## References


[B] J. M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier (Grenoble) 19 (1) (1969), 277–304.

[D] E. Damek, *Left-invariant degenerate elliptic operators on semidirect extensions of homogeneous groups*, Studia Math. 89 (1988), 169–196.

[DH] E. Damek and A. Hulanicki, *Maximal functions related to subelliptic operators invariant under an action of a solvable Lie group*, ibid. 101 (1991), 34–68.

[FKP] R. A. Fefferman, C. E. Kenig and J. Pipher, *The theory of weights and the Dirichlet problem for elliptic equations*, Ann. of Math. 134 (1991), 65–124.

[FS] G. B. Folland and E. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, 1982.

[GT] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1983.

[HS] W. Hebisch and A. Sikora, *A smooth subadditive homogeneous norm on a homogeneous group*, Studia Math. 96 (1990), 231–236.

[H1] A. Hulanicki, *Subalgebra of $L_1(G)$ associated with Laplacian on a Lie group*, Colloq. Math. 31 (1974), 259–287.

[H2] —, *A class of convolution semi-groups of measures on a Lie group*, in: Lecture Notes in Math. 828, Springer, 1980, 82–101.

[HJ] A. Hulanicki and J. Jenkins, *Nilpotent groups and summability of eigenfunction expansions of Schrödinger operators*, Studia Math. 80 (1984), 235–244.

[Ko] J. J. Kohn, *Pseudo-differential operators and non-elliptic problems*, in: Pseudo-differential Operators, CIME Conference, Stresa 1968, Ed. Cremonese, Roma 1969, 157–165.

[St] E. M. Stein, *Boundary behavior of harmonic functions on symmetric spaces: maximal estimates for Poisson integrals*, Invent. Math. 74 (1983), 63–83.

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

[S] D. Stroock, *Lectures on Stochastic Analysis: Diffusion Theory*, Cambridge Univ. Press, 1987.

[SV] D. Stroock and S. R. Varadhan, *Multidimensional Diffusion Processes*, Springer, 1979.

[T] J. C. Taylor, *Skew products, regular conditional probabilities and stochastic differential equations: a remark*, preprint.

[Z] J. Zienkiewicz, *A maximal operator related to a nonsymmetric semigroup generated by a left-invariant operator on a Lie group*, submitted to Studia Math.


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# Pick–Nevanlinna interpolation on finitely-connected domains

by

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**Abstract.** Let $\Omega$ be a domain in the complex plane bounded by $m+1$ disjoint, analytic simple closed curves and let $z_0, \ldots, z_n$ be $n+1$ distinct points in $\Omega$. We show that for each $(n+1)$-tuple $(w_0, \ldots, w_n)$ of complex numbers, there is a unique analytic function $B$ such that: (a) $B$ is continuous on the closure of $\Omega$ and has constant modulus on each component of the boundary of $\Omega$; (b) $B$ has $n$ or fewer zeros in $\Omega$; and (c) $B(z_j) = w_j$, $0 \le j \le n$.

**Introduction.** The classical interpolation result of G. Pick and R. Nevanlinna referred to in the title is this. Let $z_0, \ldots, z_n$ be distinct points of the open unit disc $\Delta$ in the complex plane. Then, for each nonzero $(n+1)$-tuple of complex numbers $(w_0, \ldots, w_n)$, there is an analytic function of the form

$$B(z) = \lambda \prod_{j=1}^{r} \frac{z - a_j}{1 - \overline{a}_j z}, \quad |\lambda| = 1,\ a_j \in \Delta,\ r \le n, \tag{1}$$

and a positive real number $\varrho$ such that

$$\varrho B(z_k) = w_k, \quad k = 0, \ldots, n. \tag{2}$$

Moreover, $\varrho, \lambda, r$ and $a_1, \ldots, a_r$ are uniquely determined by the $n+1$ equations in (2). A proof of this result may be found in, for instance, [F].

The Pick–Nevanlinna theorem has found applications in diverse areas: approximation theory, most especially in the theory of $n$-widths [FM1], [FM2]; in circuit theory [D]; it is also a part of geometric function theory [A].

A function $B$ of the form (1) is called a *Blaschke product* (of degree $r$). It is a standard and easily proved matter that $|B(z)| = 1$ for all $z$ with $|z| = 1$. Conversely, it is also easily established that if $F$ is analytic on $\Delta$, continuous on $\{z : |z| \le 1\}$, and $|F(z)| = 1$ for all $z$ with $|z| = 1$, then $F$ is a Blaschke product (of some finite degree).

