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Stability of stochastic processes defined by integral functionals

by

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Abstract. The paper is devoted to the study of integral functionals $\int_0^\infty f(X(t,\omega))\,dt$ for continuous nonincreasing functions f and nonnegative stochastic processes $X(t,\omega)$ with stationary and independent increments. In particular, a concept of stability defined in terms of the functionals $\int_0^\infty f(aX(t,\omega))\,dt$ with $a\in(0,\infty)$ is discussed.

1. Preliminaries and notation. In the sequel $X=\{X(t,\omega):t\ge 0\}$ will always denote a nonnegative stochastic process with stationary and independent increments, right-continuous sample functions and satisfying the initial condition $X(0,\omega)=0$. Denote by $\pi(X,t,\cdot)$ the probability distribution of the random variable $X(t,\omega)$. The family $\pi(X,t,\cdot)$ $(t\ge 0)$ forms a convolution semigroup:

$$\pi(X,t,\cdot)*\pi(X,u,\cdot)=\pi(X,t+u,\cdot) \tag{1.1}$$

and

$$\pi(X,t,\{0\})=e^{-q(X)t} \tag{1.2}$$

where $q(X)\in[0,\infty]$. Moreover,

$$\pi(X,t,[0,a))>0 \tag{1.3}$$

for any $a\in(0,\infty)$ and sufficiently small t. All processes under consideration in the sequel will tacitly be assumed to be nondegenerate, i.e. $q(X)>0$.

It is well-known that for nondegenerate processes the potential

$$\varrho(X,A)=\int_0^\infty \pi(X,t,A)\,dt$$

is finite on bounded Borel subsets A of the half-line $[0,\infty]$ ([1], Proposition 14.1). In view of (1.2) we have

$$\varrho(X,\{0\})=q(X)^{-1}. \tag{1.4}$$

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By δ_a we shall denote the probability measure concentrated at $a \in [0,\infty)$. The support of a measure μ will be denoted by $\operatorname{supp}\mu$.

Inequality (1.3) yields

$$0 \in \operatorname{supp}\varrho(X,\cdot)\,. \tag{1.5}$$

A stochastic process X is said to be *deterministic* if $X(t,\omega) = bt$ with probability 1 or equivalently $\pi(X,t,\cdot) = \delta_{bt}$ for all $t \in [0,\infty)$ and for some positive constant b. In this case we have $\varrho(X, dy) = b^{-1}\,dy$.

LEMMA 1.1. *If* $\operatorname{supp}\pi(X,t_0,\cdot)$ *is bounded for some positive* t_0, *then the process* X *is deterministic.*

Proof. By (1.1) the probability distribution $\pi(X,t_0,\cdot)$ is infinitely divisible. Consequently, by Theorem 2.6.3 in [5], the boundedness of its support yields $\pi(X,t_0,\cdot) = \delta_{bt_0}$ for some positive constant b. Applying the semigroup property (1.1) we get $\pi(X,t,\cdot) = \delta_{bt}$ for all $t \in [0,\infty)$, which completes the proof.

A stochastic process X is said to be a *compound Poisson process* if $\varrho(X,\{0\}) > 0$. In this case there exist a positive constant q and a probability measure Q on the half-line $[0,\infty)$ with $Q(\{0\}) = 0$ such that

$$\pi(X,t,\cdot) = e^{-qt}\sum_{n=0}^{\infty}\frac{(qt)^n}{n!}Q^{*n} \tag{1.6}$$

where Q^{*n} for $n \ge 1$ is the nth convolution power of Q and $Q^{*0} = \delta_0$. The set of processes X satisfying (1.6) will be denoted by $\operatorname{Poiss}(q,Q)$. It is easy to check that

$$q(X) = q \tag{1.7}$$

and

$$\varrho(X,\cdot) = q^{-1}\sum_{n=0}^{\infty}Q^{*n} \tag{1.8}$$

for $X \in \operatorname{Poiss}(q,Q)$.

Throughout this paper P_s will denote the exponential probability distribution with parameter $s > 0$, i.e. $P_s(dy) = se^{-sy}\,dy$ on the half-line $[0,\infty)$. We shall often refer to the following representation of processes from $\operatorname{Poiss}(q,Q)$ ([4], Chapter IV, 2):

$$X(t,\omega) = 0 \qquad \text{for } t \in [0,\vartheta_0)\,, \tag{1.9}$$

$$X(t,\omega) = \sum_{j=1}^{k}\xi_j \qquad \text{for } t \in \Big[\sum_{j=0}^{k-1}\vartheta_j, \sum_{j=0}^{k}\vartheta_j\Big) \tag{1.10}$$

for $k \ge 1$ where the random variables $\vartheta_0,\vartheta_1,\ldots,\xi_1,\xi_2,\ldots$ are independent, ϑ_j $(j = 0,1,\ldots)$ have probability distribution P_q, and ξ_j $(j = 1,2,\ldots)$ have probability distribution Q.

From (1.8) by simple calculations we get

$$\varrho(X,dy) = q^{-1}(\delta_0(dy) + s\,dy) \tag{1.11}$$

for $X \in \operatorname{Poiss}(q,P_s)$.

