Proof. By Lemma 2.1 we have

$$EI^n(X,h) = n!\prod_{j=1}^{n}\int_0^\infty e^{-ju}\,\varrho(X,du) \quad (n=1,2,\ldots).$$

Hence for $I(X,h)$ and $I(Y,h)$ identically distributed,

$$\int_0^\infty e^{-nu}\,\varrho(X,du) = \int_0^\infty e^{-nu}\,\varrho(Y,du) \quad (n=1,2,\ldots),$$

which yields $\varrho(X,\cdot)=\varrho(Y,\cdot)$ ([3], Chapter XIII, 1). Thus $X\sim Y$.

THEOREM 3.3. *A process X is h-stable if and only if either X is deterministic or $X\in \mathrm{Poiss}(q,P_s)$ for some positive q and s.*

Proof. The sufficiency follows from Examples 3.1 and 3.2. To prove the necessity suppose that X is a nondeterministic h-stable process. By Theorem 3.1 and Proposition 3.1 the random variable $I(X,h)$ has gamma distribution with parameters $(\varrho(X,\{0\})^{-1},w)$ for some $w>1$. Put $q=\varrho(X,\{0\})^{-1}$ and $s=w-1$. It was shown in Example 3.2 that for $Y\in \mathrm{Poiss}(q,P_s)$ the random variable $I(Y,h)$ has gamma distribution with parameters $(\varrho(X,\{0\})^{-1},w)$. Hence and from Lemma 3.2 it follows that $X\sim Y$, which yields $X\in\mathrm{Poiss}(q,P_s)$. This completes the proof.

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INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND

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Maximal functions related to subelliptic operators invariant under an action of a nilpotent Lie group

by

EWA DAMEK (Wrocław)

Abstract. On the domain $\Omega_a=\{(x,b): x\in N, b\in\mathbb{R}^+, b>a\}$, where N is a simply connected nilpotent Lie group and $a\ge 0$, certain N-invariant second order subelliptic operators L are considered. Every bounded L-harmonic function F is the Poisson integral

$$F(x,b)=f*\check{\mu}_a^b(x)$$

for an $f\in L^\infty(N)$. The main theorem of the paper asserts that under some assumptions the maximal functions

$$M_1f(x)=\sup_{b\ge a+1}|f*\check{\mu}_a^b(x)|,\qquad M_2f(x)=\sup_{a<b\le a+1}|f*\check{\mu}_a^b(x)|$$

are of weak type $(1,1)$. Some results about moments of the harmonic measures $\check{\mu}_a^b$ are also included.

1. Introduction. The aim of this paper is to study some maximal functions naturally associated with differential operators invariant under an action of a nilpotent Lie group N and defined on $N\times\mathbb{R}^+$. Suppose that for every $a\in\mathbb{R}^+$ we have left-invariant vector fields $Y_1(a),\ldots,Y_k(a)$, $Y(a)$, depending smoothly on a, such that $Y_1(a),\ldots,Y_k(a)$ generate $\mathfrak{n}$ as a Lie algebra and for every a, $Y_1(a),\ldots,Y_k(a)$ belong to the same linear subspace $\mathfrak{v}$ of $\mathfrak{n}$. We consider the operator

$$(1.1)\qquad Lf(x,a)=\Big(\sum_{i=1}^{k}Y_i(a)^2+Y(a)+\alpha\partial_a^2-\kappa\partial_a\Big)f(x,a)$$

on the domain

$$\Omega_{a_0}=\{(x,a): x\in N, a>a_0\},\qquad a_0\ge 0,$$

and so we go a step further than in [DH], where operators invariant with respect to a solvable group structure on $N\times\mathbb{R}^+$ have been considered.

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Let $\check{\mu}_a^b$ be the family of harmonic measures associated with L, i.e. the function

$$F(x,b) = f * \check{\mu}_{a_0}^b(x)$$

solves the Dirichlet problem for Ω_{a_0} with the boundary data $f \in C_b(N)$. We prove that under some natural assumptions on L the maximal functions

$$(1.2)\quad M_1 f(x) = \sup_{b\ge a_0+1} |f * \check{\mu}_{a_0}^b(x)|, \qquad M_2 f(x) = \sup_{a_0<b\le a_0+1} |f * \check{\mu}_{a_0}^b(x)|$$

are of weak type $(1,1)$.

Maximal functions of this type have been studied by many people in different contexts. The classical example is, of course, a half space $\mathbb{R}^n \times \mathbb{R}^+$ and the Laplace operator [SW]. Also they have been considered in the case of rank one symmetric spaces (for a review see [St]). Then $a_0 = 0$, $\Omega_0 = N \times \mathbb{R}^+$ is the group $S = NA$ coming from the Iwasawa decomposition $G = NAK$ of a semisimple Lie group and L is the Laplace–Beltrami operator.

Recently the maximal functions (1.2) have been studied in the context of a left-invariant operator L on $S = N \times \mathbb{R}^+$ which is a semidirect product of a nilpotent group N and $\mathbb{R}^+ = A$ acting on N by dilations

$$\delta_a(x_1,\dots,x_n) = (a^{d_1}x_1,\dots,a^{d_n}x_n)$$

where $x = \exp(\sum_{j=1}^n x_j X_j)$ and $X_1,\dots,X_n$ is a basis of the Lie algebra $\mathfrak{n}$ ([D], [DH]). Such an operator L can be written in the form (1.1) with

$$Y_i(a) = \sum_{j=1}^k \alpha_{ij} a^{d_j} X_j, \qquad i = 1,\dots,k, \qquad Y(a) = \sum_{j=1}^n \alpha_j a^{d_j} X_j\,.$$

Both when $a_0 = 0$ (the whole group situation) and $a_0 > 0$ a number of results concerning Poisson integrals on $\partial\Omega_{a_0}$ have been established. In particular, if $a_0 = 0$ and $\kappa > 0$ then both $M_1 f$, $M_2 f$ are of weak type (1,1) [D]. If $a_0 > 0$ and the group structure is broken, the situation is slightly different. M_1 is of weak type (1,1) whenever $\kappa > 0$ but for weak type (1,1) of M_2, in addition to $\kappa \ge 0$, we have to assume that

$$Y(a) \in \operatorname{lin}\{X_1,\dots,X_k,[X_i,X_j] : i,j \le k\}$$

([DH]; for a counterexample see [Z]). One of the main tools used in [DH] to prove weak type $(1,1)$ when $a_0 > 0$ is the decomposition of the diffusion generated by L into the "vertical component" $\mathbf{a}(t)$ generated by $\alpha\partial_a^2 - \kappa\partial_a$ and the "horizontal component" for which the transition probability conditioned on the trajectory $\mathbf{a}(t)$ of the vertical component satisfies the evolution equation

$$(1.3)\qquad \partial_t u(x,t) = \Big(\sum_{i=1}^k Y_i(\mathbf{a}(t))^2 + Y(\mathbf{a}(t))\Big)u(x,t)\,.$$

This holds in a much more general situation (cf. e.g. [T]). For this and other reasons it seems very likely that the fact that $A = \mathbb{R}^+$ acts on N by dilations and that L is left-invariant on S is not very essential. What is really crucial for this analysis is that the space Ω_{a_0} is foliated by the action of N and that $\partial\Omega_{a_0}$ can be identified with N. This observation and the idea of dropping A-invariance belongs to A. Hulanicki, to whom the author is grateful for the suggestion to study the operators (1.1).

In the present paper we assume that if $X_1,\dots,X_k$ is a basis of $\mathfrak{v}$, $X_1,\dots,X_k,\dots,X_n$ a basis of $\mathfrak{n}$ and L is written in the form

$$L = \sum_{i,j=1}^k \beta_{ij}(a) X_i X_j + \sum_{j=1}^n \beta_j(a) X_j + \alpha\partial_a^2 - \kappa\partial_a$$

then the quotient of the upper bound of the coefficients $\beta_{ij}(a)$, $\beta_j(a)$ by the lowest eigenvalue of $[\beta_{ij}(a)]$ is bounded whenever $a > 0$. If $Y(a) = 0$ this is enough to prove (1.2) (Section 6) and in fact this case is fairly easy. For $Y(a) \ne 0$ the method we use requires some additional assumptions (see Theorems (6.3) and (6.5)). Throughout the paper most complications come from the fact that we do not want to restrict ourselves to the case $Y(a) = 0$ and so estimates on the related evolution both for large and small times become more difficult.

One should perhaps mention that without any group invariance the maximal function M_1 is not bounded even on L^2 in general. In [FKP] it is shown that for uniformly elliptic operators in divergence form M_1 may be unbounded on L^2 even if the coefficients of the operator are bounded together with all derivatives.

The results about the maximal functions are based on some estimates on the evolution (1.3) formulated in Section 3 and coming from [DH]. Also most of the methods we make use of have already been applied in [DH]. Sections 4 and 5 deal with properties of the harmonic measures $\check{\mu}_a^b$ (introduced in Section 2). We prove that the measures $\check{\mu}_a^b$ have some moments; the results depend on the growth of coefficients at ∞ (bounded, polynomial or exponential). In the case of bounded coefficients the moments we obtain are the best possible in this generality.