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In this paper I take up an extension of the Pick–Nevanlinna result when the unit disc $\Delta$ is replaced by a bounded domain $\Omega$ whose boundary $\Gamma$ consists of $m+1$ disjoint analytic simple closed curves. One such extension [G] is already known and is this. Let $z_0, \ldots, z_n$ be $n+1$ distinct points of $\Omega$. Then for each nonzero $(n+1)$-tuple of complex numbers $(w_0, \ldots, w_n)$, there is an analytic function $B$ on $\Omega$ and a positive real number $\varrho$ such that

(3)
(i) $\varrho B(z_k) = w_k,\ k = 0, \ldots, n$,
(ii) $B$ is continuous on $\Omega \cup \Gamma$ and $|B(z)| = 1$ if $z \in \Gamma$,
(iii) $B$ has at most $n+m$ zeros and at least $m+1$ zeros on $\Omega$ (unless $B$ is constant).

Moreover, $\varrho B$ is the unique solution of the extremum problem:

$$\inf\{\|f\|_\infty : f \in H^\infty(\Omega),\ f(z_k) = w_k,\ k = 0, \ldots, n\}. \tag{4}$$

$H^\infty(\Omega)$ is, as usual, the space of bounded analytic functions on $\Omega$. For a proof of this result, see [G] or [F]. Although (3)–(4) is an exact extension of (1)–(2), it possesses the drawback that the number of zeros of the interpolant depends very directly on the connectivity of the domain $\Omega$.

Here I set forth an alternate way to extend (1) and (2). This extension retains control on the number of zeros of the interpolant ($n$ or fewer) at the sacrifice of losing some control on the boundary values; we will see that the interpolant may be chosen to have constant modulus on each component of $\Gamma$, but the value of the modulus on one component need not equal that on another.

An essential role will be played throughout this work by a factorization theorem of R. Coifman and G. Weiss [CW]. Since this factorization and its properties are not as widely known as the inner-outer factorization, I give an exposition of it, as modified for my needs, in Section 1; Section 2 contains the main result and its proof. And, because the proof makes no use of the classical Pick–Nevanlinna theorem on the disc, the proof here provides (yet another) proof of this result.

**1. The Coifman–Weiss factorization.** The boundary of $\Omega$ is composed of $m+1$ disjoint analytic simple closed curves $\Gamma_0, \ldots, \Gamma_m$. We shall assume that $\Gamma_0$ is the "outer" component; that is, $\Gamma_0$ is the boundary of the unbounded component of the complement of $\Omega$. There is no loss of generality in assuming that $\Gamma_0$ is the circle $\{|z| = 1\}$ and we shall do so without further comment. We once and forever fix a point $t_0 \in \Omega$ to use as our base point. The following is from [CW].

For $k = 1, \ldots, m$, let $w_k(z)$ be the harmonic function on $\Omega$ with boundary values 1 on $\Gamma_k$ and zero on $\Gamma \setminus \Gamma_k$. Let

$$a_{kj} := \frac{1}{2\pi} \int_{\Gamma_k} \frac{\partial w_j}{\partial n}\, ds, \qquad j, k = 1, \ldots, m.$$

The matrix $A = (a_{jk})$ is negative definite; let $(\pi_{jk})$ be the inverse of $A$.

There is a function $\mathcal{P}(z;\zeta)$, $z \in \Omega$, $\zeta \in \Gamma$, which is analytic as a function of $z$ for each $\zeta \in \Gamma$ and for which

$$h(z) = \int_\Gamma \mathcal{P}(z;\zeta)(\operatorname{Re} h)(\zeta)\, ds(\zeta) + i \operatorname{Im} h(t_0) \tag{5}$$

whenever $h$ is analytic on $\Omega$ and continuous on $\Omega \cup \Gamma$.

For $a \in \Omega$ define

$$B(z;a) := (z-a)\exp\Big\{-\int_\Gamma \mathcal{P}(z;\zeta)\log|\zeta - a|\, ds(\zeta)\Big\}. \tag{6}$$

Then $B(a;a) = 0$ and $B$ is a conformal map of $\Omega$ onto the open unit disc $\Delta$ with $m$ circular slits removed. These slits are located on circles about the origin of radii

$$\varrho_j(a) := \exp\Big\{\sum_{k=1}^m \pi_{jk} w_k(a)\Big\}, \qquad j = 1, \ldots, m. \tag{7}$$

Further, $B(z;a)$ carries $\Gamma_0$ onto the unit circle.

