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Hausdorff and conformal measures for expanding piecewise monotonic maps of the interval

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Abstract. Let A be a topologically transitive invariant subset of an expanding piecewise monotonic map on $[0,1]$ with the Darboux property. We investigate existence and uniqueness of conformal measures on A and relate Hausdorff and conformal measures on A to each other.

1. Introduction. A map $T:[0,1]\to[0,1]$ is called *piecewise monotonic* if there is a finite subset $\{a_0,a_1,\ldots,a_N\}$ of $[0,1]$ with $0=a_0<a_1<\ldots<a_N=1$ such that $T|(a_{i-1},a_i)$ is strictly monotone and continuous for $1\le i\le N$. The aim of this paper is to compare Hausdorff measures and conformal measures on T-invariant subsets of $[0,1]$. These questions are motivated by similar investigations for Julia sets of rational maps on the Riemannian sphere (cf. [2], [3]).

Throughout the paper T will denote a piecewise monotonic map on $[0,1]$ and $\mathcal{Z}$ will denote the family $\{(0,a_1),(a_1,a_2),\ldots,(a_{N-1},1)\}$ of intervals on which T is continuous and monotone. Furthermore, we always assume that the derivative of T exists on (a_{i-1},a_i) and can be extended to a continuous function on $[a_{i-1},a_i]$ for $1\le i\le N$. This implies that T has bounded derivative.

We shall investigate measures which have no atoms. Hence it does not matter to neglect countable sets. In order to avoid discontinuities of T, we disregard the partition points $a_0,a_1,\ldots,a_N$ and their inverse images. To this end set $R_1=\bigcup_{Z\in\mathcal{Z}}Z$, and for $k\ge2$ set $R_k=\bigcap_{l=0}^{k-1}T^{-l}(R_1)$, which is the set on which $(T^k)'$ is defined. Finally, set

$$R_{\mathcal{Z}}=\bigcap_{i=0}^{\infty}R_i=\bigcap_{l=0}^{\infty}T^{-l}(R_1)\,.$$

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We consider a closed T-invariant subset A of $R_{\mathcal{Z}}$ (closed with respect to the relative topology of $R_{\mathcal{Z}}$). For $t>0$ we want to investigate the absolute continuity of the t-dimensional Hausdorff measure and of a t-conformal measure on A with respect to each other.

We recall the definition of these measures. For $B\subset[0,1]$ let $\mathcal{C}(B,\delta)$ be the set of all finite or countable covers of B by intervals of length less than δ. Then

$$\nu(B):=\lim_{\delta\to 0}\inf_{\mathcal{U}\in\mathcal{C}(B,\delta)}\sum_{U\in\mathcal{U}}|U|^t$$

is called the *t-dimensional Hausdorff measure* of B, where $|U|$ denotes the length of the interval U. For fixed t, it defines a measure on the Borel sets of $[0,1]$. The *Hausdorff dimension* $\mathrm{HD}(B)$ of B is defined as $\inf\{t>0:\nu(B)=0\}$, which equals $\sup\{t>0:\nu(B)=\infty\}$ (cf. [4]). A Borel probability measure m concentrated on A is said to be a *t-conformal measure* on A if

$$m(T(Y))=\int_Y|T'|^t\,dm\quad\text{ for all } Y\subset A \text{ contained in some } Z\in\mathcal{Z}. \tag{1.1}$$

The map T is said to be *expanding* if there is a $k\ge 1$ such that $\inf_{R_k}|(T^k)'|>1$, and *piecewise Hölder differentiable* if the derivative of $T|Z$ is Hölder continuous for all $Z\in\mathcal{Z}$. The set A is *topologically transitive* if it contains a dense orbit, and A has the *Darboux property* if

$$T(Z\cap A)=T(Z)\cap A\quad\text{ for all } Z\in\mathcal{Z}. \tag{1.2}$$

In [7] the nonwandering set of a piecewise monotonic transformation is decomposed into topologically transitive components. The components in this decomposition, which have positive entropy, are closed invariant topologically transitive subsets which have the Darboux property. Hence, when restricted to $R_{\mathcal{Z}}$, they can serve as examples of the sets A investigated in this paper.

Finally, we say that T satisfies the *Misiurewicz condition* if the set

$$\bigcup_{i=1}^{N}\{\lim_{x\uparrow a_i}T^j(x):j\ge 1\}\cup\bigcup_{i=0}^{N-1}\{\lim_{x\downarrow a_i}T^j(x):j\ge 1\}$$

has empty intersection with $\bigcup_{i=0}^{N-1}(a_i,a_i+\varepsilon)\cup\bigcup_{i=1}^{N}(a_i-\varepsilon,a_i)$ for some $\varepsilon>0$.

We begin in Section 2 with some preparatory lemmas. In Section 3 we investigate t-conformal measures on A for $t>0$. For an expanding piecewise monotonic map T which is piecewise Hölder differentiable, and for a closed T-invariant topologically transitive subset A of $R_{\mathcal{Z}}$ which has the Darboux property, we show that there is a unique t-conformal measure on A if $t=\mathrm{HD}(A)>0$, and that there is no t-conformal measure on A if $t\ne\mathrm{HD}(A)$ and $t>0$. The unique $\mathrm{HD}(A)$-conformal measure is ergodic, has no atoms and is positive on open subsets of A.

In Section 4 we consider the t-dimensional Hausdorff measure ν on A for $t=\mathrm{HD}(A)>0$. For an expanding piecewise monotonic map T which is piecewise Hölder differentiable, and for a closed T-invariant topologically transitive subset A of $R_{\mathcal{Z}}$ which has the Darboux property, we show that $\nu=cm$ for some $c\in[0,\infty)$, where m is the unique $\mathrm{HD}(A)$-conformal measure on A. If T additionally satisfies the Misiurewicz condition, then $c\in(0,\infty)$.

2. Volume lemmas.

Set

$$\mathcal{Z}_n=\bigvee_{i=0}^{n-1}T^{-i}\mathcal{Z}=\Big\{\bigcap_{i=0}^{n-1}T^{-i}(Z_i)\ne\emptyset:Z_i\in\mathcal{Z}\Big\},$$

a family of open intervals, on each of which T^n is continuous and monotone. The aim of this section is to estimate the length and the measure of the intervals in $\mathcal{Z}_n$. We have $R_n=\bigcup_{Z\in\mathcal{Z}_n}Z$.

LEMMA 1. *Let T be a piecewise monotonic map and let A be a closed invariant subset of $R_{\mathcal{Z}}$. Let m be a t-conformal measure on A. Then*

(i) *$\int_{T(Y)}g\,dm=\int_Y|T'|^t g\circ T\,dm$ if $Y\subset A\cap Z$ for some $Z\in\mathcal{Z}$ and if g is a nonnegative measurable function,*

(ii) *$m(T^n(Y))=\int_Y|(T^n)'|^t\,dm$ for $n\ge 1$ if $Y\subset A\cap Z$ for some $Z\in\mathcal{Z}_n$.*

Proof. (i) follows from (1.1) first for step functions g and then for all $g\ge 0$.