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Preliminaries. We follow the notation of [DH]. Throughout the paper N is a nilpotent Lie group and τ a left-invariant Riemannian distance from the identity. If f is a function on N then we will write

$$\|f\| = \sup_{x\in N} |f(x)|\,.$$

$B_R(x)$ denotes the ball of radius R and centre x, i.e.

$$B_R(x) = \{y : \tau(x^{-1}y) < R\}.$$

We will also use some smoothing of τ. If $\Phi \in C_c^\infty(B_1(e))$ is a nonnegative function such that $\int \Phi(x)\,dx = 1$ then $\tau * \Phi$ has the following properties [H1]:

$$\tau(x) - 1 \le \tau * \Phi(x) \le \tau(x) + 1, \qquad x \in N,$$
$$\|X(\tau * \Phi)\| < \infty, \qquad \|XY(\tau * \Phi)\| < \infty.$$

Moreover, τ is subadditive, and $(1+\tau)^\xi$, e^τ are submultiplicative (for definitions see [H1]). For every subadditive function Ψ on N we have (see [H1])

$$\Psi(x) \le C(\tau(x) + 1), \qquad x \in N.$$

Analogously if Ψ is submultiplicative then

$$\Psi(x) \le e^{C(\tau(x)+1)}, \qquad x \in N.$$

Sometimes our group N will be *homogeneous* [FS]. This means that there is a basis $X_1, \ldots, X_n$ of $\mathfrak{n}$ and a sequence of numbers $1 = d_1 \le \ldots \le d_n$ such that for every $a \in \mathbb{R}^+$ the mapping

$$X_j \to a^{d_j} X_j$$

extends to an automorphism δ_a of $\mathfrak{n}$. A *homogeneous norm* on N is a function

$$N \ni x \longrightarrow |x| \in \mathbb{R}^+$$

which is C^∞ outside $x = e$, $|\delta_a x| = a|x|$, and $|x| = 0$ if and only if $x = e$. There is always a subadditive homogeneous norm [HS], i.e. such that

$$|xy| \le |x| + |y|. \tag{1.4}$$

2. The Dirichlet problem and harmonic measures.

In this section we study the Dirichlet problem for subelliptic second order N-invariant operators

$$L = Y_1^2 + \ldots + Y_m^2 + Y_0 \tag{2.1}$$

on domains

$$\Omega_{a_0} = \{(x, a) : x \in N,\ a > a_0\} \subset N \times \mathbb{R}^+.$$

That is, we assume that

(2.2) $Y_1, \ldots, Y_m$ generate the tangent space to S at every $s \in S$,

and for every $x \in N$,

$$Y_j f(x, a) = Y_j({}_x f)(e, a)$$

where ${}_x f(y, a) = f(xy, a)$.

Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{n}$ and $X_0 = \partial/\partial a$. Then by N-invariance

$$L = \sum_{i,j=0}^n \beta_{ij}(a) X_i X_j + \sum_{i=0}^n \beta_i(a) X_i \tag{2.3}$$

with smooth β_{ij}, β_i and with the matrix $[\beta_{ij}(a)]$ being positive semidefinite. By (2.2) for every $a > 0$, $\beta_{00}(a) > 0$, so in what follows we assume

$$\beta_{00}(a) = 1.$$

Also, we assume that

$$\beta_0(a) \le 0 \qquad \text{for } a > 0.$$

The following maximum principle is valid for L being degenerate elliptic (without Hörmander's condition) so we formulate it as follows:

(2.4) THEOREM (MAXIMUM PRINCIPLE). *Let L defined on Ω_{a_0} be as in* (2.3) *with $\beta_{00}(a) = 1$. Given $R, r > 0$ we define a domain $D(R, r)$ by*

$$D(R, r) = \{(x, b) : \tau(x) < R,\ a_0 < b < a_0 + r\}.$$

For every $b > a_0$ and $\varepsilon > 0$ there are $R, r > 0$ such that if $F \in C^2(D(R, r)) \cap C(\overline{D}(R, r))$, $|F| \le 1$, $LF \ge 0$ and $F(x, a_0) \le 0$ for $x \in B_R(e)$ then

$$F(e, b) \le \varepsilon.$$

Proof. Let

$$G(x, a) = -\varepsilon(a - a_0)^\gamma/(b - a_0)^\gamma - (\varepsilon^{2/\gamma}\gamma(1-\gamma)/(2CM(b-a_0)^2))(\tau * \Phi(x) + 1),$$

$x \in N$, $a > a_0$, where $0 < \gamma < 1$ and

$$M = \max\{|\beta_{ij}(a)|, |\beta_i(a)| : a_0 \le a \le a_0 + (b - a_0)\varepsilon^{-1/\gamma}, 0 \le i, j \le n\},$$

$$C = \sum_{i,j=1}^n \|X_i X_j(\tau * \Phi)\| + \sum_{i=1}^n \|X_i(\tau * \Phi)\|.$$

First we notice that $LG(x, a) > 0$ whenever $x \in N$ and $a_0 < a < a_0 + (b - a_0)\varepsilon^{-1/\gamma}$. Indeed,

$$(\partial_a^2 + \beta_0(a)\partial_a) G(x, a) \ge \gamma(1-\gamma)\varepsilon^{2/\gamma}(b - a_0)^{-2}$$

and

$$\|L(\tau * \Phi + 1)\| \le MC.$$

Now let

$$R = 2CM(b - a_0)^2/(\varepsilon^{2/\gamma}\gamma(1-\gamma)), \qquad r = (b - a_0)\varepsilon^{-1/\gamma}.$$

Then $G(x, a_0) \le 0$ and $G \le -1$ on the remaining part of the boundary ∂D of D. Hence $F + G \le 0$ on ∂D. The weak maximum principle for degenerate elliptic operators (Proposition 1.1 in [B]) implies $F + G \le 0$ in D and the proof is complete. ∎

(2.5) COROLLARY. *If $F \in C^2(\Omega_a) \cap C(\overline{\Omega}_a)$, $LF \geq 0$ and F is bounded then for every $b > a$ and $x \in N$,*

$$F(x,b) \leq \sup_{y\in N} F(y,a). \quad \blacksquare$$

(2.6) COROLLARY. *For every $b > a$ and $\varepsilon > 0$ there is a compact set $K(\varepsilon, b, a)$ such that if $F \in C^2(\Omega_a) \cap C(\overline{\Omega}_a)$, $LF = 0$, F is bounded and $F(x,a) \geq 1$ for $x \in K(\varepsilon, b, a)$ then $F(b) \geq 1 - \varepsilon$.* $\blacksquare$

Now we are going to show that if $Y_1, \ldots, Y_m$ generate the tangent space to S at every $s \in S$ then for every a the Dirichlet problem for Ω_a has a unique solution.

(2.7) THEOREM. *For every bounded continuous function f on N there is a unique bounded L-harmonic function F (i.e. $LF = 0$) on Ω_a, continuous on $\overline{\Omega}_a$ and such that $F(x,a) = f(x)$ for $x \in N$.*

For operators (2.1) Bony's version of Harnack's inequality [B] is available and also the Dirichlet problem can be solved in every set from a basis $\mathcal{R}$ of open sets in S. Therefore the proof of Theorem (2.7) is a standard application of Perron's method ([GT], see also [DH]) provided we establish the following two facts.

(2.8) THE MAXIMUM PRINCIPLE FOR Ω_a. *Let $F : \Omega_a \to [-\infty, \infty)$ be a subharmonic function in Ω_a, upper semicontinuous and bounded on $\overline{\Omega}_a$. Then*

$$\sup_{(x,b)\in\Omega} F(x,b) \leq \sup_{x\in N} F(x,a).$$

(2.8) can be proved in the same way as Theorem (2.4). We just notice that $F + G$ is subharmonic and apply Bony's maximum principle [B] for subharmonic functions on compact sets. $\blacksquare$

(2.9) *Existence of a barrier function at every point of $\partial\Omega_a$.* A barrier function at $(x_0, a) \in \partial\Omega_a$ can be constructed for example as in [DH] by means of the function

$$W(x,b) = b^\gamma + (\gamma(1-\gamma)/(MC))\Psi(x_0^{-1}x)$$

where Ψ is a *Hunt function* on N, i.e. Ψ, $X_j\Psi$, $X_iX_j\Psi$ are bounded, $\Psi(e) = 0$ and $\Psi(x) > 0$ for $x \neq e$ (cf. e.g. [H2]), $0 < \gamma < 1$,

$$C = \sum_{i,j=1}^{n} \|X_iX_j\Psi\| + \sum_{i=1}^{n} \|X_i\Psi\|,$$

and

$$M = \max\{|\beta_{ij}(b)|, |\beta_i(b)| : a \leq b \leq a+1,\ i,j = 1,\ldots,n\}.$$

It is easy to notice that $W \in C(\overline{\Omega}_a)$, $W > 0$ if $x \neq x_0$ or $b \neq a$, $W(x_0,a) = 0$ and $LW \leq 0$ in $\{(x,b) : a < b < a+1\}$. $\blacksquare$

Let $\mathcal{H}_a$ be the space of bounded harmonic functions on Ω_a, continuous on $\overline{\Omega}_a$. By the previous theorem and Corollary (2.5) for every $s = (x,b)$ in Ω_a the mapping

$$m_s(f) = F(s), \qquad F \in \mathcal{H}_a,\ F|_N = f,$$

is a well defined continuous functional on $C_b(N)$ with $\|m_s\| = 1$ and by Corollary (2.6)

$$\sup\{|m_s(f)| : f \in C_c(N),\ \|f\| = 1\} = 1.$$

Hence there exists a probability measure $\mu_a^{x,b}$ on N such that

$$F(x,b) = \langle f, \mu_a^{x,b}\rangle, \qquad x \in N,\ a,b \in \mathbb{R}^+,\ a < b,$$

for f in $C_b(N)$. Since L commutes with left translations we see that

$$F(x,b) = \int_N f(xy)\,d\mu_a^b(y) = f * \check{\mu}_a^b(x), \tag{2.10}$$

where $\mu_a^b = \mu_a^{e,b}$, $d\check{\mu}(x) = d\mu(x^{-1})$. Then it follows immediately from (2.10) that

$$\check{\mu}_a^b = \check{\mu}_a^c * \check{\mu}_c^b \quad \text{for } a < c < b. \tag{2.11}$$

Moreover, $\check{\mu}_a^b$ is an approximate identity as $b \to a$.

(2.12) PROPOSITION. *For every right-invariant differential operator ∂ on N, $\partial\mu_a^b \in L^2(N)$ and consequently μ_a^b is smooth and $\partial\mu_a^b$ is bounded.*

This follows from Sobolev's lemma (cf. e.g. [D]). By Proposition (2.12),

$$\check{\mu}_a^b \in L^p(N), \qquad p \geq 1,$$

and if $f \in L^p(N)$, $p \geq 1$, then $f * \check{\mu}_a^b$ is L-harmonic on Ω_a.