Two processes X and Y are said to be *equivalent*, in symbols $X \sim Y$, whenever $\pi(X,t,\cdot) = \pi(Y,t,\cdot)$ for all $t \in [0,\infty)$. By Proposition 15.21 in [1], $X \sim Y$ if and only if $\varrho(X,\cdot) = \varrho(Y,\cdot)$. We write $X \prec Y$ if $\varrho(X,[0,a)) \le \varrho(Y,[0,a))$ for all $a \in (0,\infty)$. This order relation is reflexive and transitive. Moreover, $X \prec Y$ and $Y \prec X$ yield $X \sim Y$.

Denote by $\mathcal{F}$ the set of all nonnegative, continuous and nonincreasing functions defined on $[0,\infty)$.

Integrating by parts we get the following simple result.

LEMMA 1.2. *If* $X \prec Y$, *then*

$$\int_0^\infty f(u)\,\varrho(X,du) \le \int_0^\infty f(u)\,\varrho(Y,du)$$

for all $f \in \mathcal{F}$.

In the sequel we shall lean heavily on the following statement.

LEMMA 1.3. *For every process* X *and* $a \in (0,\infty)$ *there exists a number* $s \in (0,\infty)$ *such that* $X \prec Y$ *for* $Y \in \operatorname{Poiss}(\varrho(X,[0,a))^{-1},P_s)$.

Proof. Setting $V(\cdot) = \int_0^\infty e^{-t}\pi(X,t,\cdot)\,dt$ we get a probability measure concentrated on $[0,\infty)$. By (1.2) and (1.4),

$$V(\{0\}) = \frac{\varrho(X,\{0\})}{1+\varrho(X,\{0\})} < 1\,,$$

which yields $m = \int_0^\infty u\,V(du) \in (0,\infty]$. Put $W = \sum_{n=0}^\infty V^{*n}$. Applying the Renewal Theorem ([3], XI) we have

$$\lim_{y\to\infty} W([0,y))/y = 1/m\,.$$

Since, by Proposition 13.7 in [1], $\varrho(X,[0,y)) = 1 + W([0,y))$, we have

$$\lim_{y\to\infty}\varrho(X,[0,y))/y = 1/m\,.$$

Consequently, for any $a \in (0,\infty)$ the supremum

$$s = \sup\{\varrho(X,[0,y))/(y\varrho(X,[0,a))) : y \ge a\}$$

is finite. The inequality $\varrho(X,[0,y)) \le \varrho(X,[0,a))(1+sy)$ for all $y \in (0,\infty)$ is obvious. Observe that, by (1.11), the right-hand side of the last inequality is equal to $\varrho(Y,[0,y))$ for any $Y\in\mathrm{Poiss}(\varrho(X,[0,a))^{-1},P_s)$, which completes the proof.

2. Random integral functionals. This section is devoted to the study of the probability distribution of random functionals

$$I(X,f)=\int\limits_0^\infty f(X(t,\omega))\,dt$$

for $f\in\mathcal{F}$. The following simple formulae for the moments $EI^n(X,f)$ will be needed below.

LEMMA 2.1. *For any $f\in\mathcal{F}$ and $n\geq 1$,*

$$EI^n(X,f)=n!\int\limits_0^\infty\ldots\int\limits_0^\infty\prod_{j=1}^n f(y_1+\ldots+y_j)\,\varrho(X,dy_1)\ldots\varrho(X,dy_n)\,.$$

Proof. We start from the formula

$$EI^n(X,f)=E\int\limits_0^\infty\ldots\int\limits_0^\infty\prod_{j=1}^n f(X(t_j,\omega))\,dt_1\ldots dt_n$$
$$=n!E\int\limits_0^\infty\ldots\int\limits_0^\infty\prod_{j=1}^n f(X(u_1+\ldots+u_j,\omega))\,du_1\ldots du_n\,.$$

Since f is nonnegative, we can change the order of integration to get

$$EI^n(X,f)=n!\int\limits_0^\infty\ldots\int\limits_0^\infty E\prod_{j=1}^n f(X(u_1+\ldots+u_j,\omega))\,du_1\ldots du_n\,.$$

Now, by the stationarity and independence of the increments of X, it is easy to check the formula

$$E\prod_{j=1}^n f(X(u_1+\ldots+u_j,\omega))$$
$$=\int\limits_0^\infty\ldots\int\limits_0^\infty\prod_{j=1}^n f(y_1+\ldots+y_j)\,\pi(X,u_1,dy_1)\ldots\pi(X,u_n,dy_n)\,,$$

which together with the previous one yields the assertion of the lemma.

The inequality $f(y_1+\ldots+y_j)\leq f(y_j)$ $(j=1,\ldots,n)$ for $f\in\mathcal{F}$ and Lemma 2.1 imply the following statement.

COROLLARY 2.1. *For every $f\in\mathcal{F}$ and $n\geq 1$,*

$$EI^n(X,f)\leq n!\,(EI(X,f))^n\,.$$

Further, the following result is an immediate consequence of Lemmas 1.2 and 2.1.

COROLLARY 2.2. *If $X\prec Y$, then $EI^n(X,f)\leq EI^n(Y,f)$ for every $f\in\mathcal{F}$ and $n\geq 1$.*

LEMMA 2.2. *If $f\in\mathcal{F}$ and $\int_0^\infty f(y)\,dy<\infty$, then for every process X the expectation $EI(X,f)$ is finite.*

Proof. By Lemma 1.3 and Corollary 2.2 it suffices to prove the assertion for $X\in\mathrm{Poiss}(q,P_s)$. But for these processes we have, by (1.11) and Lemma 2.1,

$$EI(X,f)=\int\limits_0^\infty f(y)\,\varrho(X,dy)=q^{-1}\Big(f(0)+s\int\limits_0^\infty f(y)\,dy\Big)\,,$$

which completes the proof.