(ii) follows by induction, using (i) and the chain rule in the induction step and observing that $T(Y)$ is a subset of an element of $\mathcal{Z}_{k-1}$ if Y is a subset of some element of $\mathcal{Z}_k$. ∎

Recall that the length of an interval I is denoted by $|I|$.

LEMMA 2. *Let T be an expanding piecewise monotonic map and let A be a closed invariant subset of $R_{\mathcal{Z}}$. Let m be a t-conformal measure on A. Then there are constants $c>0$ and $\beta>1$ such that*

(i) *$\inf_{R_n}|(T^n)'|\ge c\beta^n$ for $n\ge 1$,*

(ii) *$\sup_{Z\in\mathcal{Z}_n}|Z|<(1/c)\beta^{-n}$ for $n\ge 1$,*

(iii) *$\sup_{Z\in\mathcal{Z}_n}m(Z)<(1/c^t)\beta^{-nt}$ for $n\ge 1$.*

Proof. (i) Since T is expanding, there is a k such that $\beta:=(\inf_{R_k}|(T^k)'|)^{1/k}>1$. By the chain rule we get $\inf_{R_{lk}}|(T^{lk})'|\ge\beta^{lk}$ for all $l\ge 1$. Set $d=\inf_{R_1}|T'|$. If $d=0$ then $\inf_{R_k}|(T^k)'|=0$ by the chain rule, as T' is bounded, a contradiction. Hence $d>0$. Set $c=\min\{(d/\beta)^i:0\le i<k\}$. The chain rule then gives the desired result.

(ii) If $Z \in \mathcal{Z}_n$, then $|Z| \le |T^n(Z)| \sup_Z(1/|(T^n)'|)$ by the mean value theorem. Hence the desired result follows from (i) and $|T^n(Z)| \le 1$.

(iii) follows in the same way as (ii) using Lemma 1(ii), since $m(Z) = m(Z \cap A)$. ∎

LEMMA 3. *Let T be an expanding piecewise monotonic map. Suppose that T is piecewise Hölder differentiable. Then there is a constant $d > 0$ such that*

$$e^{-d} \le |(T^n)'(x)/(T^n)'(y)| \le e^d$$

for all $n \ge 1$ if x and y are in the same element of $\mathcal{Z}_n$.

Proof. We get $\inf_{R_1} |T'| > 0$ from Lemma 2(i). Therefore $\varphi := -\log|T'|$ is defined. As T is piecewise Hölder differentiable, $\varphi|Z$ is Hölder continuous for all $Z \in \mathcal{Z}$, which means that there are an $\alpha \in \mathbb{R}$ and an $\varepsilon > 0$ such that $|\varphi(x) - \varphi(y)| \le \alpha|x-y|^\varepsilon$ if x and y are in the same element of $\mathcal{Z}$. If now $Z \in \mathcal{Z}_n$ and $x, y \in Z$ then $T^i(x)$ and $T^i(y)$ are in the same element of $\mathcal{Z}_{n-i}$ for $0 \le i \le n-1$. Hence

$$\left| \log \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \right| \le \sum_{i=0}^{n-1} |\varphi(T^i(x)) - \varphi(T^i(y))| \le \sum_{i=0}^{n-1} \alpha |T^i(x) - T^i(y)|^\varepsilon ,$$

which is bounded by $\sum_{i=0}^{n-1} \alpha c^{-\varepsilon} \beta^{-\varepsilon(n-i)}$ from Lemma 2(ii). As $\beta > 1$, the desired result follows with $d = \alpha c^{-\varepsilon} \sum_{j=1}^{\infty} \beta^{-\varepsilon j}$. ∎

LEMMA 4. *Let T be a piecewise monotonic map and let A be a closed invariant subset of $R_{\mathcal{Z}}$ with the Darboux property. If V is a subinterval of some element of $\mathcal{Z}_n$ then $T^n(V \cap A) = T^n(V) \cap A$.*

Proof. For $n = 1$ this follows from (1.2), since $T|Z$ is strictly monotone for all $Z \in \mathcal{Z}$. We proceed by induction. Suppose that the assertion is shown for $n \le k$. Let V be a subinterval of some element of $\mathcal{Z}_{k+1}$. Then $T(V)$ is a subinterval of some element of $\mathcal{Z}_k$. The induction hypothesis implies that $T^k(T(V) \cap A) = T^k(T(V)) \cap A$ and that $T(V) \cap A = T(V \cap A)$. This gives $T^{k+1}(V \cap A) = T^{k+1}(V) \cap A$, finishing the proof. ∎

Now we introduce the main tool of this section, the so-called *Markov diagram* of a piecewise monotonic transformation T (cf. [7]). Let C be an open subinterval of some element of $\mathcal{Z}$. We say that D is a *successor* of C if $D \ne \emptyset$ and if $D = T(C) \cap Z$ for some $Z \in \mathcal{Z}$. Remark that D is an open subinterval of $Z \in \mathcal{Z}$, since $T|C$ is strictly monotone. We write $C \to D$ if D is a successor of C. Set $\mathcal{D}_0 = \mathcal{Z}$. If $\mathcal{D}_i$ is defined, set $\mathcal{D}_{i+1} = \mathcal{D}_i \cup \{D :$ there is a $C \in \mathcal{D}_i$ with $C \to D\}$. These sets are finite, since $\mathcal{Z}$ is finite and since the number of successors of an interval C is bounded by the number of elements of $\mathcal{Z}$. Finally, set $\mathcal{D} = \bigcup_{i=0}^{\infty} \mathcal{D}_i$, which consists of subintervals of elements of $\mathcal{Z}$. We thus get a finite or countable oriented graph $(\mathcal{D}, \to)$.

Remember that $\mathcal{Z}_n$ is a partition of R_n into intervals. For each $x \in R_n$ let $Z_n(x)$ be the unique element of $\mathcal{Z}_n$ which contains x. Since $\mathcal{Z}$ is a partition of R_1, for $C \in \mathcal{D}$, we see that $T(C) \cap R_1$ is the disjoint union of the successors of C. Hence, if $y \in C \cap R_2$, there is a unique $D \in \mathcal{D}$ with $C \to D$ and $T(y) \in D$. Furthermore, each $x \in R_1$ is contained in exactly one $Z \in \mathcal{Z} = \mathcal{D}_0$, namely in $Z_1(x)$. If now $x \in R_{\mathcal{Z}}$, then $T^i(x) \in R_{\mathcal{Z}}$ for all i and there is a unique path $D_0(x)D_1(x)D_2(x)\ldots$ in $(\mathcal{D}, \to)$ with $D_0(x) = Z_1(x)$ and $T^i(x) \in D_i(x)$ for $i \ge 0$.