As another application of the maximum principle we obtain the following characterization of bounded L-harmonic functions on Ω_a.

(2.13) THEOREM. *Let $F \in C(\overline{\Omega}_a) \cap C^2(\Omega_a)$ be a bounded L-harmonic function. Then*

$$F(x,b) = f * \check{\mu}_a^b(x) \tag{2.14}$$

for a unique $f \in L^\infty(N)$.

Proof. Let $F_b(x) = F(x,b)$. There is a sequence $a_m \to a$ and $f \in L^\infty(N)$ such that F_{a_m} converges weak* to f. In particular, for $\varphi \in C_c(N)$,

$$F^m(x,b) = \varphi * F_{a_m} * \check{\mu}_a^b(x) \to \varphi * f * \check{\mu}_a^b(x)$$

as $a_m \to a$. Thus it is sufficient to show that

$$F^m(e,b) \to \varphi * F_b(e).$$

For that we evaluate

$$(2.15)\qquad |F^m(x,a_m)-\varphi*F_{a_m}(x)| \le \int_N |\varphi*F_{a_m}(xy^{-1})-\varphi*F_{a_m}(x)|\,d\mu_a^{a_m}(y)$$

when x belongs to a compact set K. Let U be a neighbourhood of e in N. Then

$$\int_{N\setminus U} |\varphi*F_{a_m}(xy^{-1})-\varphi*F_{a_m}(x)|\,d\mu_a^{a_m}(y) \le \|\varphi\|_{L^1}\|F\|_{L^\infty}\mu_a^{a_m}(N\setminus U),$$

which is smaller than $\delta/2$ for a_m sufficiently close to a. For every $\delta>0$ we can find U such that for all a_m

$$|\varphi*F_{a_m}(xy^{-1})-\varphi*F_{a_m}(x)|<\delta/2$$

whenever $x\in K$, $y\in U$. Hence the right-hand side of (2.15) is not greater than δ. Applying now the Maximum Principle (2.4) to the harmonic functions $F^m(x,b)-\varphi*F_b(x)-\delta$ and $\varphi*F_b(x)-F^m(x,b)-\delta$ on Ω_{a_m} and to K sufficiently large we obtain

$$|F^m(e,b)-\varphi*F_b(e)|\le\varepsilon,$$

which proves (2.14). Uniqueness of f follows from the fact that $\check{\mu}_a^b$ is an approximate identity. ∎

3. Evolutions associated with operators with homogeneous second order part.

Let N be a homogeneous group [FS] and let $X_1,\ldots,X_n$ be a homogeneous basis of the Lie algebra $\mathfrak{n}$ of N, i.e. for the group of dilations δ_r, $r>0$, we have

$$\delta_r X_j = r^{d_j}X_j,\qquad j=1,\ldots,n.$$

Let $Q=d_1+\ldots+d_n$. We assume that $X_1,\ldots,X_k$ generate $\mathfrak{n}$ and $d_1=\ldots=d_k$. Let

$$Y_i(t)=\sum_{j=1}^k \alpha_{ij}(t)X_j,\qquad i=1,\ldots,k,\qquad Y(t)=\sum_{j=1}^n\alpha_j(t)X_j$$

with α_{ij}, α_i continuous and with the matrix $A(t)=[\alpha_{ij}(t)]$ being nonsingular for every t. We consider the operator

$$(3.1)\qquad L=L_0+L_1$$

on $N\times\mathbb{R}^+$, where

$$L_0f(x,t)=\sum_{i=1}^k Y_i(t)^2f(x,t),\qquad L_1f(x,t)=Y(t)f(x,t).$$

Let

$$(3.2)\qquad \eta=\max\{|\alpha_j(t)|:t>0,\ j=1,\ldots,n\},$$

$$(3.3)\qquad \Lambda=\max\{|\alpha_{ij}(t)|:t>0,\ i,j=1,\ldots,k\},$$

$$(3.4)\qquad \lambda=\inf\{\|A(t)v\|:t>0,\ \|v\|=1\},$$

where $\|\ \|$ is a fixed euclidean norm in $\mathfrak{n}$. We assume that $\eta,\Lambda<\infty$ and $\lambda>0$. Let

$$(3.5)\qquad m=\max(1+\eta,\Lambda^2)\cdot k^2\lambda^{-2}+1,\qquad M=\max(\eta,\Lambda^2).$$

Remark. Every operator

$$L=\sum_{i,j=1}^k\beta_{ij}(t)X_iX_j+\sum_{j=1}^k\beta_j(t)X_j$$

with continuous $\beta_{ij}(t)$, $\beta_j(t)$ and positive definite matrix $[\beta_{ij}(t)]$ can be written in the form (3.1). Then λ^2 and Λ^2 are the smallest and the largest eigenvalues of the matrix $[\beta_{ij}(t)]$, i.e.

$$\lambda^2|\xi|^2\le\sum_{i,j=1}^k\beta_{ij}(t)\xi_i\xi_j\le\Lambda^2|\xi|^2,\qquad \xi\in\mathbb{R}^k.$$

For a multiindex $I=(i_1,\ldots,i_n)$ we shall write

$$|I|=d_1i_1+\ldots+d_ni_n.$$

Let $X^I=X_1^{i_1}\ldots X_n^{i_n}$ denote a left-invariant and $\check{X}^I=\check{X}_1^{i_1}\ldots\check{X}_n^{i_n}$ a right-invariant operator. We start with the following pointwise estimate.

(3.6) THEOREM. *Let $P(s,t,x)$ be the fundamental solution of $L-\partial_t$ (transition probability function [SV]). Then for every multiindex I there are constants C, K, $\alpha\ge -Q/2-|I|/2$ and $\beta\le -Q/2-|I|/2$ such that*

$$(3.7)\qquad \|X^IP(s,t)\|_{L^\infty}\le\begin{cases}Cm^K(t-s)^\alpha & \text{for } t-s\ge1,\\ Cm^K(t-s)^\beta & \text{for } t-s\le 1.\end{cases}$$

Moreover,

$$(3.8)\qquad \|X^IP(s,t)\|_{L^\infty}\le Cm^K(t-s)^{-Q/2-|I|/2}\qquad \text{for } t-s\ge1$$

provided $\alpha_j(t)=0$ for $j=1,\ldots,k$, and

$$(3.9)\qquad \|X^IP(s,t)\|_{L^\infty}\le Cm^K(t-s)^{-Q/2-|I|/2}\qquad \text{for } t-s\le1$$

provided $\alpha_j(t)=0$ for j such that

$$X_j\notin\operatorname{lin}\{X_1,\ldots,X_k,[X_j,X_p]:1\le j,p\le k\}.$$

Remark. Unlike (3.8) and (3.9), the estimate (3.7) is not optimal but we write it out because we will need it later. In what follows we will say that L satisfies *condition* $\mathfrak{U}$ if $\alpha_j(t)=0$ for $j=1,\ldots,k$, and respectively, *condition*

$\mathfrak{V}$ if $\alpha_j(t) = 0$ for j such that $X_j \notin \operatorname{lin}\{X_1, \ldots, X_k, [X_j, X_p] : 1 \le j, p \le k\}$. In particular, when L is homogeneous ($Y(t) \in \operatorname{lin}\{[X_i, X_j] : 1 \le i, j \le k\}$) then (3.8), (3.9) hold.

The proof of Theorem (3.6) is contained in the proofs of Theorems (5.3) and (5.14) of [DH]. But since we want to write in a more explicit way the dependence of the relevant constants on η, Λ, λ, we formulate here a few lemmas as the main steps of the proof.

Let

$$B = \{(x,t) : |x| < 1, 1/2 < t < 1\},$$
$$\langle f, g\rangle = \int f(x,t) g(x,t)\, dx\, dt, \qquad \|f\|_{L^2}^2 = \langle f, f\rangle .$$

We denote by $C^{\infty,1}(B)$ the set of functions defined in B which are C^∞ with respect to x and once continuously differentiable in t.

(3.10) Lemma ([DH]). *Let $\varphi \in C_c^\infty(B)$, $0 \le \varphi \le 1$. There is a $C = C(\varphi)$ such that for every $u \in C^{\infty,1}(B)$ satisfying $(L - \partial_t)u = 0$ in a neighbourhood of the support of φ we have*

$$|\langle L_0(\varphi u), \varphi u\rangle| \le C \max(1 + \eta, \Lambda^2) \|u\|_{L^2(B)} . \ \blacksquare$$

(3.11) Lemma ([DH]). *Let $\varphi \in C_c^\infty(B)$, $0 \le \varphi \le 1$. For every I there are constants $C = C(\varphi, I)$ and $K = K(I)$ such that if $u \in C^{\infty,1}(B)$ and $(L - \partial_t)u = 0$ in a neighbourhood of the support of φ then*

$$\|X^I(\varphi u)\|_{L^2} \le C m^K \|u\|_{L^2(B)} , \tag{3.12}$$

and the same with $\check{X}^I$ in place of X^I.