Observe that for the deterministic process $X(t,\omega)=t$ we have $I(X,f)=\int_0^\infty f(t)\,dt$. Consequently, Lemma 2.2 yields the following corollary.

COROLLARY 2.3. *Suppose that $f\in\mathcal{F}$. The random functional $I(X,f)$ is finite with probability 1 for every process X if and only if $\int_0^\infty f(y)\,dy<\infty$.*

Denote by $\mathcal{F}_0$ the subset of $\mathcal{F}$ consisting of functions f satisfying $\int_0^\infty f(y)\,dy<\infty$. Given $f\in\mathcal{F}_0$ we denote by $L(X,f,\cdot)$ the Laplace transform of the probability distribution of $I(X,f)$, i.e.

$$L(X,f,z)=Ee^{-zI(X,f)}$$

for $z\in[0,\infty)$. Further, for every $a\in[0,\infty)$ we denote by T_a the shift operator defined by the formula $(T_af)(y)=f(y+a)$ on $\mathcal{F}_0$.

For any $f\in\mathcal{F}_0$ using the representation (1.9)–(1.10) of a process X from $\mathrm{Poiss}(q,Q)$ we have

$$I(X,f)=f(0)\vartheta_0+\sum_{k=1}^\infty f(\xi_1+\ldots+\xi_k)\vartheta_k\,. \tag{2.1}$$

Taking two independent random variables ϑ and ξ with probability distributions P_q and Q respectively such that the pair (ϑ,ξ) is independent of the process X we conclude, by (2.1), that the random variables $I(X,f)$ and $f(0)\vartheta+I(X,T_\xi f)$ are identically distributed. Hence we get the following result.

LEMMA 2.3. *For $f\in\mathcal{F}_0$ and $X\in\mathrm{Poiss}(q,Q)$,*

$$L(X,f,z)=q(q+f(0)z)^{-1}\int\limits_0^\infty L(X,T_yf,z)\,Q(dy)\quad(z\in[0,\infty))\,.$$

LEMMA 2.4. *For $f\in\mathcal{F}_0$ and $X\in\mathrm{Poiss}(q,P_s)$,*

$$L(X,f,z)=q(q+f(0)z)^{-1}\exp\Big(-sz\int\limits_0^\infty\frac{f(u)}{q+f(u)z}\,du\Big)\quad(z\in[0,\infty))\,.$$

Proof. Replacing f by $T_u f$ ($u \in [0,\infty)$) and Q by P_s in Lemma 2.3 we get

$$L(X, T_u f, z) = sq(q + f(u)z)^{-1} \int_0^\infty L(X, T_{u+y} f, z) e^{-sy}\, dy\,. \tag{2.2}$$

Since $T_u f \le f$, we have $\lim_{u\to\infty} I(X, T_u f) = 0$ with probability 1 by the bounded convergence theorem. Consequently,

$$\lim_{u\to\infty} L(X, T_u f, z) = 1 \tag{2.3}$$

for all $z \in [0,\infty)$. Setting $H(u,z) = e^{-su} L(X, T_u f, z)$ for $u, z \in [0,\infty)$, we get, by (2.2) and (2.3), the equation

$$H(u,z) = sq(q + f(u)z)^{-1} \int_u^\infty H(y,z)\, dy \tag{2.4}$$

with the limit condition

$$\lim_{u\to\infty} e^{su} H(u,z) = 1\,. \tag{2.5}$$

The general solution of (2.4) is of the form

$$H(u,z) = G(z) sq(q + f(u)z)^{-1} \exp\Big(-sq \int_0^u \frac{dy}{q + f(y)z}\Big). \tag{2.6}$$

Multiplying both sides by e^{su} and letting $u \to \infty$ we get, by (2.5),

$$1 = G(z) s \exp\Big(sz \int_0^\infty \frac{f(y)}{q + f(y)z}\, dy\Big),$$

which together with (2.6) for $u = 0$ implies the assertion of the lemma.

From Corollary 2.1 and Lemma 2.2 it follows that the function $L(X, f, \cdot)$ can be extended to an analytic function in the circle $|z| EI(X, f) < 1$ with the power series representation

$$L(X, f, z) = \sum_{n=0}^\infty \frac{(-z)^n EI^n(X, f)}{n!}\,.$$

Denote by $r(X, f)$ the convergence radius of the above series, i.e.

$$r(X, f)^{-1} = \lim_{n\to\infty} (EI^n(X, f)/n!)^{1/n}\,. \tag{2.7}$$

Observe that, by Lemma 2.4, for $X \in \mathrm{Poiss}(q, P_s)$ the function $L(X, f, \cdot)$ is analytic in the circle $f(0)|z| < q$. Hence we get the following estimate.

LEMMA 2.5. *If $f \in \mathcal{F}_0$ and $X \in \mathrm{Poiss}(q, P_s)$, then $r(X, f) \ge q/f(0)$.*

As an immediate consequence of Corollary 2.2 and formula (2.7) we get the following statement.

LEMMA 2.6. *If $X \prec Y$, then $r(X, f) \ge r(Y, f)$ for every $f \in \mathcal{F}_0$.*

The following theorem will be useful later.

