $f = (F_1, \ldots, F_n) \in U_n(B_\infty)$ which is not reducible in $B_\infty$. This shows that $\text{sr} B_\infty \geq n$. Since $n$ can be chosen arbitrarily, $\text{sr} B_\infty = \infty$. $lacksquare$

**Remarks.** 1. Using the footnote, we see that our proof for $n = \infty$ is independent of Treil’s result.

2. In general, the algebras $B_n$ do not contain the disk algebra. In fact, let $Y$ be the weak-* closure of the interpolating sequence $z_n = 1 - 2^{-n}$, $n \in \mathbb{N}$, in $M(H^\infty)$. Then the Gelfand transform $\hat{x}$ of $x$ is constantly one on $\{z_n\} \setminus \{z_n\}$, which is homeomorphic to $\beta N \setminus N$. Now assume that $x \in B_n$. Because $\tau$ maps $\beta N \setminus N$ onto $X = T^n$ and $X$ is the Shilov boundary of $A(\mathbb{D}^n)$, we see that $\mathcal{S}(z_n) \equiv 1$, which is of course absurd.

However, if we modify a bit the definition of the algebras $B_n$ by setting

$$\tilde{B}_n = \{ f \in H^\infty : f|_Y \in \tau^* A(\mathbb{D}^n) \},$$

where $Y = \{z_n\} \setminus \{z_n\}$ ($z_n = 1 - 2^{-n}$), then $A(\mathbb{D}) \subseteq \tilde{B}_n$ ($n = 1, 2, \ldots, \infty$). Also in this case, $\text{sr} \tilde{B}_n = [n/2] + 1$ (just replace the interpolating Blaschke product by a suitable function $b \in H^\infty$ vanishing on $Y^*$).

S. Scheinberg [13] showed that for none of these algebras $\tilde{B}_n$ ($n \geq 2$) does the corona theorem hold; i.e., the unit disk is not dense in the spectrum of $\tilde{B}_n$. It is an open problem whether there exist natural subalgebras $A$ of $H^\infty$ whose stable rank is not one, but for which $\mathbb{D}$ is dense in $M(A)$.

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**References**


MATHEMATISCHES INSTITUT
UNIVERSITÄT KARLSRUHE
POSTFACH 6539
D-7500 KARLSRUHE 1, GERMANY

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