Proof. For $\varepsilon \ge 0$ we define a Sobolev norm on functions supported in B putting

$$\|f\|_\varepsilon^2 = \|(1 + \Delta)^{\varepsilon/2} f\|_{L^2}^2, \qquad f \in C_c^{\infty,1}(B),$$

where $\Delta = \check{X}_1^2 + \ldots + \check{X}_n^2$. By Kohn's lemma [Ko] there are ε and $C = C(\varepsilon)$ such that for every $f \in C_c^{\infty,1}(B)$

$$\|f\|_\varepsilon^2 \le C\Big(\sum_{j=1}^k \|X_j f\|_0^2 + \|f\|_0^2\Big),$$

and so by (3.4)

$$\|X_i f\|_0^2 \le \lambda^{-2} k \Big(\sum_{j=1}^k \int |Y_j(t) f(x,t)|^2\, dx\, dt\Big).$$

Therefore

$$\|f\|_\varepsilon^2 \le C\Big(\lambda^{-2} k^2 \sum_{j=1}^k \|Y_j(t) f\|_0^2 + \|f\|_0^2\Big).$$

Taking $f = \varphi u$, by Lemma (3.10) we have for a $C = C(\varepsilon, \varphi)$,

$$\|\varphi u\|_\varepsilon^2 \le C m \|u\|_{L^2(B)} .$$

Proceeding further as in [DH] we deduce that for every N, φ, ε there is a $C = C(\varphi, N, \varepsilon)$ such that if u is as above then

$$\|(1 + \Delta)^{N\varepsilon}(\varphi u)\|_{L^2} \le C m^N \|u\|_{L^2(B)} .$$

Therefore (3.12) follows. $\blacksquare$

(3.13) Lemma. *Let $B_1 \subset B$, and let I be a multiindex. There are $C = C(I, B_1, B)$, $K = K(I)$ such that if $u \in C^{\infty,1}(B)$ and $(L - \partial_t)u = 0$ in B then*

$$\sup_{(x,t)\in B_1} |X^I u(x,t)| \le C m^K M \|u\|_{L^2(B)} , \tag{3.14}$$

and the same for $\check{X}^I u$.

Proof. Let $\varphi \in C_c^\infty(B \cap (N \times (1/2 + \delta, 1 - \delta)))$, $0 \le \varphi \le 1$. Obviously it is enough to prove (3.14) for $\check{X}^I$. $\check{X}^I u$ is $(L - \partial_t)$-harmonic and so

$$\begin{aligned}
|\varphi \check{X}^I u(x,t)| &\le \int_0^1 |\partial_t(\varphi \check{X}^I u)(x,t)|\, dt \\
&\le \int_0^1 |\partial_t \varphi \cdot X^I u(x,t)|\, dt + \int_0^1 |\varphi L_t X^I u(x,t)|\, dt \\
&\le \|\delta_t \varphi\|_{L^\infty} \int_{1/2+\delta}^{1-\delta} |\check{X}^I u(x,t)|\, dt \\
&\quad + \Lambda^2 \int_{1/2+\delta}^{1-\delta} \sum_{i,j=1}^k |X_i X_j \check{X}^I u(x,t)|\, dt \\
&\quad + \eta \int_{1/2+\delta}^{1-\delta} \sum_{i=1}^n |X_i \check{X}^I u(x,t)|\, dt .
\end{aligned}$$

Now applying the Sobolev inequality with respect to the x variable and the previous theorem we obtain (3.14). $\blacksquare$

Proof of Theorem (3.6). Let u be $(L - \partial_t)$-harmonic and $D_r(x,t) = (\delta_r x, r^2 t)$. Then

$$r^2((L - \partial_t)u) \circ D_r = (L^r - \partial_t)(u \circ D_r),$$

where

$$L^r = \sum_{i=1}^k Y_i(r^2 t)^2 + \sum_{j=1}^n \alpha_j(r^2 t) r^{2-d_j} X_j . \tag{3.15}$$

Therefore $(L^r - \partial_t)(u \circ D_r) = 0$ and applying the previous lemma to $u \circ D_r$, for some $K = K(I)$ and $C = C(B_1, B, I)$, we get

$$\sup_{B_1} |(X^I u) \circ D_r| \le C m^K M C(r) r^{-|I|} \|u \circ D_r\|_{L^2(B)},$$

where

$$C(r) = \begin{cases} r^{\alpha_1}, & r \ge 1, \\ r^{\beta_1}, & r \le 1, \end{cases}$$

with $\alpha_1 = \alpha_1(I)$ nonnegative and $\beta_1 = \beta_1(I)$ nonpositive. Moreover, $\alpha_1 = 0$ (respectively $\beta_1 = 0$) if the operator satisfies condition $\mathfrak{U}$ (respectively $\mathfrak{V}$). Now substituting

$$u = f * P_t(x), \quad f \in L^p, \ 1 \le p \le \infty,$$

where $P_t = P(0,t)$ and proceeding as in [DH] we obtain

$$\text{(3.16)} \quad |X^I(f * P_r)(e)| \le C m^{K(I)} M r^{-|I|/2 - Q/4} C(r^{1/2}) \Big(\frac{1}{2r} \int_{2r\varepsilon}^{2r} \int_{|x| < r^{1/2}} |f * P_t|^2 \, dx \, dt \Big)^{1/2}$$

and so

$$\text{(3.17)} \quad |X^I(f * P_r)(e)| \le C m^{K(I)} M r^{-|I|/2 - Q/4} C(r^{1/2}) \|f\|_{L^2}.$$

In particular,

$$\text{(3.18)} \quad \|P_r\|_{L^2} \le C m^{K(0)} M r^{-Q/4} C(r^{1/2}).$$

Assume now that L satisfies condition $\mathfrak{U}$ (and $r \ge 1$) or $\mathfrak{V}$ (and $r \le 1$). Then by (3.18) there is a C such that

$$\text{(3.19)} \quad \frac{1}{2r} \int_{2r\varepsilon}^{2r} \|f * P_t\|_{L^2}^2 \, dt \le C m^{2K(0)} M^2 r^{-Q/2} \|f\|_{L^2}^2.$$

Finally, putting (3.16) and (3.19) together we get

$$|X^I(f * P_r)(e)| \le C m^{K(I)+K(0)} M^2 r^{-|I|/2 - Q/2} \|f\|_{L^1}$$

and replacing f by ${}_x f$,

$$|f * X^I P_r(x)| \le C m^{K(I)+K(0)} M^2 r^{-|I|/2 - Q/2} \|f\|_{L^1}.$$

Taking an approximate identity in L^1 we arrive at (3.8) and (3.9) for $s = 0$ and if we now consider the operator L' with

$$\alpha'_{ij}(t) = \alpha_{ij}(s+t), \quad i,j = 1, \dots, k,$$
$$\alpha'_j(t) = \alpha_j(s+t), \quad j = 1, \dots, n,$$

we obtain (3.8) and (3.9). For (3.7) we have to take into account $C(r^{1/2})$ and proceed as above. ■

Later we will use the following simple property of the evolution [SV]:

$$\text{(3.20)} \quad P(s,t) = P(s,u) * P(u,t) \quad \text{for every } s < u < t,$$

and also the fact that there is a $C = C(X_1, \dots, X_n)$ such that if $s < t$ and $R \ge CM(t-s)$ then

$$\text{(3.21)} \quad P(s,t,B_R(e)^c) \le 2 \dim N \, e^{-R^2/(C^2 M(t-s))}$$

([S], see also [DH]).

The following lemma has been proved in [DH] (Theorem (5.16)) for $(1+\tau)^K$ instead of $e^{\tau(x)^2/\xi}$ but the proof is essentially the same.

(3.22) Lemma. *Let M, m be as in (3.5), and C as in (3.21). Then for every multiindex I there are $\beta = \beta(I) \ge |I|/2$ and $C(M,m)$ such that*

$$\langle e^{\tau^2/(4C^2 M)}, |X^I P(s,t)| \rangle \le C(M,m)(t-s)^{-\beta} \quad \textit{for } t - s \le 1.$$

Moreover, when L satisfies condition $\mathfrak{V}$ or $I = (0, \dots, 0)$ then $\beta = |I|/2$. The same holds with a subadditive homogeneous norm $|\cdot|$ instead of τ and with another constant $C(M,m)$. ■

From now on we will assume that our homogeneous norm $|\cdot|$ is subadditive. Let $P^r(s,t,x)$ be the evolution associated with L^r and let η_r be defined as in (3.2) for L^r. Then

$$\text{(3.23)} \quad X^I P^r(0,t,x) = r^{Q+|I|}(X^I P)(0, r^2 t, \delta_r x), \quad x \in N, \ t > 0.$$

If L satisfies conditions $\mathfrak{U}$ or $\mathfrak{V}$ then η_r is bounded independently of r for $r \ge 1$ in case $\mathfrak{U}$ and for $r \le 1$ in case $\mathfrak{V}$. Therefore by Lemma (3.22) for $1/2 \le t - s \le 1$,

$$\text{(3.24)} \quad \langle e^{|\cdot|^2/(4C^2 M)}, |X^I P^r(s,t)| \rangle$$

is bounded independently of r for $r \ge 1$ in case $\mathfrak{U}$ and $r \le 1$ in case $\mathfrak{V}$ respectively.

(3.25) Theorem. *Let L satisfy condition $\mathfrak{U}$ or $\mathfrak{V}$. There is a $C(M,m)$ such that*

$$\text{(3.26)} \quad |X^I P(s,t,x)| \le C(M,m)(t-s)^{-Q/2 - |I|/2} e^{-|x|^2/(16 C^2 M(t-s))}$$

for $t - s \ge 1$ in case $\mathfrak{U}$, and $t - s \le 1$ in case $\mathfrak{V}$.