THEOREM 2.1. *If $f \in \mathcal{F}_0$, then $r(X, f) = (f(0)\varrho(X, \{0\}))^{-1}$ for every process X.*

Proof. For every positive number a there exist, by Lemma 1.3, a positive number s and a process $Y \in \mathrm{Poiss}(\varrho(X, [0, a))^{-1}, P_s)$ such that $X \prec Y$. Applying Lemmas 2.5 and 2.6 we get $r(X, f) \ge (f(0)\, \varrho(X, [0, a)))^{-1}$. Letting $a \to 0$ we have

$$r(X, f) \ge (f(0)\varrho(X, \{0\}))^{-1}\,, \tag{2.8}$$

which completes the proof in the case $\varrho(X, \{0\}) = 0$. In the remaining case X is a compound Poisson process and, by (2.1), $I(X, f) \ge f(0)\vartheta_0$ with probability 1 where the random variable ϑ_0 has probability distribution P_q and, by (1.4) and (1.7), $q^{-1} = \varrho(X, \{0\})$. Thus

$$EI^n(X, f) \ge f^n(0) E\vartheta_0^n = f^n(0) n! \varrho^n(X, \{0\})\,.$$

Hence and from (2.7) we get $r(X, f)^{-1} \ge f(0)\varrho(X, \{0\})$, which together with (2.8) yields the assertion of the theorem.

THEOREM 2.2. *If $I(X, f) \ge a$ with probability 1 for a function f from $\mathcal{F}_0$ and a positive constant a, then the process X is deterministic.*

Proof. Denote by $\mathcal{B}_u$ the σ-field of random events generated by the values $X(t, \omega)$ for $t \le u$. Suppose that the process X satisfies the assumption of the theorem for some $f \in \mathcal{F}_0$ and $a \in (0, \infty)$. We define the random variable τ by setting

$$\tau(\omega) = \sup\Big\{v : \int_0^v f(X(t, \omega))\, dt < \frac{a}{2}\Big\}.$$

It is clear that τ is finite with probability 1,

$$\int_0^{\tau(\omega)} f(X(t, \omega))\, dt = \frac{a}{2} \tag{2.9}$$

with probability 1 and $\{\tau(\omega) \le u\} \in \mathcal{B}_u$ for all $u \in (0, \infty)$, i.e. the random variable τ is a stopping time. Consequently, setting $\eta = X(\tau, \omega)$ and $Y(t, \omega) = X(t + \tau, \omega) - X(\tau, \omega)$ for $t \in [0, \infty)$ we conclude that $X \sim Y$ and the process Y and the random variable η are independent ([4], Chapter I, 4, Theorem 7). By (2.9) we have

$$I(Y, T_\eta f) = \int_{\tau(\omega)}^\infty f(X(t, \omega))\, dt \ge \frac{a}{2}$$

with probability 1. Consequently, denoting by λ the probability distribution of η and taking into account the independence of η and Y we get $I(Y, T_y f) \geq a/2$ with probability 1 for all $y \in \operatorname{supp} \lambda$. Since $\lim_{y\to\infty} I(Y, T_y F) = 0$ with probability 1, we conclude that $\operatorname{supp} \lambda$ is bounded. Thus $X(\tau, \omega) \leq b$ with probability 1 for some constant b. On the other hand, by (2.9), $f(0)\tau \leq a/2$ with probability 1. Put $t_0 = a/(2f(0))$. Since the sample functions of the process X are nondecreasing, we infer that $X(t_0, \omega) \leq b$ with probability 1. Applying Lemma 1.1 we get the assertion of the theorem.

3. f-stable processes. Suppose that to every process X there corresponds a random variable $J(X)$. Denote by $N(X, J)$ the set of probability distributions of the random variables $J(aX)$ with $a \in (0, \infty)$. A process X is said to be *J-stable* whenever $N(X, J)$ is closed under convolution. Observe that for the mapping $J_0(X) = X(t_0, \cdot)$ for some positive number t_0 the J_0-stability coincides with the usual one. The purpose of this section is to study J-stable processes X in the case $J(X) = I(X, f)$ for some $f \in \mathcal{F}_0$. Of course, we exclude the trivial case $f(0) = 0$ for which $I(X, f) = 0$ with probability 1. Denote by $\mathcal{F}_+$ the subset of $\mathcal{F}_0$ consisting of functions taking a positive value at the origin. Let $\lambda(X, f, \cdot)$ denote the probability distribution of $I(X, f)$ and $\Lambda(X, f) = \{\lambda(aX, f, \cdot) : a \in (0, \infty)\}$. Further, for any $a \in (0, \infty)$ we put $(U_a f)(x) = f(ax)$. It is clear that $\mathcal{F}_+$ is invariant under all transformations U_a and

$$\lambda(aX, f, \cdot) = \lambda(X, U_a f, \cdot) . \tag{3.1}$$

We shall say "f-stable" instead of "$I(X, f)$-stable" for short. Thus X is f-stable if and only if $\Lambda(X, f)$ is closed under convolution or, equivalently, the set of Laplace transforms $\{L(X, U_a f, \cdot) : a \in (0, \infty)\}$ is closed under pointwise multiplication.

To begin with, we give some examples of f-stable processes.