For $p \in \Gamma_k,\ k = 1, \ldots, m$, let

$$B(z;p) := (z-p)\exp\Big\{-\int_\Gamma \mathcal{P}(z;\zeta)\log|\zeta - p|\, ds(\zeta)\Big\}. \tag{8}$$

Then $B(z;p)$ is a conformal map of $\Omega$ onto the annular region $\{w : \exp(\pi_{kk}) < |w| < 1\}$ with $m-1$ circular slits removed. These slits are located on circles about the origin of radii $\varrho_j = \exp\{\pi_{jk}\}$, $j = 1, \ldots, m$, $j \neq k$. Further, $B(z;p)$ maps $\Gamma_0$ onto the unit circle.

With all of this background, we are ready for the Coifman–Weiss factorization theorem.

*Let $f$ be analytic on a neighborhood of $\Omega \cup \Gamma$ and not equal to zero on $\Gamma$. Then $f$ has a unique factorization*

$$f = BG \tag{9}$$

*where*

$$B(z) = \gamma_1 \prod_{j=1}^r B(z;a_j) \prod_{j=1}^m \{B(z;p_j)\}^{l_j}, \tag{10}$$

$$G(z) = \gamma_2 \exp\Big\{\int_\Gamma \mathcal{P}(z;\zeta)\log|F(\zeta)|\, ds(\zeta)\Big\}. \tag{11}$$

Here $|\gamma_1| = |\gamma_2| = 1$; $\gamma_2$ is chosen to make

(12) $$G(t_0) > 0;$$

$a_1, \ldots, a_r$ are the zeros of $f$ in $\Omega$;

(13) $$p_j \in \Gamma_j, \quad j = 1, \ldots, m;$$

(14) $l_1, \ldots, l_m$ are integers (positive, zero or negative) which are uniquely determined by $f$.

All of the above is from the paper of Coifman and Weiss; see particularly Theorems I, IV and formula (4.13). The kernel $\mathcal{P}(z;\zeta)$ is that of the title of their paper. I shall refer to (9) as "the CW factorization of $f$".

The following continuity result is needed in Section 2.

PROPOSITION 1. *Let $\{a_k\}$ be a sequence of points in the plane with $a_k \to a$ as $k \to \infty$. Let*

$$z - a_k = B_k G_k \quad \text{and} \quad z - a = BG$$

*be the CW factorizations of $z - a_k$ and $z - a$, respectively. Then*

$$B_k \to B \quad \text{and} \quad G_k \to G \quad \text{as } k \to \infty$$

*uniformly on compact subsets of $\Omega$.*

Proof. There are a number of separate cases to be considered:

(A) $a \in \Omega$ (and hence so is $a_k$ for all large $k$),

(B) $a$ is in the unbounded component of the complement of $\overline{\Omega}$ (and hence so is $a_k$ for all large $k$),

(C) $a$ is in one of the bounded components of the complement of $\overline{\Omega}$ (and hence so is $a_k$ for all large $k$),

(D) $a \in \Gamma_0$,

(E) $a \in \Gamma_j$ for some $j \in \{1, \ldots, m\}$.

Case (A) is the easiest. From [CW] we have

(15) $$z - a_k = B(z; a_k) \exp\Big\{ \int_\Gamma \mathcal{P}(z;\zeta) \log|\zeta - a_k|\, ds(\zeta) \Big\}$$

and the conclusion is evident.

For case (B) we use Theorem 4.10 of [CW] to conclude that

(16) $$z - a_k = \gamma_k \exp\Big\{ \int_\Gamma \mathcal{P}(z;\zeta) \log|\zeta - a_k|\, ds(\zeta) \Big\}$$

that is, the $B$-factor is a unimodular constant. This is because the integers $l_1, \ldots, l_m$ in (10) are the winding numbers of $z - a_k$ about $\Gamma_1, \ldots, \Gamma_m$, respectively. From this, the conclusion is again immediate.



