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Comparison of Orlicz–Lorentz spaces

by

S. J. MONTGOMERY-SMITH (Columbia, Mo.)

I dedicate this paper to my Mother and Father,
who as well as introducing me to mathematics
at an early age, shared the arduous task of bringing me up

Abstract. Orlicz–Lorentz spaces provide a common generalization of Orlicz spaces and Lorentz spaces. They have been studied by many authors, including Mastyło, Maligranda, and Kamińska. In this paper, we consider the problem of comparing the Orlicz–Lorentz norms, and establish necessary and sufficient conditions for them to be equivalent. As a corollary, we give necessary and sufficient conditions for a Lorentz–Sharpley space to be equivalent to an Orlicz space, extending results of Lorentz and Raynaud. We also give an example of a rearrangement invariant space that is not an Orlicz–Lorentz space.

1. Introduction. The most well known examples of Banach spaces are the L_p spaces. Their definition is very well known: if $(\Omega, \mathcal{F}, \mu)$ is a measure space, and $1 \le p \le \infty$, then for any measurable function $f : \Omega \to \mathbb{C}$, the L_p-norm is defined to be $\|f\|_p = (\int_\Omega |f(\omega)|^p \, d\mu(\omega))^{1/p}$ for $p < \infty$, and $\|f\|_\infty = \operatorname{ess\,sup}_{\omega \in \Omega} |f(\omega)|$ for $p = \infty$. Then we define the Banach space $L_p(\Omega, \mathcal{F}, \mu)$ to be the vector space of all measurable functions $f : \Omega \to \mathbb{C}$ for which $\|f\|_p$ is finite.

It is natural to search for generalizations of these L_p spaces. The first examples are the *Orlicz spaces*. These were first studied by Orlicz [O] and Luxemburg [L]. We say that $F : [0, \infty) \to [0, \infty)$ is an *Orlicz function* if F is nondecreasing and convex with $F(0) = 0$. Now we define the *Luxemburg norm* by

$$\|f\|_F = \inf\Big\{c : \int_\Omega F(|f(\omega)|/c) \, d\mu(\omega) \le 1\Big\},$$

whenever f is a measurable function, and define the *Orlicz space* $L_F(\Omega, \mathcal{F}, \mu)$ to be those measurable functions f for which $\|f\|_F$ is finite. The Orlicz space

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L_F is a true generalization of L_p, at least for $p < \infty$: if $F(t) = t^p$, then $L_F = L_p$ with equality of norms.

The other examples are the *Lorentz spaces*. These were introduced by Lorentz [Lo1], [Lo2]. If f is a measurable function, we define the *nonincreasing rearrangement* of f to be

$$f^*(x) = \sup\{t : \mu(|f| \ge t) \ge x\}.$$

If $1 \le q < \infty$, and if $w : (0,\infty) \to (0,\infty)$ is a nonincreasing function, we define the *Lorentz norm* of a measurable function f to be

$$\|f\|_{w,q} = \Big(\int_0^\infty w(x) f^*(x)^q\, dx\Big)^{1/q}.$$

We define the *Lorentz space* $\Lambda_{w,q}(\Omega,\mathcal{F},\mu)$ to be the space of those measurable functions f for which $\|f\|_{w,q}$ is finite. These spaces also represent a generalization of the L_p spaces: if $w(x) = 1$ for all $0 \le x < \infty$, then $\Lambda_{w,p} = L_p$ with equality of norms.

There is one, rather peculiar, choice of the function w which turns out to be rather useful. If $1 \le q \le p < \infty$, we define the spaces $L_{p,q}$ to be $\Lambda_{w,q}$ with $w(x) = (q/p)x^{q/p-1}$. A good reference for a description of these spaces is Hunt [H]. By a suitable change of variables, the $L_{p,q}$ norm may also be defined in the following fashion:

$$\|f\|_{p,q} = \Big(\int_0^\infty f^*(x^{p/q})^q\, dx\Big)^{1/q}.$$

Thus $L_{p,p} = L_p$ with equality of norms. The reason for this definition is that for any measurable set $A \in \mathcal{F}$, we have $\|\chi_A\|_{p,q} = \|\chi_A\|_p = \mu(A)^{1/p}$. Thus $L_{p,q}$ is a space identical to L_p for characteristic functions, but "glued" together in an L_q fashion.

In all the spaces defined above, if we only desire to study quasi-Banach spaces rather than Banach spaces, we may remove some of the restrictions placed upon the defining parameters. Thus with the L_p spaces, we need only have $p > 0$. With the Orlicz spaces L_F and the Lorentz space $\Lambda_{w,q}$, we may weaken the restrictions that F be convex and that w be nonincreasing (we omit details). The spaces $\Lambda_{w,q}$ so obtained were studied by Sharpley [S], and so we might call them Lorentz–Sharpley spaces. With the $L_{p,q}$ spaces, we need only have $0 < p < \infty$ and $0 < q \le \infty$, where if $q = \infty$, we define the Lorentz norm by

$$\|f\|_{p,\infty} = \sup_{x\ge 0} x^{1/p} f^*(x).$$

Now we come to the object of the paper, the *Orlicz–Lorentz spaces*. These are a common generalization of the Orlicz spaces and the Lorentz spaces. They have been studied by Mastyło (see Part 4 of [My]), Maligranda [Ma], and Kamińska [Ka1], [Ka2], [Ka3]. If G is an Orlicz function, and if $w : [0,\infty) \to [0,\infty)$ is a nonincreasing function, we define the *Orlicz–Lorentz norm* of a measurable function f to be

$$\|f\|_{w,G} = \inf\Big\{c : \int_0^\infty w(x) G(f^*(x)/c)\, dx \le 1\Big\}.$$

We define the *Orlicz–Lorentz space* $\Lambda_{w,G}(\Omega,\mathcal{F},\mu)$ to be the vector space of measurable functions f for which $\|f\|_{w,G}$ is finite. If we do not require that the space be a Banach space, but only a quasi-Banach space, we may weaken the restrictions placed upon G and w as we did for L_F and $\Lambda_{w,p}$ above.