Example 3.1. Deterministic processes are f-stable for all $f \in \mathcal{F}_+$. In fact, if $X(t, \omega) = bt$ with probability 1 for some constant $b \in (0, \infty)$, then

$$I(aX, f) = \frac{1}{ab} \int_0^\infty f(y)\, dy \quad \text{for all } a \in (0, \infty)$$

and, consequently, $\Lambda(X, f) = \{\delta_c : c \in (0, \infty)\}$.

Throughout this section $\gamma(c, w, \cdot)$ will denote the gamma probability distribution with positive parameters c and w, i.e. $\gamma(c, w, dy) = \Gamma(w)^{-1} c^w \times y^{w-1} e^{-cy}\, dy$ on the half-line $[0, \infty)$. The Laplace transform of $\gamma(c, w, \cdot)$ is equal to $c^w (c + z)^{-w}$ and

$$\gamma(c, v, \cdot) * \gamma(c, w, \cdot) = \gamma(c, v + w, \cdot) . \tag{3.2}$$

Example 3.2. Let $h(x) = e^{-x}$. The processes from $\operatorname{Poiss}(q, P_s)$ for $q, s \in (0, \infty)$ are h-stable. In fact, by Lemma 2.4,

$$L(X, U_a h, z) = \frac{q}{q+z} \exp\Big(- sz \int_0^\infty \frac{e^{-au}}{q + e^{-au} z}\, du \Big) = \Big(\frac{q}{q+z} \Big)^{1 + a^{-1} s} ,$$

which, by (3.1), yields $\lambda(aX, h, \cdot) = \gamma(q, 1 + a^{-1}s, \cdot)$. Consequently, $\Lambda(X, h) = \{\gamma(q, w, \cdot) : w \in (1, \infty)\}$, which, by (3.2), shows that X is h-stable.

We now establish some properties of the Laplace transform $L(X, U_a f, \cdot)$ for $f \in \mathcal{F}_+$ which will be needed below. Since $U_a f \geq U_b f$ for $a \leq b$, we infer that for every $z \in [0, \infty)$ the mapping $a \to L(X, U_a f, z)$ is nondecreasing and, by the bounded convergence theorem, continuous. Moreover, $\lim_{a\to 0} I(X, U_a f) = \infty$ with probability 1, which yields

$$\lim_{a\to 0} L(X, U_a f, z) = 0 \quad \text{for } z \in (0, \infty) . \tag{3.3}$$

Applying Lemma 2.1 we have $\lim_{a\to\infty} EI(X, U_a f) = f(0)\varrho(X, \{0\})$, which yields $\lim_{a\to\infty} I(X, U_a f) = 0$ with probability 1 in the case $\varrho(X, \{0\}) = 0$. In the remaining case taking the representation (2.1) we get $\lim_{a\to\infty} I(X, U_a f) = f(0)\vartheta_0$ in probability where the random variable ϑ_0 has probability distribution P_q and, by (1.4) and (1.7), $q^{-1} = \varrho(X, \{0\})$. Hence

$$\lim_{a\to\infty} L(X, U_a f, z) = (1 + f(0)\varrho(X, \{0\}) z)^{-1} \tag{3.4}$$

for $z \in [0, \infty)$.

Proposition 3.1. *Suppose that $f \in \mathcal{F}_+$ and the set $\Lambda(X, f)$ consists of gamma distributions. Then X is a compound f-stable Poisson process,*

$$\Lambda(X, f) = \{\gamma(f(0)^{-1} \varrho(X, \{0\})^{-1}, w, \cdot) : w \in (1, \infty)\} \tag{3.5}$$

and $\operatorname{supp} \varrho(X, \cdot) = [0, \infty)$.

Proof. By the assumption for every $a \in (0, \infty)$ there exist positive numbers $c(a)$ and $w(a)$ such that

$$L(X, U_a f, z) = c(a)^{w(a)} (c(a) + z)^{-w(a)} . \tag{3.6}$$

Hence the convergence radius $r(X, f)$ is equal to $c(a)$, which, by Theorem 2.1, yields

$$c(a) = f(0)^{-1} \varrho(X, \{0\})^{-1} . \tag{3.7}$$

This shows that $\varrho(X, \{0\}) > 0$ and, consequently, X is a compound Poisson process. Moreover, $w(\cdot)$ is a nonincreasing continuous function satisfying, by (3.3) and (3.4), the limit conditions

$$\lim_{a\to 0} w(a) = \infty \quad \text{and} \quad \lim_{a\to\infty} w(a) = 1 . \tag{3.8}$$

Further, $X \in \text{Poiss}(q, Q)$ for some q and Q, which, by (1.8), yields

$$\varrho(X, \cdot) = q^{-1} \sum_{n=0}^{\infty} Q^n . \tag{3.9}$$

In particular, $q^{-1} = \varrho(X, \{0\})$. Now applying Lemma 2.3 we get from (3.6) and (3.7) the equation

$$(1 + f(0)\varrho(X, \{0\})z)^{1-w(a)} = \int_0^{\infty} L(X, T_y U_a f, z)\, Q(dy) \tag{3.10}$$

for $z \in [0, \infty)$. Observe that the convergence radius of the right-hand side is at least $\inf\{r(X, T_y U_a f) : y \in \operatorname{supp} Q\}$. By Theorem 2.1 this infimum is $f(ay_0)^{-1}\varrho(X, \{0\})^{-1}$ where $y_0 = \min \operatorname{supp} Q$. If $w(a) > 1$, then the convergence radius of the left-hand side of (3.10) is $f(0)^{-1}\varrho(X, \{0\})^{-1}$. Consequently, $f(ay_0) \geq f(0)$. Since f is nonincreasing, we have