For $x \in R_{\mathcal{Z}}$ we show by induction that

$$D_k(x) = T^k(Z_{k+1}(x)) \quad \text{for } k \ge 0 . \tag{2.1}$$

For $k = 0$ this is the definition of $D_0(x)$. Suppose that (2.1) is shown for $k = n-1$. We have $Z_{n+1}(x) = Z_n(x) \cap T^{-n}(Z_1(T^n(x)))$, because the right hand side is an element of $\mathcal{Z}_{n+1}$ and contains x. Hence $T^n(Z_{n+1}(x)) = T^n(Z_n(x)) \cap Z_1(T^n(x)) = T(D_{n-1}(x)) \cap Z_1(T^n(x))$. Since $T(D_{n-1}(x)) \cap Z_1(T^n(x))$ is a successor of $D_{n-1}(x)$ which contains $T^n(x)$, it equals $D_n(x)$.

We show first that certain exceptional sets are small.

LEMMA 5. *Let T be an expanding piecewise monotonic map and let A be a closed invariant subset of $R_{\mathcal{Z}}$. Define $L_j = \{x \in R_{\mathcal{Z}} : D_r(x) \in \mathcal{D}_j$ only for finitely many $r\}$. Let $\beta > 1$ be as in Lemma 2. If $t > (\log 2)/(j \log \beta)$, then*

(i) *$\nu(L_j) = 0$, where ν is the t-dimensional Hausdorff measure,*

(ii) *$m(L_j) = 0$ for any t-conformal measure m on A.*

Proof. For $l \ge 1$ and $E \in \mathcal{D}$ let $M_{l,E}$ be the set of all $x \in R_{\mathcal{Z}}$ which satisfy $D_l(x) = E$ and $D_r(x) \notin \mathcal{D}_j$ for $r > l$. Then $L_j = \bigcup_{l \ge 1} \bigcup_{E \in \mathcal{D}} M_{l,E}$. As $\mathcal{D}$ is countable, it suffices to show that

$$\nu(M_{l,E}) = 0 \quad \text{and} \quad m(M_{l,E}) = 0 \tag{2.2}$$

for all $l \ge 1$ and all $E \in \mathcal{D}$. For $n > l$ let $\mathcal{P}_n$ be the set of all paths $D_0 D_1 \ldots D_{n-1}$ of length n in $(\mathcal{D}, \to)$ with $D_0 \in \mathcal{D}_0 = \mathcal{Z}$, $D_l = E$ and $D_k \notin \mathcal{D}_j$ for $k > l$. Set

$$\mathcal{X}_n = \Big\{ \bigcap_{k=0}^{n-1} T^{-k}(D_k) : D_0 D_1 \ldots D_{n-1} \in \mathcal{P}_n \Big\} .$$

It follows from the definition of $M_{l,E}$ that $D_0(x)D_1(x)\ldots D_{n-1}(x) \in \mathcal{P}_n$ for each $n > l$ if $x \in M_{l,E}$. Since also $x \in \bigcap_{k=0}^{n-1} T^{-k}(D_k(x))$, we deduce that $\mathcal{X}_n$ covers $M_{l,E}$ for each $n > l$.

The graph $(\mathcal{D}, \to)$ is investigated in [7]. Lemma 9 of [7] says that each $D \in \mathcal{D} \setminus \mathcal{D}_j$ has at most two successors in $\mathcal{D} \setminus \mathcal{D}_j$, and if it has exactly two, then at each of these there starts only one path of length j which is in $\mathcal{D} \setminus \mathcal{D}_j$. This implies that the number of paths in $(\mathcal{D} \setminus \mathcal{D}_j, \to)$ of length

$n-l$ starting at E is bounded by $2^{((n-l)/j)+1}$. Furthermore, the number of successors of any $D \in \mathcal{D}$ is bounded by card $\mathcal{Z}$. This implies that

$$\text{card}\,\mathcal{X}_n \le \text{card}\,\mathcal{P}_n \le (\text{card}\,\mathcal{Z})^l 2^{((n-l)/j)+1}. \tag{2.3}$$

As each element of $\mathcal{D}$ is contained in an element of $\mathcal{Z}$, we find that each element of $\mathcal{X}_n$ is contained in an element of $\mathcal{Z}_n$. Let $\mathcal{U}_n$ be the set of all $Z \in \mathcal{Z}_n$ which contain an element of $\mathcal{X}_n$. Then $\mathcal{U}_n$ is a cover of $M_{l,E}$ by intervals satisfying card $\mathcal{U}_n \le$ card $\mathcal{X}_n$. Now (2.3) and Lemma 2(ii) imply

$$\sum_{Z \in \mathcal{U}_n} |Z|^t \le (\text{card}\,\mathcal{Z})^l 2^{((n-l)/j)+1} c^{-t} \beta^{-tn}.$$

The right hand side tends to zero as $n \to \infty$, since $(\log 2)/(j \log \beta) < t$. Also $\lim_{n\to\infty} \max_{Z\in\mathcal{U}_n} |Z| = 0$ by Lemma 2(ii). The definition of Hausdorff measure implies that $\nu(M_{l,E}) = 0$, which is the first part of (2.2).

Similarly, using Lemma 2(iii) we get

$$m(M_{l,E}) \le \sum_{Z\in\mathcal{U}_n} m(Z) \le \sum_{Z\in\mathcal{U}_n} c^{-t}\beta^{-nt} \le (\text{card}\,\mathcal{Z})^l 2^{((n-l)/j)+1} c^{-t}\beta^{-tn}.$$

Again the right hand side tends to zero as $n \to \infty$, and the second part of (2.2) follows. ■

Now we can show the main lemma. For $x \in R_{\mathcal{Z}}$ and $n \ge 1$ remember that $Z_n(x)$ is the unique element of $\mathcal{Z}_n$ which contains x.

LEMMA 6. *Let T be an expanding piecewise monotonic map which is piecewise Hölder differentiable, and let A be a closed invariant subset of $R_{\mathcal{Z}}$ with the Darboux property. For every $t > 0$ there is a set $L \subset A$ satisfying the following*:

(i) $\nu(L) = 0$ *for the t-dimensional Hausdorff measure ν*,

(ii) $m(L) = 0$ *for any t-conformal measure m on A*,

(iii) *for each t-conformal measure m on A we have*

$$\inf_{x\in A}\inf_{k\ge 2} |(T^{k-1})'(x)|^{-t}/m(Z_k(x)) > 0,$$

(iv) *for each $x \in A \setminus L$ there is an infinite subset I of $\mathbb{N}$, depending on x, such that*

$$\sup_{x\in A\setminus L}\sup_{k\in I} |(T^{k-1})'(x)|^{-t}/m(Z_k(x)) < \infty$$

for each t-conformal measure m on A which is positive on open subsets of A.

Proof. Fix j such that $(\log 2)/(j\log\beta) < t$, where $\beta > 1$ is as in Lemma 2. Let L_j be as in Lemma 5 and set $L = L_j \cap A$. By the choice of j, (i) and (ii) follow from Lemma 5.