We shall not work with this definition of the Orlicz–Lorentz space, however, but with a different, equivalent definition that bears more resemblance to the spaces $L_{p,q}$. This definition is given in the following section.

2. Definitions. First we define φ-functions. These replace the notion of Orlicz functions in our discussions.

DEFINITION. A *φ-function* is a function $F : [0,\infty) \to [0,\infty)$ such that

(i) $F(0) = 0$;
(ii) $\lim_{t\to\infty} F(t) = \infty$;
(iii) F is strictly increasing;
(iv) F is continuous.

We will say that a φ-function F is *dilatory* if for some $c_1, c_2 > 1$ we have $F(c_1 t) \ge c_2 F(t)$ for all $0 \le t < \infty$. We will say that F satisfies the *Δ_2-condition* if F^{-1} is dilatory.

If F is a φ-function, we define $\widetilde{F}(t)$ to be $1/F(1/t)$ if $t > 0$, and 0 if $t = 0$.

The definition of a φ-function is slightly more restrictive than that of an Orlicz function in that we insist that F be strictly increasing. The notion of dilatory replaces the notion of convexity. The Orlicz spaces generated by dilatory functions are only quasi-Banach spaces, in contrast to those generated by Orlicz functions, which are Banach spaces. The Δ_2-condition appears widely in literature about Orlicz spaces.

DEFINITION. If $(\Omega,\mathcal{F},\mu)$ is a measure space, and F is a φ-function, then we define the *Luxemburg functional* by

$$\|f\|_F = \inf\Big\{c : \int_\Omega F(|f(\omega)|/c)\, d\mu(\omega) \le 1\Big\},$$

for every measurable function f. We define the *Orlicz space*, $L_F(\Omega,\mathcal{F},\mu)$ (or $L_F(\mu)$, $L_F(\Omega)$ or L_F for short), to be the vector space of measurable functions f for which $\|f\|_F < \infty$, modulo functions that are zero almost everywhere.

Now we define the Orlicz–Lorentz spaces.

DEFINITION. If $(\Omega, \mathcal{F}, \mu)$ is a measure space, and F and G are φ-functions, then we define the *Orlicz–Lorentz functional* of a measurable function f by

$$\|f\|_{F,G} = \|f^* \circ \widetilde{F} \circ \widetilde{G}^{-1}\|_G .$$

We define the *Orlicz–Lorentz space*, $L_{F,G}(\Omega, \mathcal{F}, \mu)$ (or $L_{F,G}(\mu)$, $L_{F,G}(\Omega)$ or $L_{F,G}$ for short), to be the vector space of measurable functions f for which $\|f\|_{F,G} < \infty$, modulo functions that are zero almost everywhere. Similarly, by means of the (*weak-*)*Orlicz–Lorentz functional*

$$\|f\|_{F,\infty} = \sup_{x \geq 0} \widetilde{F}^{-1}(x) f^*(x) ,$$

we define the *Orlicz–Lorentz space* $L_{F,\infty}(\Omega, \mathcal{F}, \mu)$.

We see that $L_{F,F} = L_F$ with equality of norms, and that if $F(t) = t^p$ and $G(t) = t^q$, then $L_{F,G} = L_{p,q}$, and $L_{F,\infty} = L_{p,\infty}$, also with equality of norms. Thus, if $F(t) = t^p$, we shall write $L_{p,G}$ for $L_{F,G}$, and $L_{G,p}$ for $L_{G,F}$. Also, if A is any measurable set, then $\|\chi_A\|_{F,G} = \|\chi_A\|_{F,\infty} = \|\chi_A\|_F = \widetilde{F}^{-1}(\mu(A))$.

The Orlicz–Lorentz spaces defined here are equivalent to the definition given in the introduction, as we now describe.

DEFINITION. A *weight function* is a function $w : (0,\infty) \to (0,\infty)$ such that $W(t) = \int_0^t w(s)\,ds$ is a φ-function.

Then if w is a weight function, and G is a φ-function, then $\Lambda_{w,G} = L_{\widetilde{W}^{-1}\circ G, G}$, where $W(t) = \int_0^t w(s)\,ds$.

Now let us provide some examples. We define the *modified logarithm* and the *modified exponential* functions by

$$\operatorname{lm}(t) = \begin{cases} 1 + \log t & \text{if } t \geq 1, \\ 1/(1+\log(1/t)) & \text{if } 0 < t < 1, \\ 0 & \text{if } t = 0; \end{cases}$$

$$\operatorname{em}(t) = \operatorname{lm}^{-1}(t) = \begin{cases} \exp(t-1) & \text{if } t \geq 1, \\ \exp(1-(1/t)) & \text{if } 0 < t < 1, \\ 0 & \text{if } t = 0. \end{cases}$$

These functions are designed so that for large t they behave like the logarithm and the exponential functions, so that $\operatorname{lm} 1 = 1$ and $\operatorname{em} 1 = 1$, and so that $\widetilde{\operatorname{lm}} = \operatorname{lm}$ and $\widetilde{\operatorname{em}} = \operatorname{em}$. Then the functions $t^p(\operatorname{lm} t)^\alpha$ and $\operatorname{em}(t^p)$ are φ-functions whenever $0 < p < \infty$ and $-\infty < \alpha < \infty$. If the measure space is a probability space, then the Orlicz spaces created using these functions are also known as *Zygmund spaces*, and the Orlicz–Lorentz spaces $L_{t^p(\operatorname{lm} t)^\alpha,q}$ and $L_{\operatorname{em}(t^p),q}$ are known as *Lorentz–Zygmund spaces* (see, for example, [B–S]).