$$f(0) = f(ay_0) \quad \text{for } w(a) > 1 . \tag{3.11}$$

By (3.6)–(3.8) in order to prove (3.5) it suffices to show that $w(a) > 1$ for all $a \in (0, \infty)$. Suppose the contrary and put $a_0 = \inf\{a : w(a) = 1\}$. Since $w(\cdot)$ is nonincreasing, we have, by (3.8), $w(b) = 1$ for $b > a_0$. Hence and from (3.10) it follows that

$$\int_0^{\infty} L(X, T_y U_b f, z) Q(dy) = 1 \quad \text{for } b > a_0 .$$

Since the integrand is not greater than 1, we conclude that $L(X, T_y U_b f, z) = 1$ for all $y \in \operatorname{supp} Q$, $b > a_0$ and $z \in [0, \infty)$ and, consequently, $r(X, T_y U_b f) = \infty$. On the other hand, by Theorem 2.1, $r(X, T_y U_b f) = f(by)^{-1}\varrho(X, \{0\})^{-1}$, which implies that $f(by) = 0$ for $y \in \operatorname{supp} Q$ and $b > a_0$. By the continuity of f we get $f(a_0 y_0) = 0$. On the other hand, by (3.11), $f(a_0 y_0) = f(0)$, which contradicts the assumption $f \in \mathcal{F}_+$. The inequality $w(a) > 1$ for all $a \in (0, \infty)$ is thus proved.

Of course, from (3.5) it follows immediately that the process X is f-stable. Moreover, by (3.11), $y_0 = 0$ because functions from $\mathcal{F}_+$ are not constant. Since $Q(\{0\}) = 0$, we infer that $\operatorname{supp} Q$ contains arbitrarily small positive numbers. Consequently, the closed additive semigroup containing $\operatorname{supp} Q$ coincides with $[0, \infty)$. Hence and from (3.9) we get $\operatorname{supp} \varrho(X, \{0\}) = [0, \infty)$, which completes the proof.

Let $\mathcal{K}$ be the space of all real-valued continuous functions defined on the open half-line $(0, \infty)$ with the topology of uniform convergence on every compact subset of $(0, \infty)$. It is clear that $\mathcal{K}$ is metrizable. In what follows for the sake of brevity we shall use the notation

$$\begin{gathered} \Phi(0, z) = 0, \quad \Phi(a, z) = L(X, U_a f, z) , \\ \Phi(\infty, z) = (1 + f(0)\varrho(X, \{0\}))^{-1} \end{gathered} \tag{3.12}$$

where $a, z \in (0, \infty)$. It is easy to verify, by (3.3) and (3.4), that the mapping $\varphi(a) = \Phi(a, \cdot)$ from the compactified half-line $[0, \infty]$ into $\mathcal{K}$ is continuous. Put $\mathcal{K}_\Phi = \varphi([0, \infty])$. It is clear that $\mathcal{K}_\Phi$ is compact and connected, i.e. $\mathcal{K}_\Phi$ is a continuum.

LEMMA 3.1. *For every continuum C contained in $\mathcal{K}_\Phi$ the inverse image $\varphi^{-1}(C)$ is also a continuum.*

Proof. Since $[0, \infty]$ is compact, we infer, by Theorem 3.1.12 in [2], that the mapping φ is closed. Consequently, the fibres $\varphi^{-1}(\Phi(a, \cdot))$ are compact. Put $a^- = \min \varphi^{-1}(\Phi(a, \cdot))$ and $a^+ = \max \varphi^{-1}(\Phi(a, \cdot))$. Suppose that $b \in [a^-, a^+]$. Since the function $a \to \Phi(a, \cdot)$ is nondecreasing, we have

$$\Phi(a, \cdot) = \Phi(a^-, \cdot) \leq \Phi(b, \cdot) \leq \Phi(a^+, \cdot) = \Phi(a, \cdot) ,$$

which yields $\Phi(b, \cdot) = \Phi(a, \cdot)$ and, consequently, $b \in \varphi^{-1}(\Phi(a, \cdot))$. Thus $\varphi^{-1}(\Phi(a, \cdot)) = [a^-, a^+]$, which shows that all fibres $\varphi^{-1}(\Phi(a, \cdot))$ are connected. Now our assertion is a direct consequence of Theorem 6.1.28 in [2].

A set M of probability measures is said to be *closed under convolution powers* if $\mu^{*n} \in M$ for every $n \geq 2$ whenever $\mu \in M$.

PROPOSITION 3.2. *Suppose that $f \in \mathcal{F}_+$ and $\Lambda(X, f)$ is closed under convolution powers. Then either $\Lambda(X, f)$ consists of gamma distributions or the process X is deterministic.*