We have included (iii) in this lemma, although it does not depend on L, since it can be proved together with (iv). To this end set $a = \min\{m(D\cap A) : D \in \mathcal{D}_j,\ D\cap A \ne \emptyset\}$. Since $\mathcal{D}_j$ is finite, since each $D \in \mathcal{D}$ is a nonempty open interval and since m is positive on open subsets of A, we get $a > 0$. Fix $x \in A\setminus L = A\setminus L_j$. By definition of L_j the set $J := \{k : D_k(x) \in \mathcal{D}_j\}$ is infinite. Since $D_k(x) = T^k(Z_{k+1}(x))$ by (2.1) and since $A \cap T^k(Z_{k+1}(x))$ contains $T^k(x)$ and is therefore nonempty, for $k \in J$ we get $a \le m(T^k(Z_{k+1}(x))\cap A)$. Furthermore, $m(T^k(Z_{k+1}(x))\cap A) \le 1$ holds trivially for all $k \ge 1$ and all $x \in R_{\mathcal{Z}}$. From Lemma 1(ii) we get

$$\begin{aligned} m(T^k(Z_{k+1}(x)\cap A)) &\inf_{Z_{k+1}(x)} |(T^k)'|^{-t} \\ &\le m(Z_{k+1}(x)\cap A) \le m(T^k(Z_{k+1}(x)\cap A)) \sup_{Z_{k+1}(x)} |(T^k)'|^{-t} \end{aligned}$$

and Lemma 4 says that $T^k(Z_{k+1}(x)\cap A) = T^k(Z_{k+1}(x))\cap A$. Because of $m(Z_{k+1}(x)\cap A) = m(Z_{k+1}(x))$, Lemma 3 implies that

$$e^{-td} \le |(T^k)'(x)|^{-t}/m(Z_{k+1}(x)) \quad \text{for } k\in\mathbb{N} \text{ and } x \in A,$$

which is (iii), and that

$$|(T^k)'(x)|^{-t}/m(Z_{k+1}(x)) \le e^{td}/a \quad \text{for } x \in A\setminus L \text{ and } k \in J,$$

which is (iv) on setting $I = \{k+1 : k \in J\}$. ■

LEMMA 7. *Let T be an expanding piecewise monotonic map which is piecewise Hölder differentiable. Then*

$$e^{-d}|(T^{k-1})'(x)|^{-1}|T^{k-1}(Z_k(x))| \le |Z_k(x)| \le e^d |(T^{k-1})'(x)|^{-1}$$

for all $x \in R_{\mathcal{Z}}$ and all $k \ge 1$, where d is as in Lemma 3.

Proof. The mean value theorem implies that

$$\begin{aligned} |T^{k-1}(Z_k(x))| &\inf_{Z_k(x)} |(T^{k-1})'|^{-1} \\ &\le |Z_k(x)| \le |T^{k-1}(Z_k(x))| \sup_{Z_k(x)} |(T^{k-1})'|^{-1}. \end{aligned}$$

As $|T^{k-1}(Z_k(x))| \le 1$ and as $Z_k(x)$ is contained in an element of $\mathcal{Z}_{k-1}$, Lemma 3 gives the desired result. ■

3. Conformal measures.

In this section we investigate the existence and properties of conformal measures.

THEOREM 1. *Let T be an expanding piecewise monotonic map and let A be a closed invariant subset of $R_{\mathcal{Z}}$ with the Darboux property. Let $t > 0$.*

(i) *If $t = \mathrm{HD}(A)$, then there is a t-conformal measure on A.*

(ii) *If m is a t-conformal measure on A then m has no atoms and assigns positive measure to open subsets of A, provided that A is topologically transitive.*

Proof. We modify $([0,1],T)$ first. Set $C=\{a_0,a_1,\ldots,a_N\}$ and $W=\{x\in(0,1):T^i(x)\in C$ for some $i\ge 0\}$. In the interval $[0,1]$ replace each $x\in W$ by two points x^- and x^+ and define $y<x^-<x^+<z$ if $y<x<z$ holds in $[0,1]$. In this way $[0,1]$ becomes a totally ordered set X which is compact with respect to the order topology. We consider $R_{\mathcal Z}=(0,1)\setminus W$ as a subset of X. As W is countable, $(0,1)\setminus W$ is dense in X, and as T and T' are continuous except on C, where one-sided limits exist, we can extend T and T' continuously from $(0,1)\setminus W$ to all of X. These extended functions are denoted again by T and T'. If $Z\in\mathcal Z$ is the interval (a_{i-1},a_i) set $\widetilde Z=[a_{i-1}^+,a_i^-]$. Then $\widetilde{\mathcal Z}:=\{\widetilde Z:Z\in\mathcal Z\}$ is a partition of X into intervals which are open and closed, on each of which T is strictly monotone, and whose images under T are again intervals in X. The partition $\widetilde{\mathcal Z}_n=\bigvee_{i=0}^{n-1}T^{-i}\widetilde{\mathcal Z}$ of X then consists of open and closed intervals, on each of which T^n is monotone. Since T is expanding, Lemma 2(i) implies that W is dense in $[0,1]$. From this it follows that $\bigcup_{l=0}^\infty T^{-l}(\widetilde C)$ is dense in X, where $\widetilde C=\{0,a_1^-,a_1^+,a_2^-,\ldots,a_{N-1}^-,a_{N-1}^+,1\}$, and that (X,T) is isomorphic to a shift space and hence expansive (cf. [5] and [6]).

Since $A\subset R_{\mathcal Z}$ we have $A\subset X$. Let $\widetilde A$ be the closure of A in X. Then $T(\widetilde A)\subset\widetilde A$ and $\widetilde A\setminus A\subset X\setminus R_{\mathcal Z}$. Hence $\widetilde A\setminus A$ is countable, since A is closed in $R_{\mathcal Z}$. Furthermore,

$$T(V\cap\widetilde A)=T(V)\cap\widetilde A\quad\text{if } V \text{ is a subinterval of some } Z\in\widetilde{\mathcal Z}. \tag{3.1}$$

This follows from Lemma 4 for $n=1$, since A is dense in $\widetilde A$ and since $\widetilde T|Z$ is strictly monotone for all $Z\in\widetilde{\mathcal Z}$.

Now we can show (i). We apply the method used in [1]. For $n\ge1$ we define subsets E_n of $\widetilde A$ satisfying

$$\text{if } Z\in\widetilde{\mathcal Z}_n \text{ and } Z\cap\widetilde A\ne\emptyset \text{ then } \operatorname{card}(E_n\cap Z)=1, \tag{3.2}$$

$$T^{-1}(E_n)\subset E_{n+1} \text{ and } \operatorname{card}(E_{n+1}\setminus T^{-1}(E_n))\le 2N. \tag{3.3}$$

Choose an arbitrary set E_1 such that (3.2) holds for $n=1$. Suppose that E_n is defined and satisfies (3.2). Since T maps a $Z\in\widetilde{\mathcal Z}_{n+1}$ injectively to a subinterval of an element of $\widetilde{\mathcal Z}_n$, each $Z\in\widetilde{\mathcal Z}_{n+1}$ contains at most one element of $T^{-1}(E_n)$. If $Z\in\widetilde{\mathcal Z}_{n+1}$ satisfies $Z\cap\widetilde C=\emptyset$, then $T(Z)\in\widetilde{\mathcal Z}_n$, and using also (3.1), we deduce that Z contains exactly one element of $T^{-1}(E_n)$, provided that $Z\cap\widetilde A\ne\emptyset$. For each $Z\in\widetilde{\mathcal Z}_{n+1}$ with $Z\cap\widetilde A\ne\emptyset$ and $Z\cap T^{-1}(E_n)=\emptyset$ choose an arbitrary element of Z. Add these elements to $T^{-1}(E_n)$ and call the resulting set E_{n+1}. Since $\operatorname{card}\widetilde C=2N$, it follows that (3.2) and (3.3) are satisfied.