Remark. In particular, (3.26) holds if L is homogeneous, i.e.

$$Y(t) \in \operatorname{lin}\{[X_i, X_j] : 1 \le i, j \le k\}.$$

Proof of Theorem (3.25). Let $\xi = 4C^2 M$. By our assumptions on L and Theorem (3.6) there is a C' such that

$$\text{(3.27)} \quad \|X^I P^r(s,t)\|_{L^\infty} \le C' m^K M^2$$

for $1/2 \le t - s \le 1$ and every r. Therefore by (1.4),

$$e^{|x|^2/(4\xi)}|X^I P^r(0,1,x)| = \int_N e^{|x|^2/(4\xi)} P^r(0,1/2,xy^{-1})|X^I P^r(1/2,1,y)|\,dy$$
$$\le \int_N e^{|xy^{-1}|^2/(2\xi)} P^r(0,1/2,xy^{-1}) e^{|y|^2/(2\xi)}|X^I P^r(1/2,1,y)|\,dy$$
$$\le \Big(\int_N e^{|xy^{-1}|^2/\xi}(P^r(0,1/2,xy^{-1}))^2\,dy\Big)^{1/2}$$
$$\times \Big(\int_N e^{|y|^2/\xi}|X^I P^r(1/2,1,y)|^2\,dy\Big)^{1/2} \le C''.$$

Now by (3.27) and Lemma (3.22) we obtain (3.26) for $s = 0$, and considering

$$L' = \sum Y_i^2(s+t) + Y(s+t)$$

for every s. ■

(3.28) COROLLARY. *If L satisfies condition $\mathfrak{U}$ or $\mathfrak{V}$ then there is a $C(M,m)$ such that*

$$P(s,t,x) \le C(M,m)(t-s)^{-Q/2}(1+|\delta_{(t-s)^{1/2}}x|)^{-Q-1}$$

for $t - s \ge 1$ or $t - s \le 1$ respectively. ■

(3.29) THEOREM. *For every $K > 0$ there are constants $C = C(K,M,m)$, $\alpha = \alpha(K) \ge 0$ and $\beta = \beta(K) \ge 0$ such that*

$$P(s,t,x) \le \begin{cases} C(t-s)^{\alpha}(1+|x|)^{-K}, & t-s \ge 1, \\ C(t-s)^{-\beta}(1+|x|)^{-K}, & t-s \le 1. \end{cases} \tag{3.30}$$

Moreover, $C(K,M,m)$ depends polynomially on M, m.

Proof. Let $t - s \le 1$. Then there is a $C = C(K,M)$ such that

$$\int_N (1+|x|)^K P^r(s,t,x)\,dx \le C\begin{cases} r^{\alpha'} & r \ge 1, \\ r^{-\beta'}, & r \le 1, \end{cases}$$

for some $\alpha' = \alpha'(K) > 0$, $\beta' = \beta'(K) > 0$ ([S], see also [DH]). Using estimate (3.7) for $\|P^r(s,t)\|_{L^\infty}$ and proceeding as in the previous theorem we obtain (3.30). ■

In what follows we will need some estimates on $\langle e^{\xi\tau}, X^I P(s,t)\rangle$ and $\langle \tau^\xi, X^I P(s,t)\rangle$ as formulated in the following two theorems.

(3.31) THEOREM. *For every $\xi > 0$ and every multiindex I there are $\beta = \beta(I) \ge |I|/2$, C_2, $C_1 = C_1(\xi,M,m)$ such that*

$$\langle e^{\xi\tau}, |X^I P(s,t)|\rangle \le C_1(t-s)^{-\beta} e^{C_2(t-s)M(\xi+\xi^2)}, \tag{3.32}$$

and the same with $\check{X}^I$ in place of X^I. Moreover, when L satisfies condition $\mathfrak{V}$ or $I = (0,\ldots,0)$ then $\beta = |I|/2$.

Proof. For $t-s \le 1$, (3.32) follows from Lemma (3.22). If $I = (0,\ldots,0)$ then (3.32) is a consequence of (3.21). Since $e^{\xi\tau}$ is submultiplicative [H1], in view of (3.20), we have

$$\langle e^{\xi\tau}, |\check{X}^I P(s,t)|\rangle \le \langle e^{\xi\tau}, |\check{X}^I P(s,s+1)|\rangle\langle e^{\xi\tau}, P(s+1,t)\rangle$$

for $t - s > 1$ and (3.32) follows. ■

Remark. If $L = \partial_x^2 - \partial_t$ and $P(0,t) = t^{-1/2}e^{-x^2/(4t)}$ a simple calculation shows that

$$\langle e^{\xi\tau}, |\partial^n P(0,t)|\rangle \ge c(\xi,n)t^{-n/2}e^{\xi^2 t}.$$

Analogously using τ^ξ in place of $e^{\xi\tau}$ we derive from Theorem (5.16) of [DH] the following

(3.33) THEOREM. *For every $\xi > 0$ there is a $C = C(M)$ such that*

$$\langle \tau^\xi, P(s,t)\rangle \le C\max(t^{\xi/2}, t^\xi). \tag{3.34}$$

Moreover, for every multiindex $I \ne 0$ there are $\beta = \beta(I) \ge |I|/2$ and $C_1 = C_1(\xi,M,m)$ depending polynomially on M, m such that

$$\langle \tau^\xi, |X^I P(s,t)|\rangle \le \begin{cases} C_1(t-s)^{-\beta}, & t-s \le 1, \\ C_1(t-s)^{\xi}, & t-s \ge 1, \end{cases} \tag{3.35}$$

and the same for $\check{X}^I$ in place of X^I. If L satisfies condition $\mathfrak{V}$ then $\beta = |I|/2$.

For our maximal functions we will also consider operators (3.1) with unbounded coefficients. Let

$$L = \sum_{i=1}^k Y_i(t)^2 + Y(t)$$

and

$$\eta(t) = \max\{|\alpha_j(t)| : j = 1,\ldots,n\},$$
$$\Lambda(t) = \max\{|\alpha_{ij}(t)| : i,j = 1,\ldots,k\},$$
$$\lambda(t) = \inf\{\|A(t)v\| : \|v\| = 1\}.$$

We assume that there is a $D < \infty$ such that

$$\max(\Lambda(t)^2/\lambda(t)^2, \eta(t)/\lambda(t)^2) \le D. \tag{3.36}$$

Let

$$\widetilde{\lambda}(t) = \int_0^t \lambda(u)^2\,du.$$

$\widetilde{\lambda}$ is an increasing bijection of $[0,\infty)$ onto $[0,T]$, where $T=\int_0^\infty \lambda(u)^2\,du$. The image of $\lambda(t)^{-2}(L-\partial_t)$ via the map $(x,t)\to(x,\widetilde{\lambda}(t))=(x,s)$ is the operator

$$(3.37)\qquad (\widetilde{L}-\partial_s)f(x,s) = \lambda(\widetilde{\lambda}^{-1}(s))^{-2}\Big(\sum_{i=1}^k Y_i(\widetilde{\lambda}^{-1}(s))^2+Y(\widetilde{\lambda}^{-1}(s))-\partial_s\Big)f(x,s)$$

on $N\times(0,T)$. For the fundamental solution $\widetilde{P}(u,s,x)$, $0\le u<s<T$, of $\widetilde{L}-\partial_s$ we have the estimates given by Theorems (3.25) and (3.29). Moreover,

$$(3.38)\qquad P(u,t,x)=\widetilde{P}(\widetilde{\lambda}(u),\widetilde{\lambda}(t),x)$$

for $0\le u<t$, $x\in N$.

Assume now that there is a d such that $\lambda(t)^2\ge d$ for every $t>0$. Then

$$(3.39)\qquad \widetilde{\lambda}(t)>dt,\qquad t>0\,,$$

$T=\infty$, and Theorem (3.29) implies

(3.40) Corollary. *If L satisfies (3.36) and (3.39) then for every $K>0$ there are constants $C=C(K,D,d)$, $\beta=\beta(K)\ge 0$ such that*

$$P(s,t,x)\le C(t-s)^{-\beta}(1+|x|)^{-K}\qquad \textit{for } t-s\le 1\,.$$

4. Moments of harmonic measures (bounded coefficients case). In this chapter we are going to prove some properties of the harmonic measures $\check{\mu}_a^b$ (defined in (2.10)) corresponding to the operator

$$(4.1)\qquad L=\sum_{i=1}^k Y_i(a)^2+Y(a)+\alpha\partial_a^2-\kappa\partial_a$$

defined on $N\times\mathbb{R}^+$, where

$$Y_i(a)=\sum_{j=1}^k \alpha_{ij}(a)X_j,\qquad Y(a)=\sum_{j=1}^n \alpha_j(a)X_j\,,$$

$X_1,\dots,X_n$ is an arbitrary basis of $\mathfrak{n}$ such that $X_1,\dots,X_k$ generate $\mathfrak{n}$ (with no assumptions on homogeneity of $\mathfrak{n}$ and $X_1,\dots,X_k$), $\alpha_{ij}(a)$, $\alpha_j(a)$ are smooth functions and the matrix $A(a)=[\alpha_{ij}(a)]$ is positive definite. We assume that

$$\eta=\max_{a>0}\{|\alpha_j(a)|:j=1,\dots,n\},\qquad \Lambda=\max_{a>0}\{|\alpha_{ij}(a)|:i,j=1,\dots,k\}$$

are finite and

$$\lambda=\inf_{a>0,\|v\|=1}\|A(a)v\|>0\,.$$

As before

$$M=\max(\eta,\Lambda^2),\qquad m=\max(1+\eta,\Lambda^2)\lambda^{-2}k^2+1\,.$$

Let first $\alpha=0$ and consider the free group G with the Lie algebra generated by X_j, $j=1,\dots,k$. To avoid any confusion we will denote these vector fields on G by $\mathcal{X}_j$, i.e. we have the homomorphism $\sigma:G\to N$ such that

$$\sigma_*\mathcal{X}_j=X_j,\qquad j=1,\dots,k\,.$$

The dilations on G are defined by

$$\delta_r\mathcal{X}_j=r\mathcal{X}_j\,.$$

Let $\mathcal{Y}_i(a)=\sum_{j=1}^k\alpha_{ij}(a)\mathcal{X}_j$, $\mathcal{Y}(a)=\sum_{j=1}^n\alpha_j(a)\mathcal{X}_j$. If $P^G(a,b)$ and $P(a,b)$ are the fundamental solutions of

$$(4.2)\qquad \mathcal{L}=\sum_{i=1}^k\mathcal{Y}_i(a)^2+\mathcal{Y}(a)-\kappa\partial_a$$

and of L (with $\alpha=0$) respectively then for every Borel set $V\subseteq N$,

$$(4.3)\qquad P(a,b,V)=P^G(a,b,\sigma^{-1}(V))\,,$$

and in view of Theorem (3.31) we have the following

(4.4) Corollary. *For every $\xi>0$ and every multiindex I there are $C_1=C_1(\xi,M,m)$, $C_2,\beta=\beta(I)>0$ such that*

$$\langle e^{\xi\tau},|X^IP(a,b)|\rangle\le C_1(b-a)^{-\beta}e^{C_2(b-a)M(\xi+\xi^2)}\,,$$

and $\beta=0$ for $I=(0,\dots,0)$. ■

If $\alpha\ne0$ then without loss of generality we may assume that $\alpha=1$. Given a continuous function $\mathbf{a}:[0,\infty)\to\mathbb{R}^+$ we look at the operator

$$(4.5)\qquad L^{\mathbf{a}}=\sum_{i=1}^k Y_i(\mathbf{a}(t))^2+Y(\mathbf{a}(t))-\partial_t\,.$$