Finally, we define the notions of equivalence.

DEFINITION. We say that two φ-functions F and G are *equivalent* (in symbols $F \asymp G$) if for some number $c < \infty$ we have $F(c^{-1}t) \leq G(t) \leq F(ct)$ for all $0 \leq t < \infty$.

We say that two function spaces X and Y on the same measure space are *equivalent* if for some $c < \infty$ we have $f \in X \Leftrightarrow f \in Y$ with $c^{-1}\|f\|_X \leq \|f\|_Y \leq c\|f\|_X$ for all measurable functions f.

3. Survey of known comparison results. There are at least four obvious questions about Orlicz–Lorentz spaces.

(i) For which φ-functions F and G is $L_{F,G}$ equivalent to a normed space (or p-convex, or q-concave)?

(ii) What are the Boyd indices of the Orlicz–Lorentz spaces?

(iii) What are necessary and sufficient conditions for L_{F_1,G_1} and L_{F_2,G_2} to be equivalent?

(iv) Is every rearrangement invariant space equivalent to some Orlicz–Lorentz space?

The first and second questions are intimately related, and will be dealt with in another paper [Mo2]. In general, they are very hard to answer. The third question is the subject of this paper. As a corollary, we will also be able to answer the fourth question.

There have already been many comparison results for Lorentz spaces. Indeed, Lorentz himself provided one of the first in 1961 [Lo3]. He found necessary and sufficient conditions for $\Lambda_{w,1}$ to be equivalent to an Orlicz space.

DEFINITION. A weight function w is said to be *strictly monotone* if either

(i) w is strictly increasing, $w(t) \to 0$ as $t \to 0$ and $w(t) \to \infty$ as $t \to \infty$, or

(ii) w is strictly decreasing, $w(t) \to \infty$ as $t \to 0$ and $w(t) \to 0$ as $t \to \infty$.

A strictly monotone weight function is said to satisfy *condition* (L) if there is a number $c < \infty$ such that

$$\int_0^\infty \frac{dt}{w^{-1}(c\,w(t))} < \infty .$$

THEOREM 3.1. *Let $w : (0,\infty) \to (0,\infty)$ be a decreasing, strictly monotone weight function. Then the following are equivalent.*

(i) *$\Lambda_{w,1}$ is equivalent to an Orlicz space.*

(ii) *w satisfies condition (L).*

The kinds of weight functions that satisfy condition (L) are slowly increasing or slowly decreasing functions. An example that Lorentz implicitly gave is

$$w(t) = \begin{cases} t^{-1/\log(1+\log t)} & \text{if } t \ge 1, \\ t^{1/\log(1-\log t)} & \text{if } 0 < t < 1. \end{cases}$$

Recently, Raynaud [R] noticed that the above result is also true if w is strictly increasing. He then went on to show the following result.

THEOREM 3.2. *Let w be a weight function, and $0 < p < \infty$. If there are strictly monotone weight functions w_0 and w_1 satisfying condition (L) and a number $c < \infty$ such that*

$$c^{-1}\frac{W(t)}{t} \le w_0(t)w_1(t) \le c\,\frac{W(t)}{t},$$

then $\Lambda_{w,p}$ is equivalent to an Orlicz space.

It may seem that the scope of these results is limited, but this is not really the case. Using Lemmas 5.1.2 and 5.1.3 below, one can apply these results to find sufficient conditions for equivalence of two Orlicz–Lorentz spaces that are no stronger than the conditions given in this paper.

There are also results due to Bennett and Rudnick [B–R] (see also [B–S]). They proved the following results for probability spaces, but using their methods, it is not too hard to see that these results are true for all measure spaces.

THEOREM 3.3. *For every $0 < p < \infty$ and every $-\infty < \alpha < \infty$, the spaces $L_{t^p(\operatorname{lm} t)^\alpha}$ and $L_{t^p(\operatorname{lm} t)^\alpha,p}$ are equivalent.*

THEOREM 3.4. *For every $\beta > 0$, the spaces $L_{\operatorname{em}(t^\beta)}$ and $L_{\operatorname{em}(t^\beta),\infty}$ are equivalent.*

4. Comparison of Orlicz–Lorentz spaces. In this section, we state the main results of this paper, and give necessary and sufficient conditions for which, given certain restrictions upon G_1 and G_2, we have $\|f\|_{F,G_1} \le c\,\|f\|_{F,G_2}$. Thus we find necessary and sufficient conditions for L_{F_1,G_1} and L_{F_2,G_2} to be equivalent.

We first notice that $\|f\|_{p,q_1} \le \|f\|_{p,q_2}$ whenever $q_1 \ge q_2$ (see [H]). This suggests that we have a result something like: if $G_1 \circ G_2^{-1}$ is a convex function, then $\|f\|_{F,G_1} \le c\,\|f\|_{F,G_2}$. And this is indeed the case. However, more is true. For example, if $G(t) = t\operatorname{lm} t$, then it follows from Theorem 3.3 that $L_{G,1}$ is equivalent to $L_{G,G}$. Thus, it would seem that we only need to know that $G_1 \circ G_2^{-1}$ is "close", in some sense, to a convex function.