Set $f=-t\log|T'|$, which is continuous on $\widetilde A$, and

$$a_n=\log\sum_{x\in E_n}e^{S_nf(x)},\quad\text{where}\quad S_nf(x)=\sum_{i=0}^{n-1}f(T^i(x)).$$

Set $\widetilde{\mathcal Z}_0=\{X\}$ and $d(x,y)=\inf\{2^{-k}:x$ and y are in the same element of $\widetilde{\mathcal Z}_k\}$ for x and y in X. Then d is a metric on the topological space X. With respect to this metric, E_n is an $(n-l,2^{-l})$ separated and $(n-l,2^{-l-1})$ spanning set for $T|\widetilde A$ if $n>l$, and 1 is an expansive constant for $T|\widetilde A$. Hence Theorem 9.6 of [10] implies that the pressure $p(T|\widetilde A,f)$ of f on $(\widetilde A,T|\widetilde A)$ equals $\limsup_{n\to\infty}a_n/n$. Since $t=\mathrm{HD}(A)$, Theorem 2 and Lemma 2 of [9] imply that $p(T|\widetilde A,f)=0$. Hence $\limsup_{n\to\infty}a_n/n=0$. By Lemma 3.1 of [1] there is a sequence $(b_n)_{n\ge1}$ of positive reals with $\lim_{n\to\infty}b_n/b_{n+1}=1$ such that

$$M_s:=\sum_{n=1}^\infty b_ne^{a_n-ns}$$

satisfies $M_s<\infty$ for $s>0$ and $\lim_{s\downarrow0}M_s=\infty$.

For $s>0$ define probability measures

$$m_s=\frac{1}{M_s}\sum_{n=1}^\infty\sum_{x\in E_n}b_ne^{S_nf(x)-ns}\delta_x,$$

where δ_x is the unit measure concentrated at x. Since $f=-t\log|T'|$ and $R_{\mathcal Z}$ is dense in X, Lemma 2(i) implies that $\sup e^{S_nf}\le c^{-t}\beta^{-nt}$ with $\beta>1$. Since $\lim_{n\to\infty}b_n/b_{n+1}=1$, there is a constant d with $b_n\le d\beta^{nt/2}$ for $n\ge1$. By (3.3) this implies

$$\sum_{x\in E_{n+1}\setminus T^{-1}(E_n)}b_ne^{S_nf(x)-ns}\le 2Ndc^{-t}\beta^{-nt/2}\quad\text{for all } s\ge0.$$

Since $\lim_{s\downarrow0}M_s=\infty$, we get

$$\lim_{s\downarrow0}\frac{1}{M_s}\sum_{n=1}^\infty\sum_{x\in E_{n+1}\setminus T^{-1}(E_n)}b_ne^{S_nf(x)-ns}=0.$$

This and the computation in [1] under the heading "construction principle" show that

$$\lim_{s\downarrow0}\Big(m_s(T(Z))-\int_Z e^{-f}\,dm_s\Big)=0\quad\text{for all } Z\in\bigcup_{n=1}^\infty\widetilde{\mathcal Z}_n. \tag{3.4}$$

Let $(s_i)_{i\ge1}$ be a sequence decreasing to zero and let m be a weak limit point of $(m_{s_i})_{i\geq 1}$. Then m is concentrated on A and, by (3.4),

$$(3.5)\qquad m(T(Z)) = \int_Z e^{-f}\,dm \quad \text{if } Z \in \bigcup_{n=1}^{\infty} \widetilde{\mathcal{Z}}_n \text{ and } Z \cap \widetilde{C} = \emptyset\,,$$

since in this case both Z and $T(Z)$ are open and closed in X. Furthermore, again by (3.4),

$$(3.6)\qquad m(T(Z)) \geq \int_Z e^{-f}\,dm \quad \text{if } Z \in \bigcup_{n=1}^{\infty} \widetilde{\mathcal{Z}}_n \text{ and } Z \cap \widetilde{C} \neq \emptyset\,,$$

since in this case Z is open and closed and $T(Z)$ is only closed in X.

Fix $x \in \widetilde{A}$. Choosing a sequence of intervals in $\bigcup_{n=1}^{\infty} \widetilde{\mathcal{Z}}_n$ which decreases to x, from (3.5) and (3.6) we get $m(\{T(x)\}) \geq e^{-f(x)} m(\{x\})$. Hence $m(\{T^n(x)\}) \geq e^{-S_n f(x)} m(\{x\})$ for $n \geq 1$. Since $\inf e^{-S_n f} \geq c^t \beta^{nt}$, we get $m(\{x\}) = 0$ and m has no atoms. As $\widetilde{A} \setminus A$ is countable, m is concentrated on A.

Furthermore, $m(T(Y)) = \int_Y e^{-f}\,dm$ for all subintervals Y of elements of $\widetilde{\mathcal{Z}}$, because they are disjoint unions of intervals Z used in (3.5) up to sets of m-measure zero. For an interval $Y \subset [0,1]$ contained in an element of $\mathcal{Z}$, we have $T(Y \cap A) = T(Y) \cap A$ by Lemma 4, and we get

$$m(T(Y \cap A)) = \int_{Y \cap A} e^{-f}\,dm\,,$$

since m is concentrated on A. As intervals generate Borel sets and as $f = -t\log|T'|$, this shows that m is t-conformal, and (i) is proved.

If m is a t-conformal measure on A, we get $1 \geq m(\{T^k(x)\}) \geq c^t \beta^{kt} m(\{x\})$ for all $k \geq 1$ by Lemma 1(ii) and Lemma 2(i). Since $\beta > 1$ the first part of (ii) follows.