Let $P(\mathbf{a};s,t,x)$ be the fundamental solution of $L^{\mathbf{a}}$ and $T_a(\mathbf{a})=\inf\{t:\mathbf{a}(t)\le a\}$. Then as in [DH] for $a<b$ we have

$$(4.6)\qquad \check{\mu}_a^b(V)=\int P(\mathbf{a};0,T_a,V)\,d\mathcal{W}_b(\mathbf{a})=E_bP(\mathbf{a};0,T_a,V)\,,$$

where $\mathcal{W}_b$ is the Wiener measure associated with $\partial_a^2-\kappa\partial_a$ starting at b. (For the proof of (4.6) see [T].) The distribution of T_a is given by

$$(4.7)\qquad \mathcal{W}_b\{T_a<t\}=\int_0^t(4\pi)^{-1/2}(b-a)s^{-3/2}\exp[-(b-a-\kappa s)^2/(4s)]\,ds\,.$$

We will also need the operator

$$\mathcal{L}^{\mathbf{a}} = \sum_{i=1}^{k} \mathcal{Y}_i(\mathbf{a}(t))^2 + \mathcal{Y}(\mathbf{a}(t)) - \partial_t \tag{4.8}$$

on $G \times \mathbb{R}^+$. As before

$$P(\mathbf{a}; s, t, V) = P^G(\mathbf{a}; s, t, \sigma^{-1}(V)), \tag{4.9}$$

where $P^G(\mathbf{a}; s, t)$ is the fundamental solution of (4.8).

Now we consider separately the cases $\kappa > 0$ and $\kappa = 0$.

(4.10) THEOREM. *Let $\kappa > 0$ and C_2 be as in Corollary (4.4). For every ξ satisfying $\xi + \xi^2 < \kappa^2/(4C_2M)$ there are $C = C(\xi, M, m)$ and $\beta = \beta(I)$ such that*

$$\langle |X^I \check{\mu}_a^b|, e^{\xi\tau} \rangle \le C(b-a)^{-\beta} e^{\kappa(b-a)}, \tag{4.11}$$

and the same with $\check{X}^I$ in place of X^I. If $I = (0, \ldots, 0)$ then $\beta = 0$.

Remark. If $L = \partial_x^2 + \partial_t^2 - \partial_t$ and $\eta > 1$ then for every ξ

$$\langle \check{\mu}_a^b, e^{\xi|\cdot|^\eta} \rangle = \infty. \tag{4.12}$$

Indeed, by (4.7),

$$\langle \check{\mu}_a^b, e^{\xi|\cdot|^\eta} \rangle = 2b \int_0^\infty (4\pi)^{-1/2} t^{-3/2} e^{-(b-t)^2/(4t)} \Big(\int_0^\infty (2\pi t)^{-1/2} e^{-x^2/(4t)+\xi x^\eta}\, dx \Big)\, dt$$

and

$$\int_0^\infty (2\pi t)^{-1/2} e^{-x^2/(4t)+\xi|x|^\eta}\, dx = (2/\pi)^{1/2} \int_0^\infty e^{-x^2+4\xi t^{\eta/2} x^\eta}\, dx.$$

Let

$$\begin{aligned} w_1 &= (\eta\xi)^{1/(2-\eta)} t^{\eta/(2(2-\eta))}, \\ w_2 &= (2\eta\xi)^{1/(2-\eta)} t^{\eta/(2(2-\eta))}, \\ C_1 &= \xi^{2/(2-\eta)} (4\eta^{\eta/(2-\eta)} - \eta^{2/(2-\eta)}), \\ C_2 &= \xi^{1/(2-\eta)} ((2\eta)^{1/(2-\eta)} - \eta^{1/(2-\eta)}). \end{aligned}$$

Then

$$\int_{w_1}^{w_2} e^{-x^2+4\xi t^{\eta/2} x^\eta}\, dx \ge C_2 t^{\eta/(2(2-\eta))} e^{C_1 t^{\eta/(2-\eta)}},$$

which proves (4.12). ∎

Proof of Theorem (4.10). By (4.6) and Corollary (4.4),

$$\langle e^{\xi\tau}, |X^I \check{\mu}_a^b| \rangle \le E_b \langle e^{\xi\tau}, |X^I P(\mathbf{a}; 0, T_a)| \rangle$$

$$\begin{aligned} &\le C_1 (4\pi)^{-1/2} (b-a) \int_0^\infty t^{-\beta} e^{C_2 t M(\xi+\xi^2)} t^{-3/2} e^{-(b-a-\kappa t)^2/(4t)}\, dt \\ &\le C_1 (4\pi)^{-1/2} (b-a) e^{(b-a)\kappa/2} \Big(\int_0^1 t^{-\beta-3/2} e^{MC_2(\xi+\xi^2)} e^{-(b-a)^2/(4t)} \\ &\qquad + \int_1^0 e^{C_2 t M(\xi+\xi^2)} e^{-\kappa^2 t/4}\, dt \Big). \end{aligned}$$

Now if $C_2 M(\xi + \xi^2) < \kappa^2/4$ then (4.11) follows. ∎

(4.13) THEOREM. *Let $\kappa = 0$. If $\xi < 1/2$ then there are $C = C(\xi, M, m)$ and $\beta = \beta(I)$ such that*

$$\langle |X^I \check{\mu}_a^b|, \tau^\xi \rangle \le \begin{cases} C(b-a)^{-\beta} & \text{if } b-a \le 1, \\ C(b-a)^{2\xi} & \text{if } b-a \ge 1, \end{cases} \tag{4.14}$$

and $\beta = 0$ for $I = (0, \ldots, 0)$.

Remark. If $L = \partial_x^2 + \partial_x + \partial_t^2$ and $\xi \ge 1/2$ then $\langle \check{\mu}_0^b, |\cdot|^\xi \rangle = \infty$. Indeed, by (4.7),

$$\langle \check{\mu}_0^b, |\cdot|^\xi \rangle = (4\pi)^{-1/2} b \int_0^\infty t^{-3/2} e^{-b^2/(4t)} \Big(\int_{-\infty}^\infty (2\pi t)^{-1/2} |x|^\xi e^{-(x-t)^2/(4t)}\, dx \Big)\, dt$$

and

$$\int_{-\infty}^\infty (2\pi t)^{-1/2} |x|^\xi e^{-(x-t)^2/(4t)}\, dx \ge \int_0^\infty (2\pi t)^{-1/2} (x+t)^\xi e^{-x^2/(4t)}\, dx \ge t^{\xi/2}.$$

Therefore

$$\langle \check{\mu}_0^b, |\cdot|^\xi \rangle \ge \frac{1}{2} (4\pi)^{-1/2} b \int_0^\infty t^{-3/2+\xi} e^{-b^2/(4t)}\, dt = \infty.$$

Proof of Theorem (4.13). By (4.6), (4.9) and the fact that $(1 + \tau \circ \sigma)^\xi$ is a submultiplicative function on G,

$$\begin{aligned} \langle |X^I \check{\mu}_0^b|, \tau^\xi \rangle &= E_b \langle (\tau \circ \sigma)^\xi, |\mathcal{X}^I P^G(\mathbf{a}; 0, T_a)| \rangle \\ &\le E_b(\langle (\tau \circ \sigma)^\xi, |\mathcal{X}^I P^G(\mathbf{a}; 0, T_a)| \rangle, T_a \le 1) \\ &\quad + E_b(\langle (\tau \circ \sigma)^\xi, |\mathcal{X}^I P^G(\mathbf{a}; 0, 1)| \rangle \cdot \langle (\tau \circ \sigma)^\xi, P^G(\mathbf{a}; 1, T_a) \rangle, T_a > 1). \end{aligned}$$

But in view of (4.7) and Theorem (3.33),

$$E_b(\langle (\tau \circ \sigma)^\xi, |\mathcal{X}^I P^G(\mathbf{a}; 0, T_a)| \rangle, T_a \le 1) \le C_1 (4\pi)^{-1/2} (b-a) \int_0^1 t^{-\beta-3/2} e^{-(b-a)^2/(4t)}\, dt$$

and

$$E_b(\langle(\tau\circ\sigma)^\xi, |\mathcal{X}^I P^G(\mathbf{a};0,1)\rangle\cdot\langle(\tau\circ\sigma)^\xi, P^G(\mathbf{a};1,T_a)\rangle, T_a>1)$$
$$\le C_1^2(4\pi)^{-1/2}(b-a)\int\limits_1^\infty t^{\xi-3/2}e^{-(b-a)^2/(4t)}\,dt\,.$$

Now substituting $t=(b-a)^2 s$ we obtain (4.14). ∎

5. Moments of harmonic measures (unbounded coefficients case). Now we are going to prove that the derivatives of the harmonic measures (2.10) corresponding to the operator

$$L=\sum_{i,j=1}^n \beta_{ij}(a)X_iX_j+\sum_{i=1}^n\beta_i(a)+\partial_a^2-2\kappa\partial_a\,,\tag{5.1}$$

with coefficients growing at most as $e^{\eta a}$ as $a\to\infty$, have some moments. Our main tools here are appropriate maximum principles (compare Theorems (2.5) and (2.6) in [DH]). We also consider the case when $|\beta_{ij}(a)|$, $|\beta_i(a)|$ grow at most polynomially to show that then we can, of course, obtain better moments. Corollary (5.11) of this chapter is used in proving the weak type (1,1) of the local maximal function $M_2 f$ (see Chapter 6).