In this paper, we establish precisely what this notion of closeness is. But, before stating the conditions, we first give a little bit of motivation. We note that a dilatory φ-function G is completely determined, up to equivalence, by its values $G(a^n)$, where $a > 1$ is any fixed number, and n ranges over all integers. Thus, we note that a φ-function G is equivalent to a convex function if and only if for some $a > 1$ and $N \in \mathbb{N}$, and all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, we have $G(a^{n+m}) \ge a^{m-N}G(a^n)$ (see Lemma 5.4.2 below).

In all that follows, we take the natural numbers to be $\mathbb{N} = \{1, 2, 3, \ldots\}$.

DEFINITION. Let G be a φ-function. We say that G is

(i) *almost convex* if there are $a > 1$, $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do *not* have

$$G(a^{n+m}) \ge a^{m-N}G(a^n)$$

is less than b^m;

(ii) *almost concave* if there are $a > 1$, $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do *not* have

$$G(a^{n+m}) \le a^{m+N}G(a^n)$$

is less than b^m;

(iii) *almost linear* if there are $a > 1$, $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do *not* have

$$a^{m-N}G(a^n) \le G(a^{n+m}) \le a^{m+N}G(a^n)$$

is less than b^m;

(iv) *almost constant* if there are $a > 1$, $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do *not* have

$$G(a^{n+m}) \le a^N G(a^n)$$

is less than b^m;

(v) *almost vertical* if G^{-1} is almost constant.

We will also express our results in terms of what we shall call condition (J).

DEFINITION. If F and G are φ-functions, then we say that F is *equivalently less convex* than G (in symbols $F \prec G$) if $G \circ F^{-1}$ is equivalent to a convex function. We say that F is *equivalently more convex* than G (in symbols $F \succ G$) if G is equivalently less convex than F.

A φ-function F is said to be an *N-function* if it is equivalent to a φ-function F_0 such that $F_0(t)/t$ is strictly increasing, $F_0(t)/t \to \infty$ as $t \to \infty$, and $F_0(t)/t \to 0$ as $t \to 0$.

A φ-function F is said to be *complementary* to a φ-function G if for some $c < \infty$ we have

$$c^{-1}t \le F^{-1}(t) \cdot G^{-1}(t) \le ct \quad (0 \le t < \infty).$$

If F is an N-function, we will let F^* denote a function complementary to F.

The notation F^* makes sense if F is an N-function, because then there is always a function G complementary to F, and further, if G_1 and G_2 are both complementary to F, then G_1 and G_2 are equivalent.

Our definition of a complementary function differs from the usual definition. If F is an N-function that is convex, then the complementary function is usually defined by $F^*(t) = \sup_{s\ge 0}(st - F(s))$. However, it is known that $t \le F^{-1}(t)\cdot F^{*-1}(t) \le 2t$ (see [K–R]). Thus our definition is equivalent.

DEFINITION. An N-function H is said to satisfy *condition* (J) if

$$\|1/\widetilde{H}^{*-1}\|_{H^*} < \infty .$$

The kinds of N-functions that satisfy condition (J) are slowly rising functions. These are essentially the kinds of Orlicz functions that Lorentz describes in Theorem 1 of his paper [Lo3].

We also describe our results in a third fashion. The following definitions are motivated by the fact that $G_1 \succ G_2$ if and only if for some $c < \infty$ and all $s \ge 1$ and $t > 0$ we have $G_1(st)/G_1(t) \ge c^{-1}G_2(st)/G_2(t)$ (see Lemma 5.4.2).

DEFINITION. Let G_1 and G_2 be φ-functions. We say that

(i) G_1 is *almost less convex* than G_2 if there are $a > 1$, $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do *not* have

$$\frac{G_1(a^{n+m})}{G_1(a^n)} \le a^N \frac{G_2(a^{n+m})}{G_2(a^n)}$$

is less than b^m;

(ii) G_1 is *almost more convex* than G_2 if G_2 is almost less convex than G_1;

(iii) G_1 is *almost equivalent* to G_2 if there are $a > 1$, $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do *not* have

$$a^{-N}\frac{G_2(a^{n+m})}{G_2(a^n)} \le \frac{G_1(a^{n+m})}{G_1(a^n)} \le a^N \frac{G_2(a^{n+m})}{G_2(a^n)}$$

is less than b^m.

Now we collect together the comparison results. For all these results, we will assume that the measure space is $[0,\infty)$ with Lebesgue measure. In fact, any nonatomic infinite measure space will do. There are also similar results for nonatomic probability spaces, and for $\mathbb{N}$ with the counting measure (i.e. sequence spaces). We do not give details for these cases. However, the idea is that for nonatomic probability spaces, we need only consider the properties of the relevant φ-functions $G(t)$ for large t, and for sequence spaces, their properties for small t. Obviously, if one is only interested in sufficient conditions for Orlicz–Lorentz spaces to be equivalent, one can use any measure space. (Recall that $\asymp$ means *equivalent to*, see Section 2.)