In order to show the second part of (ii), suppose that I is a nontrivial interval in A with $m(I) = 0$. If $J \subset A$ with $m(J) = 0$, we get $m(T(J)) = 0$ applying (1.1) with $Y = J \cap Z$ for all $Z \in \mathcal{Z}$. In particular, $m(T^n(I)) = 0$ for all n. Let $\widetilde{I}$ be the closure of I in $\widetilde{A}$. By (3.1) we can apply Proposition 1 of [8] to $(\widetilde{A}, T|\widetilde{A})$ and get $\bigcup_{n=0}^{\infty} T^n(\widetilde{I}) = \widetilde{A}$, since topological transitivity of A implies topological transitivity of $\widetilde{A}$. Since A is closed in $R_{\mathcal{Z}}$, we get $\widetilde{A} \setminus A \subset \bigcup_{l=0}^{\infty} T^{-l}(\widetilde{C})$. This implies that $A \setminus \bigcup_{n=0}^{\infty} T^n(I) \subset \bigcup_{l=1}^{\infty} T^l(\widetilde{C})$. As $m(T^n(I)) = 0$ for all n and as m has no atoms we get $m(A) = 0$, a contradiction. ∎

THEOREM 2. *Let T be an expanding piecewise monotonic map which is piecewise Hölder differentiable, and let A be a closed invariant topologically transitive subset of $R_{\mathcal{Z}}$ with the Darboux property. Let m be a t-conformal measure on A for some $t > 0$. Then m is ergodic.*

Proof. Let E be a subset of A satisfying $m(E) > 0$ and $T^{-1}(E) \cap A = E$. Define

$$\widehat{m}(B) = \frac{m(B \cap E)}{m(E)} \quad \text{for measurable } B \subset A\,.$$

If $Y \subset A \cap Z$ for some $Z \in \mathcal{Z}$ then

$$\widehat{m}(T(Y)) = \frac{m(T(Y) \cap E)}{m(E)} = \frac{m(T(Y \cap E))}{m(E)} = \frac{1}{m(E)} \int_{Y \cap E} |T'|^t\,dm = \int_Y |T'|^t\,d\widehat{m}\,.$$

This says that $\widehat{m}$ is a t-conformal measure on A. Since A is topologically transitive, m and $\widehat{m}$ are positive on open subsets of A by Theorem 1(ii). By Lemma 6 there is a set $L \subset A$ with $m(L) = 0$ and a $c > 0$ such that for every $x \in A \setminus L$ there is an infinite subset $I(x)$ of $\mathbb{N}$ with

$$\inf_{k \in I(x)} \frac{\widehat{m}(Z_k(x))}{m(Z_k(x))} \geq c\,.$$

Let $B \subset A \setminus L$ be a Borel set. Let $\varepsilon > 0$ be arbitrary and let U be a neighbourhood of B with $\widehat{m}(U) \leq \widehat{m}(B) + \varepsilon$. For each $x \in B$ choose $k(x) \in I(x)$ such that $Z_{k(x)}(x) \subset U$, which is possible by Lemma 2(ii). Since two elements of $\bigcup_{n=1}^{\infty} \mathcal{Z}_n$ are either disjoint or one contains the other, there is a subset $\mathcal{C}$ of $\{Z_{k(x)}(x) : x \in B\}$ consisting of pairwise disjoint intervals which cover B and are contained in U. Then

$$\widehat{m}(B) + \varepsilon \geq \widehat{m}(U) \geq \sum_{C \in \mathcal{C}} \widehat{m}(C) \geq c \sum_{C \in \mathcal{C}} m(C) \geq cm(B)\,,$$

and so $\widehat{m}(B) \geq cm(B)$. For $B = A \setminus (E \cup L)$ we have $0 = \widehat{m}(B) \geq cm(B)$. Since $m(L) = 0$, this implies $m(E) = 1$, showing that m is ergodic. ∎

THEOREM 3. *Let T be an expanding piecewise monotonic map which is piecewise Hölder differentiable, and let A be a closed invariant topologically transitive subset of $R_{\mathcal{Z}}$ with the Darboux property. If $t > 0$ equals $\mathrm{HD}(A)$ then there is a unique t-conformal measure on A. If $t > 0$ does not equal $\mathrm{HD}(A)$ then there is no t-conformal measure on A.*

Proof. Suppose that $0 < t \leq s$, that m is a t-conformal measure on A, and that $\overline{m}$ is an s-conformal measure on A. Since A is topologically transitive, m and $\overline{m}$ are positive on open subsets of A by Theorem 1(ii). By Lemma 6 there is a set $L \subset A$ with $\overline{m}(L) = 0$ and a $d > 0$ such that for every $x \in A \setminus L$ there is an infinite subset $I(x)$ of $\mathbb{N}$ with

$$\inf_{k \in I(x)} \frac{m(Z_k(x))}{\overline{m}(Z_k(x))} \geq d|(T^{k-1})'(x)|^{s-t}\,.$$

Set $\mathcal{B}_n := \{Z_k(x) : x \in A \setminus L,\ k \in I(x),\ k > n\}$.

Let $B \subset A \setminus L$ be a Borel set. Let $\varepsilon > 0$ be arbitrary and let U be a neighbourhood of B with $m(U) \le m(B) + \varepsilon$. For each $x \in B$ choose $k(x) \in I(x)$ such that $Z_{k(x)}(x) \in \mathcal{B}_n$ and $Z_{k(x)}(x) \subset U$, which is possible by Lemma 2(ii). Since two elements of $\mathcal{B}_n$ are either disjoint or one contains the other, there is a subset $\mathcal{C}$ of $\{Z_{k(x)}(x) : x \in B\}$ consisting of pairwise disjoint intervals which cover B and are contained in U.

If $s = t = \mathrm{HD}(A)$ then

$$\overline{m}(B) \le \sum_{C\in\mathcal{C}} \overline{m}(C) \le \frac{1}{d}\sum_{C\in\mathcal{C}} m(C) \le \frac{1}{d} m(U) \le \frac{1}{d}(m(B)+\varepsilon)\,,$$

and so $\overline{m}(B) \le (1/d)m(B)$ for all Borel subsets B of $A \setminus L$. As $\overline{m}(L) = 0$, we see that $\overline{m}$ is absolutely continuous with respect to m. Let h be the Radon–Nikodym derivative. If $Y \subset A \cap Z$ for some $Z \in \mathcal{Z}$, then (1.1) applied to $\overline{m}$ yields

$$\int\limits_{T(Y)} h\,dm = \int\limits_{Y} |T'|^t h\,dm\,.$$

Together with Lemma 1(i) this gives that $h = h \circ T$ m-almost everywhere. As m is ergodic, h is a constant function. Since m and $\overline{m}$ are probability measures, we get $\overline{m} = m$, which shows uniqueness of the $\mathrm{HD}(A)$-conformal measure. Its existence is shown in Theorem 1.