(5.2) THEOREM. *Let $\beta_{ij}(a)$, $\beta_i(a)$, $a>0$, be smooth functions such that the matrix $[\beta_{ij}(a)]$ is semipositive definite and let $\check{\mu}_a^b$ be the harmonic measure corresponding to L defined on $N\times\mathbb{R}^+$.*

(a) *Suppose $|\beta_{ij}(a)|, |\beta_i(a)|\le de^{\eta a}$.*

(5.3) *If $\kappa>0$ then for every $\xi<2\kappa/\eta$ there is a constant C independent of b such that*

$$\langle\tau^\xi,\check{\mu}_0^b\rangle\le Ce^{2\kappa b}\,.$$

(5.4) *If $\kappa=0$ then for every $\xi<1$ there is a constant C independent of b such that*

$$\langle\log(1+\tau)^\xi,\check{\mu}_0^b\rangle\le C(1+b^\xi)\,.$$

(b) *Suppose $|\beta_{ij}(a)|$, $|\beta_i(a)|\le d(1+a)^m$.*

(5.5) *If $\kappa>0$ then for every ξ there is a constant C independent of b such that*

$$\langle\tau^\xi,\check{\mu}_0^b\rangle\le C(1+b)^{N(m+2)}\,.$$

(5.6) *If $\kappa=0$ then for every $\xi<(2+m)^{-1}$ there is a constant C independent of b such that*

$$\langle\tau^\xi,\check{\mu}_0^b\rangle\le C(1+b)^{\xi(2+m)}\,.$$

Proof. The proofs of (5.3) and (5.4) are the same as the proofs of Theorems (2.5) and (2.6) in [DH]. For (5.3) we use the function

$$G(x,a)=-(\varepsilon e^{\sigma(a-b)}+\varepsilon^{\eta/\sigma}(\tau*\Phi(x)+1)Re^{-\eta b})^\gamma,\qquad x\in N, a>0,$$

where

$$\sigma<\eta,\quad \gamma\sigma<2\kappa,\quad \gamma\ge1,\quad \varepsilon<1,$$

$$C_1=\sum_{i,j=1}^n(\|X_iX_j(\tau*\Phi)\|+\|X_i(\tau*\Phi)\|\,\|X_j(\tau*\Phi)\|)+\sum_{i=1}^n\|X_i(\tau*\Phi)\|\,,\tag{5.7}$$

$$R=\sigma(\kappa-\gamma\sigma)/(2(\gamma+1)dC_1)\,.$$

For (5.4) the appropriate function is

$$G(x,a)=-\varepsilon a^\gamma/b^\gamma-\varepsilon^{2/\gamma}(\tau*\Phi(x)+1)Re^{-\eta\varepsilon^{-1/\gamma}b}b^{-2},\qquad x\in N, a>0\,,$$

where $\gamma<1$, $R=(1-\gamma)\gamma/(2dC_1)$ and C_1 as in (5.7).

The proof of (5.5) follows from the following maximum principle:

Let $\xi>\max(\kappa,2^{m+2}d/\kappa)$, $\varepsilon<\min(b^{-1},1)$, C_1 as in (5.7) and

$$D=\{(x,a):\tau(x)\le\varepsilon^{-m-1}\xi C_1,\ 0<a<\varepsilon^{-1}\}\,.$$

If $F\in C^2(D)\cap C(\overline{D})$, $|F|\le1$, $LF\ge0$ in D and $F(x,0)\le0$ for $\tau(x)<\varepsilon^{-m-1}\xi C_1$ then

$$F(e,b)\le e^\xi(b+\xi/\kappa+1)^\xi\,.$$

To prove this maximum principle we consider the function

$$G(x,a)=-G_0(x,a)^\xi,\qquad x\in N, a>0\,,$$

where

$$G_0(x,a)=\varepsilon(a+\xi/\kappa)+\varepsilon^{m+1}(\tau*\Phi(x)+1)/(\xi C_1)\,.$$

First we notice that $LG(x,a)>0$ whenever $x\in N$, $0<a<\varepsilon^{-1}$. Indeed, with our assumptions

$$-2\kappa\partial_a G=2\kappa\varepsilon\xi G_0^{\xi-1},\qquad |\partial_a^2 G|<\kappa\varepsilon\xi G_0^{\xi-1}\,,$$

$$\Big|\sum_{i=1}^n\beta_i(a)X_iG\Big|<\frac{\kappa\varepsilon\xi}{2}G_0^{\xi-1},\qquad \Big|\sum_{i=1}^n\beta_{ij}(a)X_iX_jG\Big|<\frac{\kappa\varepsilon\xi}{2}G_0^{\xi-1}\,.$$

Moreover, $G(x,0)\le0$ and $G\le-1$ on the remaining part of the boundary ∂D of D. Hence $F+G\le0$ on ∂D. The weak maximum principle for degenerate elliptic operators (Proposition 1.1 in [B]) implies $F+G\le0$ in D and in particular

$$F(b)\le-G(b)\le\varepsilon^\xi(b+\xi/\kappa+1)^\xi\,.$$

Hence

$$\mu_0^b\{\tau \geq \lambda\} \leq \lambda^{-\xi/(m+1)}(\xi C_1)^{\xi/(m+1)}(b+\xi/\kappa+1)^{\xi}$$

for $\lambda \geq \xi C_1 b^{m+1}$, which gives (5.5).

For the proof of (5.6) we use the function

$$G = -\varepsilon a^{\gamma}/b^{\gamma} - \varepsilon^{(2+m)/\gamma}(\tau * \Phi(x)+1)Rb^{-2-m}, \qquad x \in N, b>0,$$

where $R = \gamma(1-\gamma)/(2dC_1)$, C_1 is as in (5.7) and $\gamma < 1$. ∎

Remark. It seems that there is no hope for a better moment when $\kappa = 0$ using this method because then $-G$ must be a nonnegative, increasing function with a negative second derivative with respect to a.

(5.8) THEOREM. *Let*

$$L = \sum_{i,j=1}^{k} \beta_{ij}(a)X_iX_j + \sum_{i=1}^{n} \beta_i(a)X_i + \partial_a^2 - 2\kappa\partial_a$$

where $\beta_{ij}(a)$, $\beta_i(a)$, $a>0$, are smooth functions such that

$$|\beta_{ij}(a)|, |\beta_i(a)| < de^{\eta a}$$

and the matrix $[\beta_{ij}(a)] - cI$ is positive definite for some $c>0$. Let $\varphi = (1+\tau)^{\xi}$ or $\varphi = (1+\log(1+\tau))^{\xi}$. If $\langle \varphi, \check{\mu}_0^b\rangle < \infty$ then for every multiindex I,

$$\langle \varphi, |X^I \check{\mu}_a^b|\rangle < \infty, \tag{5.9}$$

$$\langle \varphi, |\check{X}^I \check{\mu}_a^b|\rangle < \infty. \tag{5.10}$$

Proof. (5.9) follows from Harnack's inequality that is uniform with respect to x, i.e. since $\check{\mu}_a^b(x)$ is L-harmonic as a function of x, b we have

$$|X^I \check{\mu}_a^b(x)| \leq C(a,b,I)\check{\mu}_a^b(x).$$

The proof of (5.10) is the same as the proof of Theorem (6.10) in [DH] and follows from the following facts.

a) The constant C_1 in (3.35) depends polynomially on M, m.

b) $\langle \varphi, \check{\mu}_a^b\rangle$ as a function of b is dominated by $Ce^{2\kappa b}$.

c) For a given trajectory $\mathbf{a} : [0,\infty] \to \mathbb{R}^+$,

$$\max\{|\beta_{ij}(\mathbf{a}(t))|, |\beta_p(\mathbf{a})| : i,j = 1,\ldots,k,\ p = 1,\ldots,n,\ 0 \leq t \leq 1\}$$
$$\leq d\exp(\eta\{\max \mathbf{a}(t) : 0 \leq t \leq 1\}).$$

(5.11) COROLLARY. (a) *Suppose $|\beta_{ij}(a)|, |\beta_i(a)| \leq de^{\eta a}$. Then for $\kappa > 0$, every $\xi < 2\kappa/\eta$ and every multiindex I,*

$$\langle \tau^{\xi}, |X^I \check{\mu}_a^b|\rangle, \langle \tau^{\xi}, |\check{X}^I \check{\mu}_a^b|\rangle < \infty,$$

while for $\kappa = 0$, every $\xi < 1$ and every I,

$$\langle \log(1+\tau)^{\xi}, |X^I \check{\mu}_a^b|\rangle, \langle \log(1+\tau)^{\xi}, |\check{X}^I \check{\mu}_a^b|\rangle < \infty.$$

(b) *Suppose $|\beta_{ij}(a)|, |\beta_i(a)| \leq d(1+a)^m$. Then for $\kappa > 0$, every ξ and every I,*

$$\langle \tau^{\xi}, |X^I \check{\mu}_a^b|\rangle, \langle \tau^{\xi}, |\check{X}^I \check{\mu}_a^b|\rangle < \infty,$$

while for $\kappa = 0$, every $\xi < (2+m)^{-1}$ and every I,

$$\langle \tau^{\xi}, |X^I \check{\mu}_a^b|\rangle, \langle \tau^{\xi}, |\check{X}^I \check{\mu}_a^b|\rangle < \infty.$$

6. Maximal functions. Let L be as in (4.1) and

$$\eta(a) = \max\{|\alpha_j(a)| : j = 1,\ldots,n\},$$
$$\Lambda(a) = \max\{|\alpha_{ij}(a)| : i,j = 1,\ldots,k\},$$
$$\lambda(a) = \inf\{\|A(a)v\| : \|v\| = 1\}.$$

For such operators we consider the maximal functions

$$M_1 f(x) = \sup_{b \geq a+1} |f * \check{\mu}_a^b(x)|, \qquad M_2 f(x) = \sup_{a<b\leq a+1} |f * \check{\mu}_a^b(x)|,$$
$$Mf(x) = \sup_{a<b} |f * \check{\mu}_a^b(x)| \leq M_1 f(x) + M_2 f(x), \tag{6.1}$$

where μ_a^b are the harmonic measures (2.10) corresponding to L.