Proof. Given $n \geq 2$ we put for $a \in [0, \infty]$

$$\varphi_n(a) = \varphi(a)^n = \Phi(a, \cdot)^n . \tag{3.13}$$

Obviously φ_n is a continuous nondecreasing mapping from $[0, \infty]$ into $\mathcal{K}$. Since $\Lambda(X, f)$ is closed under convolution powers, we have $\varphi_n((0, \infty)) \subset \mathcal{K}_\Phi$, which, by the compactness of $\mathcal{K}_\Phi$, yields

$$\varphi_n([0, \infty]) \subset \mathcal{K}_\Phi . \tag{3.14}$$

Moreover, $\varphi_n([0, \infty])$ is a continuum and, by Lemma 3.1, the inverse image $\varphi^{-1}(\varphi_n([0, \infty]))$ is also a continuum. Since, by (3.12), $\varphi_n(0) = 0 = \varphi(0)$, we conclude that

$$\varphi^{-1}(\varphi_n([0, \infty])) = [0, v_n] \tag{3.15}$$

where $v_n \in (0, \infty]$. As $\varphi_n(\cdot)$ is nondecreasing, we have $\varphi(v_n) = \varphi_n(\infty)$, which, by (3.12), yields

$$\Phi(v_n, z) = (1 + f(0)\varrho(X, \{0\})z)^{-n} . \tag{3.16}$$

Hence

$$v_n = \infty \quad \text{if } \varrho(X,\{0\}) = 0\,, \tag{3.17}$$

$$v_n < \infty \quad \text{if } \varrho(X,\{0\}) > 0\,. \tag{3.18}$$

First consider the case $\varrho(X,\{0\}) = 0$. By (3.15) and (3.17) there exists a number $b_n \in (0,\infty)$ such that $\varphi_n(b_n) = \varphi(1)$. Thus

$$\Phi(b_n, z) = \Phi(1,z)^{1/n} \quad (n = 2,3,\ldots)$$

for $z \in [0,\infty)$. Differentiating both sides with respect to z and taking into account the formulae

$$\frac{d}{dz}\Phi(a,z)\Big|_{z=0} = -EI(X, U_a f), \qquad \frac{d^2}{dx^2}\Phi(a,z)\Big|_{z=0} = EI^2(X, U_a f)$$

we get

$$EI(X, U_{b_n} f) = \frac{1}{n} EI(X,f)\,,$$

$$EI^2(X, U_{b_n} f) = \frac{1}{n} EI^2(X,f) + \frac{1}{n}\left(\frac{1}{n} - 1\right)(EI(X,f))^2\,.$$

Applying Corollary 2.1 we get

$$EI^2(X, U_{b_n} f) \le 2(EI(X, U_{b_n} f))^2 = \frac{2}{n^2}(EI(X,f))^2\,.$$

Consequently,

$$EI^2(X,f) + \left(\frac{1}{n} - 1\right)(EI(X,f))^2 \le \frac{2}{n}(EI(X,f))^2\,.$$

Letting $n \to \infty$ we obtain $EI^2(X,f) - (EI(X,f))^2 \le 0$, which shows that the variance of the random variable $I(X,f)$ is 0. Thus $I(X,f) = EI(X,f)$ with probability 1. Since $f(0) > 0$, we infer that $EI(X,f) > 0$. Now applying Theorem 2.2 we conclude that X is deterministic.

Now consider the case $\varrho(X,\{0\}) > 0$. By (3.18) we have $v_n < \infty$. Moreover, by (3.16), $v_{k+n} \le v_n$ for $k \ge 1$. Consequently, by (3.15), there exists a positive number $c_{k,n}$ such that $\Phi(v_{k+n},\cdot) = \Phi(c_{k,n},\cdot)^n$, which, by (3.16), yields

$$\Phi(c_{k,n}, z) = (1 + f(0)\varrho(X,\{0\})z)^{-1-k/n} \tag{3.19}$$

for $n = 2,3,\ldots$, $k = 1,2,\ldots$ and $z \in (0,\infty)$. Setting

$$\psi(a) = (1 + f(0)\varrho(X,\{0\})z)^{-1-1/a}$$

for $a \in [0,\infty]$ and $z \in (0,\infty)$ we get a continuous mapping from $[0,\infty]$ into $\mathcal{K}$. Since, by (3.19), $\psi(k/n) = \Phi(c_{k,n},\cdot) \in \mathcal{K}_\Phi$ we conclude, by the compactness of $\mathcal{K}_\Phi$, that $\psi([0,\infty]) \subset \mathcal{K}_\Phi$. Since $\psi(0) = 0 = \varphi(0)$ and $\psi(\infty) = \varphi(\infty)$, we have, by Lemma 3.1, $\varphi^{-1}(\psi([0,\infty])) = [0,\infty]$. Consequently, for any $a \in (0,\infty)$ there exists a number $b \in (0,\infty]$ such that

$$\Phi(a,z) = (1 + f(0)\varrho(X,\{0\})z)^{-1-1/b}\,,$$

which shows that the random variable $I(aX,f)$ has a gamma distribution. The proposition is thus proved.

As an immediate consequence of Propositions 3.1 and 3.2 we get the following result.