Likewise, if $a$ lies in the component of the complement of $\overline{\Omega}$ bounded by $\Gamma_j$, then $l_i = 0$ for $i \neq j$ and $l_j = 1$. Hence,

(17) $$z - a_k = B(z; p_j)\, \gamma_k \exp\Big\{ \int_\Gamma \mathcal{P}(z;\zeta) \log|z - a_k|\, ds(\zeta) \Big\}$$

for all (large) $k$ and the conclusion is again immediate. This is case (C).

For case (D), we have

$$z - a = \gamma \exp\Big\{ \int_\Gamma \mathcal{P}(z;\zeta) \log|\zeta - a|\, ds(\zeta) \Big\}$$

that is, $B(z;a)$ is a unimodular constant. If $a_k$ lies in the unbounded component of the complement of $\Omega$, then (16) holds and the conclusion is evident. If $a_k$ lies in $\Omega$, then by Theorem IV of [CW],

$$|B(z; a_k)| \to 1 \quad \text{as } a_k \to a \in \Gamma_0$$

uniformly on compact subsets of $\Omega$, and so the limit is a unimodular constant.

Finally, case (E) is completed by again invoking Theorem IV of [CW] and (17) above.

PROPOSITION 2. *Let $\{P_k\}$ be a sequence of polynomials each of degree $n$ or less and suppose that $P_k \to P$ uniformly on compact subsets on $\Omega$. Let*

$$P_k = B_k G_k \quad \text{and} \quad P = BG$$

*be the CW factorizations of $P_k$ and $P$, respectively. Then*

$$B_k \to B \quad \text{uniformly on compact subsets of } \Omega.$$

Proof. Suppose first that for all large $k$, the degree of $P_k$ equals that of $P$. Let $\{a_{k1}, \ldots, a_{kN}\}$ be the zeros of $P_k$ and $\{a_1, \ldots, a_N\}$ be those of $P$. We may suppose that the zeros are numbered so that

$$a_{kj} \to a_j \quad \text{as } k \to \infty, \quad 1 \le j \le N.$$

In this context, we apply Proposition 1 $N$ times and the conclusion follows.

The only other case is that

$$M = \text{degree } P < N_k = \text{degree } P_k, \quad k \text{ large}.$$

In this case, we can number the zeros of $P_k$ and of $P$ so that

$$a_{kj} \to a_j \quad \text{as } k \to \infty,\ 1 \le j \le M,$$

and

$$|a_{kj}| \to \infty \quad \text{as } k \to \infty,\ M + 1 \le j \le N_k.$$

We write

$$P_k(z) = c_k \prod_{j=1}^{M} (z - a_{kj}) \prod_{j=M+1}^{N_k} (z - a_{kj}).$$

Then $B_k = \widetilde{B}_k \widehat{B}_k$ where $\widetilde{B}_k$ is the $B$-factor of $\prod_{j=1}^{M}(z - a_{kj})$ and $\widehat{B}_k$ is the $B$-factor of $c_k \prod_{j=M+1}^{N_k}(z - a_{kj})$. By Proposition 1, $\widetilde{B}_k \to B$, the $B$-factor of $P$. Moreover, $\widehat{B}_k$ is a unimodular constant and we are again done.

**2. Main result.** We denote by $\mathcal{M}_n(\Omega)$ the set of those analytic functions $f$ on $\Omega$ of the form

$$f(z) = c \prod_{j=1}^{r} B(z; a_j) \prod_{k=1}^{m} \{B(z; a'_k)\}^{l_k} \tag{18}$$

where

$c$ is a nonzero complex number,
$a_1, \ldots, a_r \in \Omega$ and $r \le n$,
$a'_k \in \Gamma_k$ for $k = 1, \ldots, m$,
$l_1, \ldots, l_m$ are nonnegative integers.

Hence, each element of $\mathcal{M}_n(\Omega)$ has $n$ or fewer zeros on $\Omega$, has constant modulus on each component of $\Gamma$, and

$$|c| = \|f\|_{H^\infty(\Omega)} = |f(z)|, \quad z \in \Gamma_0;$$

that is, the maximum modulus of $f$ is attained on $\Gamma_0$.

The main result of the paper is this.