Suppose now that $0 < t < s$. As above we find a subset $\mathcal{C}$ of $\mathcal{B}_n$ of pairwise disjoint intervals which cover $A \setminus L$. Since $|(T^{k-1})'(x)|^{s-t} \ge r_n$ with $r_n = c^{s-t}\beta^{n(s-t)}$ for $k \ge n$ by Lemma 2(i), we get

$$1 \ge \sum_{C\in\mathcal{C}} m(C) \ge dr_n \sum_{C\in\mathcal{C}} \overline{m}(C) \ge dr_n\overline{m}(A\setminus L) = dr_n\overline{m}(A) = dr_n\,,$$

because $\overline{m}(L) = 0$. As $\lim_{n\to\infty} r_n = \infty$, we have arrived at a contradiction. Hence there cannot be a t-conformal measure and an s-conformal measure at the same time. Since there does exist an $\mathrm{HD}(A)$-conformal measure on A there is no t-conformal measure on A for $t \ne \mathrm{HD}(A)$ and $t > 0$. ∎

4. Hausdorff measures. First we show absolute continuity of Hausdorff measures with respect to conformal measures.

THEOREM 4. *Let T be an expanding piecewise monotonic map which is piecewise Hölder differentiable, and let A be a closed invariant topologically transitive subset of $R_{\mathcal{Z}}$ with the Darboux property. For $t = \mathrm{HD}(A) > 0$ let m be the t-conformal measure on A and let ν be the t-dimensional Hausdorff measure restricted to A. Then there is a constant a such that $\nu(B) \le am(B)$ for all Borel subsets B of A.*

Proof. Let B be a Borel subset of A. Let $\delta > 0$ and $\varepsilon > 0$ be arbitrary and let U be an open neighbourhood of B in $[0,1]$ with $m(U) < m(B) + \varepsilon$.

Since A is topologically transitive, m is positive on open subsets of A by Theorem 1(ii). By Lemma 6 there is a subset $\widetilde{B}$ of B with $\nu(\widetilde{B}) = \nu(B)$ and a constant $b > 0$ such that the set $\{k : |(T^{k-1})'(x)|^{-t} \le bm(Z_k(x))\}$ is infinite for all $x \in \widetilde{B}$. Together with Lemmas 3 and 7 this implies that for each $x \in \widetilde{B}$ there is a $k(x)$ such that $V_x := Z_{k(x)}(x)$ satisfies $V_x \subset U$, $|V_x| < \delta$ and $|V_x|^t \le e^{dt}bm(V_x)$. Set $\widetilde{\mathcal{U}} = \{V_x : x \in \widetilde{B}\}$, which covers $\widetilde{B}$. As two elements of $\bigcup_{n\ge1}\mathcal{Z}_n$ are either disjoint or one contains the other, there is a subset $\mathcal{U}$ of $\widetilde{\mathcal{U}}$ which still covers $\widetilde{B}$ and whose elements are pairwise disjoint. Since the elements V of $\mathcal{U}$ satisfy $V \subset U$ and $|V|^t \le am(V)$ with $a = e^{dt}b$, we have

$$\sum_{V\in\mathcal{U}} |V|^t \le \sum_{V\in\mathcal{U}} am(V) \le am(U) < a(m(B)+\varepsilon)\,.$$

As $\mathcal{U} \in \mathcal{C}(\widetilde{B},\delta)$ and as $\delta > 0$ was arbitrary, the definition of Hausdorff measure implies $\nu(B) = \nu(\widetilde{B}) \le a(m(B)+\varepsilon)$, and so $\nu(B) \le am(B)$. ∎

In order to show absolute continuity of conformal measures with respect to Hausdorff measures we need the following result, which is Lemma 3(i) in [8].

LEMMA 8. *Suppose that T is a piecewise monotonic map which satisfies the Misiurewicz condition. Then there is a $u > 0$ such that $|D| \ge u$ for all $D \in \bigcup_{n=0}^{\infty}\{T^n(Z) : Z \in \mathcal{Z}_{n+1}\}$.*

We define a modified t-dimensional Hausdorff measure $\widetilde{\nu}$. For $B \subset R_{\mathcal{Z}}$ let $\widetilde{\mathcal{C}}(B,\delta)$ be the set of all finite or countable covers of B by intervals of length less than δ which are elements of $\bigcup_{n\ge1}\mathcal{Z}_n$. Then set

$$\widetilde{\nu}(B) := \lim_{\delta\to0}\ \inf_{\mathcal{V}\in\widetilde{\mathcal{C}}(B,\delta)} \sum_{V\in\mathcal{V}} |V|^t\,.$$

LEMMA 9. *Let T be an expanding piecewise monotonic map which is piecewise Hölder differentiable and satisfies the Misiurewicz condition. Let ν be the t-dimensional Hausdorff measure. Then there is a constant $v > 0$ such that $\nu(B) \ge v\widetilde{\nu}(B)$ for all subsets B of $R_{\mathcal{Z}}$.*

Proof. We show first that there is a constant $w > 0$ such that

$$|Z_n(x)|/|Z_{n-1}(x)| \ge w \quad \text{for all } n \ge 2 \text{ and all } x \in R_{\mathcal{Z}}\,. \tag{4.1}$$

Lemma 7 implies that

$$|Z_n(x)| \ge e^{-d}|(T^{n-2})'(x)|^{-1}|T'(T^{n-2}(x))|^{-1}|T^{n-1}(Z_n(x))|$$

and that $|Z_{n-1}(x)| \le e^d|(T^{n-2})'(x)|^{-1}$. Using Lemma 8 we get (4.1) with $w = e^{-2d}u\inf|T'|^{-1}$.

For $B \subset R_{\mathcal{Z}}$ and $\delta > 0$ choose an arbitrary $\mathcal{U} \in \mathcal{C}(B,\delta)$. Fix $U \in \mathcal{U}$. For $x \in U \cap B$ set $n(x) = \min\{n : |Z_n(x)| \le |U|\}$, which exists by Lemma 2(ii).

We write V_x for $Z_{n(x)}(x)$. Then $|V_x| \le |U|$ and (4.1) implies that $|V_x| \ge w|U|$. Let $\mathcal{V}_U$ be a subset of $\{V_x : x \in U \cap B\}$ which covers $U \cap B$ and consists of pairwise disjoint intervals. This is possible, as two elements of $\bigcup_{n\ge1} \mathcal{Z}_n$ are either disjoint or one contains the other.

Since each $V \in \mathcal{V}_U$ satisfies $|V| \ge w|U|$, we get $\operatorname{card}\mathcal{V}_U \le 2 + 1/w$. Since each $V \in \mathcal{V}_U$ satisfies $|V| \le |U|$, we get $|U|^t \ge v\sum_{V\in\mathcal{V}_U} |V|^t$, where $v = (2+1/w)^{-1}$. Set $\mathcal{V} = \bigcup_{U\in\mathcal{U}} \mathcal{V}_U$. Then $\mathcal{V} \in \widetilde{\mathcal{C}}(B,\delta)$ and $\sum_{U\in\mathcal{U}} |U|^t \ge v\sum_{V\in\mathcal{V}} |V|^t$, yielding that $\nu(B) \ge v\widetilde{\nu}(B)$. ∎

Now we can show

THEOREM 5. *Let T be an expanding piecewise monotonic map which is piecewise Hölder differentiable and satisfies the Misiurewicz condition. Let A be a closed invariant subset of $R_{\mathcal{Z}}$ with the Darboux property. For $t = \mathrm{HD}(A) > 0$ let m be the t-conformal measure on A and let ν be the t-dimensional Hausdorff measure restricted to A. Then there is a constant b such that $m(B) \le b\nu(B)$ for all Borel subsets B of A.*

Proof. For $x \in R_{\mathcal{Z}}$ and $k \ge 2$ we get $|Z_k(x)| \ge ue^{-d}|(T^{k-1})'(x)|^{-1}$ from Lemmas 7 and 8. By Lemma 6(iii) there is a $q > 0$ with $|(T^{k-1})'(x)|^{-t} \ge qm(Z_k(x))$ for $k \ge 2$ and $x \in A$. Hence there is an $r > 0$ such that $|Z_k(x)|^t \ge rm(Z_k(x))$ for $k \in \mathbb{N}$ and $x \in A$.