PROPOSITION 4.1. *Let F_1, F_2, G_1 and G_2 be φ-functions.*

(i) *If $F_1\circ G_1^{-1} \asymp F_2\circ G_2^{-1}$ and $G_1 \asymp G_2$, and if one of G_1 or G_2 is dilatory, then L_{F_1,G_1} and L_{F_2,G_2} are equivalent.*

(ii) *If $F_1 \asymp F_2$, then $L_{F_1,\infty}$ and $L_{F_2,\infty}$ are equivalent.*

THEOREM 4.2. *Let F, G_1 and G_2 be φ-functions. Consider the following statements.*

(i) *For some $c<\infty$, we have $\|f\|_{F,G_1} \le c\|f\|_{F,G_2}$ for all measurable f.*

(ii) *$G_1\circ G_2^{-1}$ is almost convex.*

(iii) *There is an N-function H satisfying condition (J) such that $G_1\circ G_2^{-1} \succ H^{-1}$.*

(iv) *G_1 is almost more convex than G_2.*

Then, if one of G_1 or G_2 is dilatory, we have (ii)$\Rightarrow$(i). *If one of G_1 or G_2 is dilatory and G_1 satisfies the Δ_2-condition, or if G_2 satisfies the Δ_2-condition, then* (i)$\Rightarrow$(ii). *If G_2 is dilatory and satisfies the Δ_2-condition, then* (ii)$\Leftrightarrow$(iv). *We always have* (ii)$\Leftrightarrow$(iii).

THEOREM 4.3. *Let F_1, F_2, G_1 and G_2 be φ-functions such that one of G_1 or G_2 is dilatory, and that one of G_1 or G_2 satisfies the Δ_2-condition. Then the following are equivalent.*

(i) *L_{F_1,G_1} and L_{F_2,G_2} are equivalent.*

(ii) *$F_1\asymp F_2$, and $G_1\circ G_2^{-1}$ is almost linear.*

(iii) *$F_1\asymp F_2$, and there exist N-functions H and K satisfying condition (J) such that $G_1\circ G_2^{-1} = H\circ K^{-1}$.*

(iv) *$F_1\asymp F_2$, and there exist N-functions H and K satisfying condition (J) such that $G_1\circ G_2^{-1} = H^{-1}\circ K$.*

(v) *$F_1\asymp F_2$, and G_1 is almost equivalent to G_2.*

(vi) *$F_1\asymp F_2$, and there exist N-functions H and K satisfying condition (J) and a number $c<\infty$ such that $c^{-1}G_1/G_2 \le H/K \le cG_1/G_2$.*

(vii) *$F_1\asymp F_2$, and there exist strictly monotone weight functions w_0 and w_1 satisfying condition (L) and a number $c<\infty$ such that $c^{-1}G_1/G_2 \le w_0w_1 \le cG_1/G_2$.*

(viii) *$F_1\asymp F_2$, and there exists an almost linear φ-function F and a number $c<\infty$ such that $c^{-1}F(t) \le tG_1(t)/G_2(t) \le cF(t)$ for all $t>0$.*

The condition that one of the φ-functions G_1 or G_2 satisfy the Δ_2-condition is necessary, as is shown by the following example. Let $G_1(t) = \mathrm{em}\,t$ and $G_2(t) = \mathrm{em}\,t^2$. By Theorem 4.6 below, L_{1,G_1} and L_{1,G_2} are both equivalent to $L_{1,\infty}$. But it is clear that $G_1\circ G_2^{-1}$ is far from being almost linear. The author does not know whether the condition that one of G_1 or G_2 be dilatory is needed.

We are also able to obtain certain results stating that in order to compare L_{F_1,G_1} and L_{F_2,G_2}, we need only compare the norms for a certain class of test functions.

DEFINITION. Let $\mathcal{T}_1$ be the set of functions $f:[0,\infty)\to[0,\infty)$ such that for some $0=a_0<a_1<\ldots<a_n$ we have

$$f(x)=\begin{cases}1/a_i & \text{if } a_{i-1}\le x<a_i \text{ and } 1\le i\le n,\\ 0 & \text{otherwise.}\end{cases}$$

If F is a φ-function, let $\mathcal{T}_F=\{F^{-1}\circ f: f\in\mathcal{T}_1\}=\{f\circ\widetilde{F}^{-1}: f\in\mathcal{T}_1\}$.

THEOREM 4.4. *Let F, G_1 and G_2 be φ-functions. Suppose that G_2 is dilatory, and that one of G_1 or G_2 satisfies the Δ_2-condition. Then the following are equivalent.*

(i) *For some $c<\infty$ we have $\|f\|_{F,G_1}\le c\|f\|_{F,G_2}$ whenever $f^*\in\mathcal{T}_F$.*

(ii) *For some $c<\infty$ we have $\|f\|_{F,G_1}\le c\|f\|_{F,G_2}$ for all measurable f.*

THEOREM 4.5. *Let F_1, F_2, G_1 and G_2 be φ-functions. Suppose that one of G_1 or G_2 is dilatory, and that one of G_1 or G_2 satisfies the Δ_2-condition. Then the following are equivalent.*

(i) *For some $c<\infty$ we have $c^{-1}\|f\|_{F_1,G_1}\le\|f\|_{F_2,G_2}\le c\|f\|_{F_1,G_1}$ whenever $f^*\in\mathcal{T}_{F_1}$.*

(ii) *For some $c<\infty$ we have $c^{-1}\|f\|_{F_1,G_1}\le\|f\|_{F_2,G_2}\le c\|f\|_{F_1,G_1}$ whenever $f^*\in\mathcal{T}_{F_2}$.*

(iii) *L_{F_1,G_1} and L_{F_2,G_2} are equivalent.*

Finally, we give a result for the weak-Orlicz–Lorentz spaces.