THEOREM. *Let $z_0, \ldots, z_n$ be distinct points of $\Omega$ and let $w_0, \ldots, w_n$ be complex numbers, not all of which are zero. Then there is a unique element $B \in \mathcal{M}_n(\Omega)$ such that*

$$B(z_k) = w_k, \quad k = 0, \ldots, n. \tag{19}$$

Proof. Let $S^{2n+1}$ be the collection of $(n+1)$-tuples of complex numbers $\mathbf{w} = (w_0, \ldots, w_n)$ such that

$$1 = \sum_{j=0}^{n} |w_j|^2 .$$

Define $\sigma : S^{2n+1} \to \pi_n$, the space of polynomials of degree $n$ or less, by

$$\sigma(\mathbf{w}) = P \quad \text{where} \quad P(z_k) = w_k, \ k = 0, \ldots, n .$$

Then $\sigma$ is an odd mapping (i.e. $\sigma(-\mathbf{w}) = -\sigma(\mathbf{w})$) and continuous when $\pi_n$ is given the topology of uniform convergence on compact subsets of $\Omega$.



We now use $\sigma$ to construct a mapping $\tau$ from $S^{2n+1}$ into $\mathcal{M}_n(\Omega)$ in this way: given $\mathbf{x} \in S^{2n+1}$, let

$$\sigma(\mathbf{x}) = BG$$

be the $CW$ factorization of $\sigma(\mathbf{x})$; it is direct that $B \in \mathcal{M}_n(\Omega)$. Set

$$\tau(\mathbf{x}) = B .$$

Then $\tau$ is well-defined since the CW factors are uniquely determined. Further, $\tau$ is odd since $\sigma$ is odd. Finally, Proposition 2 of Section 1 implies that $\tau$ is continuous from $S^{2n+1}$ into the space of analytic functions on $\Omega$ with the topology of uniform convergence on compact subsets of $\Omega$.

Let $\mathbf{v}_0 = (w_0, \ldots, w_n)$ be a unit vector in $\mathbb{C}^{n+1}$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be $n$ other unit vectors in $\mathbb{C}^{n+1}$ such that $\mathbf{v}_0, \ldots, \mathbf{v}_n$ is an orthonormal basis of $\mathbb{C}^{n+1}$. Define $\varrho : S^{2n+1} \to \mathbb{C}^n$ by

$$\varrho(\mathbf{x}) = \{\langle \{\tau(\mathbf{x})(z_k)\}_{k=0}^{n}, \mathbf{v}_j \rangle\}_{j=1}^{n}$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{C}^{n+1}$. Since $\tau$ is odd and continuous, $\varrho$ is as well. The theorem of Borsuk [M; Theorem 68.6] then implies that $\varrho$ has a zero; that is, there is an $\mathbf{x}_0 \in S^{2n+1}$ with $\tau(\mathbf{x}_0)$ orthogonal to all of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Equivalently, there is an element $B_0$ of $\mathcal{M}_n(\Omega)$ and a complex scalar $\lambda$ with

$$B_0(z_k) = \lambda w_k, \quad k = 0, \ldots, n .$$

Clearly $\lambda$ is not zero and so $\frac{1}{\lambda} B_0$ is an element of $\mathcal{M}_n(\Omega)$ which interpolates $w_k$ at $z_k$, $k = 0, \ldots, n$. To establish uniqueness, suppose that $B_1$ is another function in $\mathcal{M}_n(\Omega)$ with

$$B_1(z_k) = B_0(z_k) = w_k, \quad k = 0, \ldots, n .$$

The difference $B_0 - B_1$ then has $n+1$ or more zeros on $\Omega$. Multiplying $B_1$ by a number slightly less than 1 will not decrease the number of zeros of $B_0 - B_1$ in $\Omega$ and so we may assume that

$$|B_1| \ne |B_0| \quad \text{on each } \Gamma_j, \ 0 \le j \le m .$$

The change in Arg $(B_0 - B_1)$ on $\Gamma_j$ is equal to the change in the argument of that $B$ which has the larger modulus on $\Gamma_j$. If $j = 1, \ldots, m$, then this change is either zero or is a negative integer multiple of $2\pi$; this is because the factor of $B_0$ (or $B_1$) of the form

$$\prod_{j=1}^{r} B(z; a_j), \quad a_1, \ldots, a_r \in \Omega ,$$

has no net change in its argument around $\Gamma_j$ while the factor of the form

$$\prod_{k=1}^{m} (B(z; a'_k))^{l_k}, \quad l_1, \ldots, l_m \ge 0 ,$$

maps $\Omega$ onto an annulus with $m-1$ circular slits removed, sending $\Gamma_0$ to the circle $|w| = 1$.