(6.2) THEOREM. *Assume that $\Lambda(a)/\lambda(a)$ is bounded and $Y(a) = 0$, i.e.*

$$L = \sum_{i=1}^{k} Y_i(a)^2 + \alpha\partial_a^2 - \kappa\partial_a.$$

Then Mf is of weak type $(1,1)$.

Proof. If $\alpha = 0$ then the theorem follows immediately from Corollary (3.28) and Proposition (2.4) in [HJ]. Obviously we may assume that a in (6.1) is 0 and $f \geq 0$. Let $P(\mathbf{a};0,t,x)$, $P^G(\mathbf{a};0,t,x)$, $\widetilde{P}^G(\mathbf{a};0,t,x)$ be the fundamental solutions associated with $L^{\mathbf{a}}$ (see (4.5)), $\mathcal{L}^{\mathbf{a}}$ (see (4.8)) and $\widetilde{\mathcal{L}}^{\mathbf{a}}$ (see (3.37)) respectively. For a function f on N, a measure μ on G and $x \in N$ we define

$$f *_G \mu(x) = \int_G f(x\sigma(y^{-1}))\,d\mu(y).$$

By (4.6),

$$Mf(x) = \sup_{b>0} E_b(f * P(\mathbf{a};0,T_0)(x)), \qquad x \in N,$$

where $T_0(\mathbf{a}) = \inf\{t : \mathbf{a}(t) = 0\}$. But in view of (4.9) and (3.38),

$$\sup_{t>0} f * P(\mathbf{a};0,t)(x) = \sup_{t>0} f *_G P^G(\mathbf{a};0,t)(x)$$
$$= \sup_{0<t<T} f *_G \widetilde{P}^G(\mathbf{a};0,t)(x),$$

where $T=\int_0^\infty \lambda(\mathbf{a}(t))^2\,dt$, and by Corollary (3.28) there is a C such that

$$\widetilde{P}^G(\mathbf{a};0,t,x)\le k_t(x)=Ct^{-Q/2}(1+|\delta_{t^{-1/2}}x|)^{-Q-1},\qquad x\in N.$$

Therefore

$$Mf(x)\le \sup_{b>0}E_b(\sup_{0<t} f*P(\mathbf{a};0,t)(x))\le \sup_{0<t<T} f*_G k_t(x),$$

which by Proposition (2.4) of [HJ] is of weak type $(1,1)$. ■

Remark. In the previous proof we in fact changed sup to E_b and so we only require that the function

$$\widetilde{\lambda}(t)=\int\limits_0^\infty \lambda(\mathbf{a}(s))^2\,ds$$

is increasing. If we also admit a first order part in the operator then we proceed differently for small and for large times and so we have to know something more about the behaviour of $\widetilde{\lambda}$. That is the reason for assumption (6.4) below.

(6.3) THEOREM. *Assume that $\Lambda(a)/\lambda(a)$, $\eta(a)/\lambda^2(a)$ are bounded and*

$$\lambda(a)^2\ge c>0. \tag{6.4}$$

If $\alpha_i(a)=0$ for $i=1,\dots,k$, i.e. $Y(a)\in\operatorname{lin}\{X_{k+1},\dots,X_n\}$, then M_1f is of weak type $(1,1)$.

Remark. If $\alpha=0$ then the theorem follows from Corollary (3.28) and Proposition (2.4) in [HJ]. Moreover, an easy calculation shows that M_1f is neither of weak type $(1,1)$ nor bounded on L^p, $p>1$, for $L=\partial_x^2+\partial_x-\partial_t$.

Proof of Theorem (6.3). We proceed as in the previous theorem. Let

$$M_1'f(x)=\sup_{b>1}E_b(f*P(\mathbf{a};0,T_0)(x),T_0\ge 1),\qquad x\in N,$$

$$M_1''f(x)=\sup_{b>1}E_b(f*P(\mathbf{a};0,T_0)(x),T_0<1),\qquad x\in N.$$

Then $M_1f(x)\le M_1'f(x)+M_1''f(x)$ for every x. In view of (4.9) and (3.38),

$$\sup_{t>1} f*P(\mathbf{a};0,t)(x)=\sup_{t>1} f*_G P^G(\mathbf{a};0,t)(x)\le \sup_{t>c} f*_G \widetilde{P}^G(\mathbf{a};0,t)(x).$$

By Corollary (3.28) there is a C such that

$$\widetilde{P}^G(\mathbf{a};0,t,x)\le k_t(x)=Ct^{-Q/2}(1+|\delta_{t^{-1/2}}x|)^{-Q-1}.$$

Therefore as before

$$M_1'f(x)\le \sup_{t>c} f*_G k_t(x),\qquad x\in N,$$

which by Proposition (2.4) in [HJ] is of weak type $(1,1)$. Obviously

$$M_1''f(x)=\sup_{b>1}E_b(f*_G P^G(\mathbf{a};0,T_0)(x),T_0<1).$$

In view of (6.4) and Corollary (3.39) there are constants C, β such that

$$P^G(\mathbf{a};0,t,x)\le Ct^{-\beta}(1+|x|)^{-Q-1}.$$

Therefore

$$f*_G P^G(\mathbf{a};0,t,x)\le Ct^{-\beta}Rf(x),$$

where

$$Rf(x)=\int\limits_G f(x\sigma(y^{-1}))(1+|y|)^{-Q-1}\,dy.$$

Obviously R is bounded on L^1 and

$$M_1''f(x)\le C\sup_{b>1}E_b(T_0^{-\beta},T_0<1)Rf(x).$$

By (4.7),

$$E_b(T_0^{-\beta},T_0<1)=(4\pi)^{-1/2}b\int\limits_0^1 t^{-3/2-\beta}\exp(-(b-\kappa t)^2/(4t))\,dt.$$

If $1<b<2\kappa$ then

$$E_b(T_0^{-\beta},T_0<1)\le (4\pi)^{-1/2}2\kappa e^{\kappa^2}\int\limits_0^1 t^{-3/2-\beta}\exp(-1/(4t))\,dt.$$

If $b\ge\max(2\kappa,1)$ then $b-\kappa t\ge b/2$ and

$$\begin{aligned}E_b(T_0^{-\beta},T_0<1)&\le (4\pi)^{-1/2}b\int\limits_0^1 t^{-3/2-\beta}\exp(-b^2/(16t))\,dt\\&=(4\pi)^{-1/2}\int\limits_0^1 t^{-3/2}(b^2t)^{-\beta}\exp(-1/(16t))\,dt\\&\le(4\pi)^{-1/2}\int\limits_0^1 t^{-3/2-\beta}\exp(-1/(16t))\,dt.\ \blacksquare\end{aligned}$$

(6.5) THEOREM. *If $\lambda(a)^2\ge c>0$, $|\alpha_{ij}(a)|,|\alpha_j(a)|\le de^{\eta a}$ for some $\eta>0$, and*

$$Y(a)\in\operatorname{lin}\{X_1,\dots,X_k,[X_r,X_p]:1\le r,p\le k\} \tag{6.6}$$

for every a, then M_2f is of weak type $(1,1)$.

Since by Corollary (5.11) all derivatives of $\check{\mu}_a^b$ are integrable, the proof is the same as that of Theorem (7.21) in [DH]. For a counterexample showing that (6.6) is essential when $\alpha=0$ see [Z].

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND

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Pick–Nevanlinna interpolation on finitely-connected domains

by

STEPHEN D. FISHER (Evanston, Ill.)

Abstract. Let Ω be a domain in the complex plane bounded by $m+1$ disjoint, analytic simple closed curves and let $z_0, \ldots, z_n$ be $n+1$ distinct points in Ω. We show that for each $(n+1)$-tuple $(w_0, \ldots, w_n)$ of complex numbers, there is a unique analytic function B such that: (a) B is continuous on the closure of Ω and has constant modulus on each component of the boundary of Ω; (b) B has n or fewer zeros in Ω; and (c) $B(z_j) = w_j$, $0 \le j \le n$.

Introduction. The classical interpolation result of G. Pick and R. Nevanlinna referred to in the title is this. Let $z_0, \ldots, z_n$ be distinct points of the open unit disc Δ in the complex plane. Then, for each nonzero $(n+1)$-tuple of complex numbers $(w_0, \ldots, w_n)$, there is an analytic function of the form

$$B(z) = \lambda \prod_{j=1}^{r} \frac{z - a_j}{1 - \overline{a}_j z}, \quad |\lambda| = 1,\ a_j \in \Delta,\ r \le n, \tag{1}$$

and a positive real number ϱ such that

$$\varrho B(z_k) = w_k, \quad k = 0, \ldots, n. \tag{2}$$

Moreover, ϱ, λ, r and $a_1, \ldots, a_r$ are uniquely determined by the $n+1$ equations in (2). A proof of this result may be found in, for instance, [F].

The Pick–Nevanlinna theorem has found applications in diverse areas: approximation theory, most especially in the theory of n-widths [FM1], [FM2]; in circuit theory [D]; it is also a part of geometric function theory [A].

A function B of the form (1) is called a *Blaschke product* (of degree r). It is a standard and easily proved matter that $|B(z)| = 1$ for all z with $|z| = 1$. Conversely, it is also easily established that if F is analytic on Δ, continuous on $\{z : |z| \le 1\}$, and $|F(z)| = 1$ for all z with $|z| = 1$, then F is a Blaschke product (of some finite degree).

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