THEOREM 4.6. *Let F_1, F_2 and G be φ-functions. Then the following are equivalent.*

(i) *$L_{F_1,G}$ and $L_{F_2,\infty}$ are equivalent.*

(ii) *$F_1\asymp F_2$, and G is almost vertical.*

(iii) *$F_1\asymp F_2$, and $\|1/\widetilde{G}^{-1}\|_G<\infty$.*

(iv) *$F_1\asymp F_2$, and $\widetilde{G}^{-1}$ satisfies condition (L).*

It is clear that all the results given in Section 3 follow from these results. We are also able to answer a question of Raynaud, and prove the converse to Theorem 3.2.

THEOREM 4.7. *Let w be a weight function, and $0<p<\infty$. If $\Lambda_{w,p}$ is equivalent to an Orlicz space, then there are strictly monotone weight functions w_0 and w_1 satisfying condition (L) and a number $c<\infty$ such that*

$$c^{-1}\frac{W(t)}{t}\le w_0(t)w_1(t)\le c\frac{W(t)}{t}.$$

Proof. This follows immediately from the implication (i)⇒(vii) in Theorem 4.3, and from the observation that a strictly monotone weight function w satisfies condition (L) if and only if w^p satisfies condition (L) for any $0<p<\infty$. ∎

5. The proof of the results of Section 4. The proofs of the results of Section 4 are rather long. We will split the proof into many lemmas that are grouped into several subsections according to their nature. Many of the lemmas, if not obvious, are at least "believable without proof", and the reader may pass over them quickly. The key results are contained in Sections 5.3, 5.5 and 5.6.

These proofs could be shortened considerably if we assumed throughout that all φ-functions were dilatory and satisfied the Δ_2-condition, but then our results would be correspondingly weaker. In particular, Theorem 6.1 below would be much less general.

5.1. *The elementary propositions.* The first result is obvious, and requires no proof.

LEMMA 5.1.1. *Let G be a φ-function.*

(i) *If G is dilatory, then for all $c_1<\infty$ there is $c_2<\infty$ such that if*

$$\int_\Omega G\circ f(\omega)\,d\mu(\omega)\le c_1,$$

then $\|f\|_G\le c_2$.

(ii) *If G satisfies the Δ_2-condition, then for all $c_1<\infty$ there is $c_2<\infty$ such that if $\|f\|_G\le c_1$ then*

$$\int_\Omega G\circ f(\omega)\,d\mu(\omega)\le c_2.$$

Now we have the first result from Section 4.

Proof of Proposition 4.1. This is a simple consequence of Lemma 5.1.1. ∎

The following results describe the basic "algebra" that the Orlicz–Lorentz spaces satisfy. Essentially, they allow one to reduce comparison of Orlicz–Lorentz spaces to the problem of comparing $L_{1,G}$ to L_1. The proofs are straightforward, so we omit them.

LEMMA 5.1.2. *Suppose that F, G_1 and G_2 are φ-functions. Then for any $c<\infty$ we have $\|f\|_{F,G_1}\le c\|f\|_{F,G_2}$ for all measurable f (respectively, $f\in\mathcal{T}_F$) if and only if $\|f\|_{1,G_1}\le c\|f\|_{1,G_2}$ for all measurable f (respectively, $f\in\mathcal{T}_1$).*

LEMMA 5.1.3. *Suppose that G_1, G_2 and H are φ-functions.*

(i) *If H is dilatory, then if for some $c_1 < \infty$ we have $\|f\|_{1,G_1} \le c_1\|f\|_{1,G_2}$ for all measurable f (respectively, $f \in \mathcal{T}_1$), then for some $c_2 < \infty$ we have $\|f\|_{1,G_1\circ H} \le c_2\|f\|_{1,G_2\circ H}$ for all measurable f (respectively, $f \in \mathcal{T}_1$).*

(ii) *If H satisfies the Δ_2-condition, then if for some $c_1 < \infty$ we have $\|f\|_{1,G_1\circ H} \le c_1\|f\|_{1,G_2\circ H}$ for all measurable f (respectively, $f \in \mathcal{T}_1$), then for some $c_2 < \infty$ we have $\|f\|_{1,G_1} \le c_2\|f\|_{1,G_2}$ for all measurable f (respectively, $f \in \mathcal{T}_1$).*

5.2. *Conditions for functions to be dilatory, etc.* Here we collect the results that pertain to when a φ-function is dilatory or satisfies the Δ_2-condition. The first result is obvious.

LEMMA 5.2.1. *Let G be a φ-function.*

(i) *If there are $a > 1$, $c_1 > 1$ and $c_2 > 1$ such that $c_1G(a^n) \le G(c_2a^n)$ except for finitely many n, then G is dilatory.*

(ii) *If there are $a > 1$, $c_1 > 1$ and $c_2 > 1$ such that $c_1G(a^n) \ge G(c_2a^n)$ except for finitely many n, then G satisfies the Δ_2-condition.*

Now we show how the property of G being dilatory or satisfying the Δ_2-condition may be captured by the properties of $L_{F,G}$.

LEMMA 5.2.2. *Suppose that G is a φ-function. Then the following are equivalent.*

(i) *G is dilatory.*

(ii) *There is $c < \infty$ such that $\|f\|_{1,G} \le c$ for all $f : \mathbb{R} \to \mathbb{R}$ of the form*

$$f(x) = \begin{cases} a^{-1} & \text{if } 0 \le x < a, \\ b^{-1} & \text{if } a \le x < b, \\ 0 & \text{otherwise}, \end{cases}$$

where $b > a > 0$.