Let B be a Borel subset of A. Let $\delta > 0$ and $\mathcal{V} \in \widetilde{\mathcal{C}}(B,\delta)$ be arbitrary. We can suppose that each $V \in \mathcal{V}$ has nonempty intersection with A and is therefore a $Z_k(x)$ for some $k \in \mathbb{N}$ and some $x \in A$. Hence

$$\sum_{V\in\mathcal{V}} |V|^t \ge r\sum_{V\in\mathcal{V}} m(V) \ge rm(B),$$

and so $\widetilde{\nu}(B) \ge rm(B)$. Lemma 9 says that $\nu(B) \ge v\widetilde{\nu}(B)$. The desired result follows with $b = 1/(vr)$. ∎

We can summarize Theorems 4 and 5 in the following

THEOREM 6. *Let T be an expanding piecewise monotonic map which is piecewise Hölder differentiable, and let A be a closed invariant topologically transitive subset of $R_{\mathcal{Z}}$ with the Darboux property. For $t = \mathrm{HD}(A) > 0$ let m be the unique t-conformal measure on A and let ν be the t-dimensional Hausdorff measure restricted to A. Then there is a $c \in [0,\infty)$ with $\nu = cm$. If T satisfies the Misiurewicz condition, then $c \ne 0$.*

Proof. Theorem 4 implies that ν is a finite measure or the zero measure on A. We show that ν satisfies (1.1). To this end let $Y \subset A$ be contained in some element of $\mathcal{Z}$. Choose $\varepsilon > 0$. Let $\mathcal{C}$ be a finite partition of R_1 into intervals which refines $\mathcal{Z}$, such that $\sup_C |T'| - \inf_C |T'| \le \varepsilon$ for all $C \in \mathcal{C}$. Set $g_C = \sup_C |T'|$ for $C \in \mathcal{C}$. Since $s := \sup |T'|$ exists, we have $|T(x) - T(y)| < s\delta$ if x and y are in the same element of $\mathcal{Z}$ and if $|x-y| < \delta$.

Fix $C \in \mathcal{C}$. Let $\mathcal{A}(\delta)$ be the set of those covers $\mathcal{U} \in \mathcal{C}(C\cap Y, \delta)$ whose elements are contained in C. Then

$$\nu(C\cap Y) = \lim_{\delta\to0} \inf_{\mathcal{U}\in\mathcal{A}(\delta)} \sum_{U\in\mathcal{U}} |U|^t.$$

Similarly let $\mathcal{B}(\delta)$ be the set of those covers $\mathcal{V} \in \mathcal{C}(T(C\cap Y),\delta)$ whose elements are contained in the interval $T(C)$. For $\mathcal{U} \in \mathcal{A}(\delta)$ set $T\mathcal{U} = \{T(U) : U \in \mathcal{U}\}$, which is then in $\mathcal{B}(s\delta)$. Furthermore, $\sum_{V\in T\mathcal{U}} |V|^t \le g_C^t \sum_{U\in\mathcal{U}} |U|^t$. This implies

$$\inf_{\mathcal{V}\in\mathcal{B}(s\delta)} \sum_{V\in\mathcal{V}} |V|^t \le g_C^t \inf_{\mathcal{U}\in\mathcal{A}(\delta)} \sum_{U\in\mathcal{U}} |U|^t.$$

As $\delta \to 0$ we get $\nu(T(C\cap Y)) \le g_C^t \nu(C\cap Y)$. By summing over $C \in \mathcal{C}$, this gives $\nu(T(Y)) \le \int_Y (|T'|^t + \varepsilon)\,d\nu$. As ε was arbitrary we get

$$\nu(T(Y)) \le \int_Y |T'|^t\,d\nu.$$

It remains to show the other inequality. Since $\inf|T'| \ge 1$, we have $|x-y| < \delta$ for x and y in the same element of $\mathcal{Z}$ with $|T(x) - T(y)| < \delta$. Hence, if $C \in \mathcal{C}$ is fixed and $\mathcal{A}(\delta)$ and $\mathcal{B}(\delta)$ are as above, then $\{(T|C)^{-1}(U) : U \in \mathcal{U}\}$ is in $\mathcal{A}(\delta)$ for every $\mathcal{U} \in \mathcal{B}(\delta)$. A similar proof to the above, choosing now $g_C = \inf_C |T'|$, shows then

$$\nu(T(Y)) \ge \int_Y |T'|^t\,d\nu.$$

This finishes the proof that ν satisfies (1.1).

Set $c = \nu(A) < \infty$. As the t-conformal measure m is unique, it follows that $\nu = cm$. Furthermore, Theorem 5 implies that c cannot be zero if T satisfies the Misiurewicz condition. ∎

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Conjugate martingale transforms

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Abstract. Characterizations of H_1, BMO and VMO martingale spaces generated by bounded Vilenkin systems via conjugate martingale transforms are studied.

1. Introduction. A theory of H_p spaces of conjugate harmonic functions on Euclidean spaces was developed by Stein [22]. In particular, $H_1(\mathbb{R}^n)$ can be characterized via the Riesz transforms:

$$(*)\qquad H_1 = \{f \in L_1 : R_j f \in L_1,\ j = 1, \ldots, m\}.$$

Chao and Taibleson (see [6]–[10], [23]) have extended this theory to local fields. Moreover, for martingale spaces, Janson and Chao ([15], [8], [5]) studied transforms with matrix operators acting on the values of the difference sequences of q-martingales.

In this paper conjugate martingale transforms with matrix operators acting on the generalized Rademacher series of the difference sequences are investigated. These transforms were first introduced by Gundy [13]. Contrary to the statement in [13] Gundy only proved $(*)$ in the case when all matrices and martingales are real. This theorem is here extended to the complex case. More exactly, a necessary and sufficient condition for the transforms is given such that $(*)$ holds whenever the martingale H_1 space is generated by a bounded Vilenkin system. Note that this space is slightly more general than the H_1 space of q-martingales. We shall prove a version of F. and M. Riesz theorem. In the simplest case when all matrices are diagonal the transforms used in this paper are called multiplier transforms. Simon's question [20] whether H_1 can be characterized via a single multiplier transform if the multiplier has two values: -1 and 1, is answered. Moreover, a necessary and sufficient condition for $(*)$ to hold for multiplier transforms is also given. A family of integrable functions for which $\|f\|_{L_1} \sim \|f\|_{H_1}$ is obtained. Similarly to [4] we also introduce a transform in the dyadic case.

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