THEOREM 3.1. *Let $f \in \mathcal{F}_+$. The following statements are equivalent*:

(i) *the process X is f-stable*,

(ii) *the set $\Lambda(X,f)$ is closed under convolution powers*,

(iii) *either $\Lambda(X,f)$ consists of gamma distributions or X is deterministic.*

THEOREM 3.2. *Suppose that $f \in \mathcal{F}_+$ and $f(c) = 0$ for some $c \in (0,\infty)$. Then f-stable processes are deterministic.*

Proof. Suppose that X is a nondeterministic f-stable process. Then, by Theorem 3.1 and Proposition 3.1, $\varrho(X,\{0\}) > 0$ and

$$\Lambda(X,f) = \{\gamma(f(0)^{-1}\varrho(X,\{0\})^{-1}, w, \cdot) : w \in (1,\infty)\}\,.$$

Consequently, for every $a \in (0,\infty)$ there exists a number $w(a) \in (1,\infty)$ such that

$$L(X, U_a f, z) = (1 + f(0)\varrho(X,\{0\})z)^{-w(a)}\,. \tag{3.20}$$

Since f is nonincreasing, we have $f(u) = 0$ for $u \ge c$. Hence

$$L(X, T_y U_a f, z) = 1 \quad \text{for } y \in [a^{-1}c,\infty) \text{ and } z \in [0,\infty)\,. \tag{3.21}$$

We may assume that $X \in \mathrm{Poiss}(q,Q)$ for some Q and $q^{-1} = \varrho(X,\{0\})$. Using Lemma 2.3 and formula (3.20) we obtain

$$(1 + f(0)\varrho(X,\{0\})z)^{1-w(a)} = \int_0^\infty L(X, T_y U_a f, z)\,Q(dy)$$

for $z \in [0,\infty)$, which, by (3.21), yields

$$(1 + f(0)\varrho(X,\{0\})z)^{1-w(a)} \ge Q([a^{-1}c,\infty))\,.$$

Letting $z \to \infty$ we get $Q([a^{-1}c,\infty)) = 0$, which, by the arbitrariness of a, implies $Q((0,\infty)) = 0$. But this is impossible because Q is a probability measure concentrated on the half-line $(0,\infty)$. The theorem is thus proved.

In what follows $h(x)$ will denote the exponential function e^{-x}. Obviously, $h \in \mathcal{F}_+$. We shall give a complete description of all h-stable processes. We begin with a uniqueness lemma.

LEMMA 3.2. *If the random variables $I(X,h)$ and $I(Y,h)$ are identically distributed, then $X \sim Y$.*

Proof. By Lemma 2.1 we have

$$EI^n(X,h) = n!\prod_{j=1}^{n}\int_0^\infty e^{-ju}\,\varrho(X,du)\quad (n=1,2,\ldots).$$

Hence for $I(X,h)$ and $I(Y,h)$ identically distributed,

$$\int_0^\infty e^{-nu}\,\varrho(X,du) = \int_0^\infty e^{-nu}\,\varrho(Y,du)\quad (n=1,2,\ldots),$$

which yields $\varrho(X,\cdot)=\varrho(Y,\cdot)$ ([3], Chapter XIII, 1). Thus $X\sim Y$.

THEOREM 3.3. *A process X is h-stable if and only if either X is deterministic or $X\in \mathrm{Poiss}(q,P_s)$ for some positive q and s.*

Proof. The sufficiency follows from Examples 3.1 and 3.2. To prove the necessity suppose that X is a nondeterministic h-stable process. By Theorem 3.1 and Proposition 3.1 the random variable $I(X,h)$ has gamma distribution with parameters $(\varrho(X,\{0\})^{-1},w)$ for some $w>1$. Put $q=\varrho(X,\{0\})^{-1}$ and $s=w-1$. It was shown in Example 3.2 that for $Y\in \mathrm{Poiss}(q,P_s)$ the random variable $I(Y,h)$ has gamma distribution with parameters $(\varrho(X,\{0\})^{-1},w)$. Hence and from Lemma 3.2 it follows that $X\sim Y$, which yields $X\in\mathrm{Poiss}(q,P_s)$. This completes the proof.

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Maximal functions related to subelliptic operators invariant under an action of a nilpotent Lie group

by

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Abstract. On the domain $\Omega_a=\{(x,b): x\in N, b\in\mathbb{R}^+, b>a\}$, where N is a simply connected nilpotent Lie group and $a\ge 0$, certain N-invariant second order subelliptic operators L are considered. Every bounded L-harmonic function F is the Poisson integral

$$F(x,b)=f*\check{\mu}_a^b(x)$$

for an $f\in L^\infty(N)$. The main theorem of the paper asserts that under some assumptions the maximal functions

$$M_1f(x)=\sup_{b\ge a+1}|f*\check{\mu}_a^b(x)|,\qquad M_2f(x)=\sup_{a<b\le a+1}|f*\check{\mu}_a^b(x)|$$

are of weak type $(1,1)$. Some results about moments of the harmonic measures $\check{\mu}_a^b$ are also included.

1. Introduction. The aim of this paper is to study some maximal functions naturally associated with differential operators invariant under an action of a nilpotent Lie group N and defined on $N\times\mathbb{R}^+$. Suppose that for every $a\in\mathbb{R}^+$ we have left-invariant vector fields $Y_1(a),\ldots,Y_k(a)$, $Y(a)$, depending smoothly on a, such that $Y_1(a),\ldots,Y_k(a)$ generate $\mathfrak{n}$ as a Lie algebra and for every a, $Y_1(a),\ldots,Y_k(a)$ belong to the same linear subspace $\mathfrak{v}$ of $\mathfrak{n}$. We consider the operator

$$Lf(x,a)=\Big(\sum_{i=1}^{k}Y_i(a)^2+Y(a)+\alpha\partial_a^2-\kappa\partial_a\Big)f(x,a)\tag{1.1}$$

on the domain

$$\Omega_{a_0}=\{(x,a): x\in N, a>a_0\},\qquad a_0\ge 0,$$

and so we go a step further than in [DH], where operators invariant with respect to a solvable group structure on $N\times\mathbb{R}^+$ have been considered.

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