Proof. First we will show that (i)⇒(ii). Given a function f of the above form, we note that

$$\int_0^\infty G \circ f^* \circ \widetilde{G}^{-1}(x)\,dx \le \widetilde{G}(a)G(a^{-1}) + \widetilde{G}(b)G(b^{-1}) = 2\,.$$

Then the result follows immediately from Lemma 5.1.1.

To show that (ii)⇒(i), we will consider functions of the form

$$f(x) = \begin{cases} G^{-1}(3^m) & \text{if } 0 \le x < 1/G^{-1}(3^m), \\ G^{-1}(3^n) & \text{if } 1/G^{-1}(3^m) \le x < 1/G^{-1}(3^n), \\ 0 & \text{otherwise}, \end{cases}$$

where $m > n$ are integers. Then $\|f\|_{1,G} \le c$, and so

$$1 \ge \int_0^\infty G(c^{-1}f^* \circ \widetilde{G}^{-1}(x))\,dx \ge 3^{-m}G(c^{-1}G^{-1}(3^m)) + \tfrac{2}{3}\cdot 3^{-n}G(c^{-1}G^{-1}(3^n))\,.$$

Therefore, for all except one $n \in \mathbb{N}$ we have

$$\tfrac{2}{3}\cdot 3^{-n}G(c^{-1}G^{-1}(3^n)) \le \tfrac{1}{2}\,,$$

that is, $c^{-1}G^{-1}(3^n) \le G^{-1}(\frac{3}{4}\cdot 3^n)$. By Lemma 5.2.1, G^{-1} satisfies the Δ_2-condition, and hence G is dilatory. ■

LEMMA 5.2.3. *If F, G_1 and G_2 are φ-functions such that G_1 is dilatory and for some $c < \infty$ we have $\|f\|_{F,G_2} \le c\|f\|_{F,G_1}$ for all $f \in \mathcal{T}_F$, then G_2 is dilatory.*

Proof. This follows immediately from Lemmas 5.1.2 and 5.2.2. ■

LEMMA 5.2.4. *Suppose that G is a φ-function. Consider the following statements.*

(i) *G satisfies the Δ_2-condition.*

(ii) *Given $c > 1$, there are $d > 1$ and $N \in \mathbb{N}$ such that $\|f\|_{1,G} \ge c$ for all $f : \mathbb{R} \to \mathbb{R}$ of the form*

$$f(x) = \begin{cases} d^{-k_i} & \text{if } d^{k_{i-1}} \le x < d^{k_i} \text{ and } 1 < i \le N, \\ 0 & \text{otherwise}, \end{cases}$$

where $k_1 < \ldots < k_N$ are integers, and $k_0 = -\infty$.

Then (ii)⇒(i). *Furthermore, if G is dilatory, then* (i)⇒(ii).

Proof. First we will show that (i)⇒(ii) when G is dilatory. Choose d so that $\widetilde{G}(d^n) \ge 2\widetilde{G}(d^{n-1})$ for all $n \in \mathbb{Z}$. Then if f is of the above form, we have

$$\int_0^\infty G \circ f^* \circ \widetilde{G}^{-1}(x)\,dx \ge \sum_{i=1}^N (\widetilde{G}(d^{k_i}) - \widetilde{G}(d^{k_i-1}))G(d^{-k_i}) \ge N/2\,.$$

Thus if G satisfies the Δ_2-condition, then by Lemma 5.1.1, there is some $N \in \mathbb{N}$ such that for all f of the above form, we have $\|f\|_{1,G} \ge c$.

To show that (ii)⇒(i), pick $c > 2$. Then for any f of the above form,

$$1 < \int_0^\infty G(2^{-1}f^* \circ \widetilde{G}^{-1}(x))\,dx \le \sum_{i=1}^N \widetilde{G}(d^{k_i})G(2^{-1}d^{-k_i})\,.$$

Therefore, the cardinality of the set of $n \in \mathbb{N}$ such that $\widetilde{G}(d^n)G(2^{-1}d^{-n}) \le N^{-1}$ is less than N. By Lemma 5.2.1, this shows that G satisfies the Δ_2-condition. ■

LEMMA 5.2.5. *Suppose that F, G_1 and G_2 are φ-functions such that one of G_1 and G_2 is dilatory. If G_1 satisfies the Δ_2-condition, and for some $c < \infty$ we have $\|f\|_{F,G_2} \ge c\|f\|_{F,G_1}$ for all $f \in \mathcal{T}_F$, then G_2 satisfies the Δ_2-condition.*

Proof. By Lemma 5.2.3, G_1 is dilatory. Now, the result follows immediately from Lemmas 5.1.2 and 5.2.4. ∎

5.3. *Comparison conditions for $L_{1,G}$.* In this subsection, we give the key lemma that demonstrates the relationship between the almost convexity of G, and the comparison between $L_{1,G}$ and L_1. As a corollary, we will also obtain results that show that in the definition of the "almost" properties we can take the value of a to be arbitrarily large.

LEMMA 5.3.1. *Suppose that G is a φ-function. Then the following are equivalent.*

(i) *For some $c < \infty$, we have $\|f_{1,G}\| \le c\|f\|_1$ for all measurable f.*
(ii) *For some $c < \infty$, we have $\|f_{1,G}\| \le c\|f\|_1$ for all $f \in \mathcal{T}_1$.*
(iii) *For all sufficiently large a, there are $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do not have*

$$G(a^{n+m}) \ge a^{m-N} G(a^n)$$

is less than b^m.
(iv) *G is almost convex.*

The proof will require the next lemma.