The net change in $\operatorname{Arg}(B_0 - B_1)$ on $\Gamma_0$ is that of $B_0$ and this in turn is no more than $2\pi n$. Hence, the total net change of $\operatorname{Arg}(B_0 - B_1)$ over all of $\Gamma$ is at most $2\pi n$. This shows that $B_0 - B_1$ could not have $n+1$ zeros without being identically zero.

EXAMPLE. Without the hypothesis that $l_1, \ldots, l_m$ are nonnegative, the uniqueness assertion of the Theorem fails. To see this, let $\Omega$ be the annulus $r_0 < |z| < R_0$, $r_0 < 1 < R_0$, and let $z_0, \ldots, z_n$ be the $(n+1)$st roots of unity. Then

$$B_0(z) := z^{n+1} \quad \text{and} \quad B_1(z) = z^{-n-1}$$

agree at $z_0, \ldots, z_n$; moreover, both have no zeros on $\Omega$ and have constant modulus on $\Gamma = \partial\Omega$. (I am indebted to S. Fedorov for this example.)

We give $\mathcal{M}_n(\Omega)$ the topology of uniform convergence on compact subsets of $\Omega$.

COROLLARY. *For each positive integer $n$ there is an odd continuous mapping of the sphere $S^{2n+1}$ into those elements of $\mathcal{M}_n(\Omega)$ which have sup norm* 1 *on $\Omega$.*

Proof. One such mapping is the function $\tau$ given in the proof of the Theorem. Another such mapping is obtained in the following way. For each $\mathbf{x} = (w_0, \ldots, w_n) \in S^{2n+1}$, the Theorem implies that there is a unique $B$ of the form (10) and a unique positive scalar $c$ such that

$$cB(z_j) = w_j, \quad 0 \le j \le n.$$

The mapping is then given by $\gamma(\mathbf{x}) = B$. To prove the continuity of $\gamma$, let $\{\mathbf{x}_k\}$ be a sequence of points in $S^{2n+1}$ with $\mathbf{x}_k \to \mathbf{x}$. We write

$$\mathbf{x}_k = (w_{0k}, \ldots, w_{nk}) \quad \text{and} \quad \mathbf{x} = (w_0, \ldots, w_n).$$

The functions $B_k := \gamma(\mathbf{x}_k)$ lie in the unit sphere of $H^\infty(\Omega)$ and hence some subsequence, again denoted by $\{B_k\}$, converges uniformly on compact subsets of $\Omega$ to a function $B$ of the form (10) with $r \le n$ and $l_1, \ldots, l_m \ge 0$. Moreover, by the Theorem, there are positive scalars $\{c_k\}$ such that

$$c_k B_k(z_j) = w_{jk}, \quad 0 \le j \le n, \ k = 1, 2, \ldots \tag{20}$$

If some subsequence of $\{c_k\}$ were unbounded, then by using (20) and letting $k \to \infty$, it would follow that $B(z_j) = 0$, $0 \le j \le n$. But this is clearly impossible since $B$ has $n$ or fewer zeros. From (20) and the fact that $\{c_k\}$ is a bounded sequence, we conclude that $c_k \to c$ and so

$$cB(z_j) = w_j, \quad 0 \le j \le n.$$



Since the interpolant from $\mathcal{M}_n$ to the data $\mathbf{x} = \{w_j\}$ is unique, it follows that $\gamma(\mathbf{x}) = B$.

## References


[A] L. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, New York 1973.

[CW] R. Coifman and G. Weiss, *A kernel associated with certain multiply connected domains and its application to factorization theorems*, Studia Math. 28 (1966), 31–68.

[D] Ph. Delsarte, Y. Genin and Y. Kamp, *The Pick–Nevanlinna problem*, Internat. J. Circuit Theory Appl. 9 (1981), 177–187.

[F] S. D. Fisher, *Function Theory on Planar Domains*, Wiley, New York 1983.

[FM1] S. D. Fisher and C. A. Micchelli, *n-widths of sets of analytic functions*, Duke Math. J. 47 (1980), 789–801.

[FM2] —, *Optimal sampling of holomorphic functions II*, Math. Ann. 273 (1985), 131–147.

[G] P. Garabedian, *Schwarz's lemma and the Szegö kernel function*, Trans. Amer. Math. Soc. 67 (1949), 1–35.

[M] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park, Calif., 1984.


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