LEMMA 5.3.2. *Let G be a φ-function. If G is almost convex, then given $a' > 1$, there are $a > 1$, $b > 1$, $c < \infty$ and $N \in \mathbb{N}$ such that $a > \max\{a', b\}$ and such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do not have*

$$G(a^{n+m}) \ge a^{m-N} G(a^n)$$

is less than cb^m.

There are similar results if G is almost concave or almost constant.

Proof. There are $a > 1$, $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set

$$A_m = \{n \in \mathbb{Z} : G(a^{n+m}) < a^{m-N} G(a^n)\}$$

is less than b^m. Pick $c \in \mathbb{N}$ such that $a^c > b$ and $a^c > a'$, and let

$$A'_m = \{n \in \mathbb{Z} : G(a^{c(n+m)}) < a^{c(m-N)} G(a^{cn})\}.$$

Then, if $n \in A'_m$, then at least one of $cn, cn+m, \ldots,$ or $cn+(c-1)m \in A_m$, and hence $|A'_m| < cb^m$. ∎

Proof of Lemma 5.3.1. Clearly, (i)⇒(ii) and (iii)⇒(iv). We will show (ii)⇒(iii). By Lemma 5.2.3, we know that G is dilatory. Thus we suppose that $a > 2$ and $G(at) \ge 2G(t)$ for all $t \ge 0$. Choose N so that $a^{N-1} > c$. We will prove the result by showing that there cannot be numbers $m \in \mathbb{N}$ and $n_1 < n_2 < \ldots < n_{a^{m-N+1}}$ such that

$$G(a^{n_i}) < a^{m-N} G(a^{n_i - m}).$$

For otherwise, consider the function

$$f(x) = \begin{cases} a^{n_i} & \text{if } a^{-n_{i+1}} \le x < a^{-n_i} \text{ and } 1 \le i \le a^{m-N+1}, \\ 0 & \text{otherwise,} \end{cases}$$

where we take $n_{a^{m-N+2}} = \infty$. Clearly, $\|f\|_1 \le a^{m-N+1}$. But also, we have the following inequalities:

$$\int_0^\infty G(a^{-m} f^* \circ \widetilde{G}^{-1}(x))\,dx \ge \sum_{i=1}^{a^{m-N+1}} \int_{\widetilde{G}(a^{-n_{i+1}})}^{\widetilde{G}(a^{-n_i})} G(a^{-m} f^* \circ \widetilde{G}^{-1}(x))\,dx$$
$$\ge \frac{1}{2} \sum_{i=1}^{a^{m-N+1}} \frac{G(a^{-m+n_i})}{G(a^{n_i})} \ge a/2 > 1,$$

where the penultimate inequality follows because $\widetilde{G}(a^{-n_i}) > 2\widetilde{G}(a^{-n_{i+1}})$. Thus $\|f\|_{1,G} \ge a^m \ge a^{N-1}\|f\|_1$, which is a contradiction.

Now we show that (iv)⇒(i). By Lemma 5.3.2, there are $a > b > 1$, $c < \infty$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set

$$A_m = \{n \in \mathbb{Z} : G(a^n) < a^{m-N} G(a^{n-m})\}$$

is less than cb^m. Let $\{k_1, k_2, \ldots\}$ be the (possible finite or empty) set of integers not in $\bigcup_{m=1}^\infty A_m$. Define the sequence of sets B_m by setting $B_1 = A_1$, and

$$B_m = \{k_{m-1}\} \cup A_m \setminus \bigcup_{m' < m} A_{m'}$$

for $m > 1$. Then $|B_m| \le cb^m$.

Choose $c_1 = (a-1)/a^4$. Suppose f is a measurable function such that

$$\|f\|_1 = \int_0^\infty f^*(x)\,dx \le c_1.$$

For each $n \in \mathbb{Z}$, let $m_n \in \mathbb{Z} \cup \{\infty\}$ be such that

$$a^{-m_n} \le a^{-n} f^*(a^{-n}) \le a^{1-m_n}.$$

Then

$$c_1 \geq \sum_{n=-\infty}^{\infty} \int_{a^{-n-1}}^{a^{-n}} f^*(x)\,dx \geq \sum_{n=-\infty}^{\infty} (a^{-n} - a^{-n-1}) f^*(a^n)$$
$$\geq \frac{a-1}{a} \sum_{n=-\infty}^{\infty} a^{-m_n}.$$

Therefore,

$$\sum_{n=-\infty}^{\infty} a^{-m_n} \leq \frac{a}{a-1} c_1 = a^{-3}.$$

In particular, we note that $m_n \geq 3$ for all $n \in \mathbb{Z}$.

Then

$$\int_0^\infty G \circ f^* \circ \widetilde{G}^{-1}(x)\,dx = \sum_{n=-\infty}^{\infty} \int_{\widetilde{G}(a^{-n-1})}^{\widetilde{G}(a^{-n})} G \circ f^* \circ \widetilde{G}^{-1}(x)\,dx$$
$$\leq \sum_{n=-\infty}^{\infty} \widetilde{G}(a^{-n}) G(f^*(a^{-n-1})) \leq \sum_{n=-\infty}^{\infty} G(a^{2+n-m_{n+1}})/G(a^n).$$

Now let $V = \{n \in \mathbb{Z} : n \notin B_m \text{ for all } m \leq m_{n+1} - 2\}$. Then

$$\int_0^\infty G \circ f^* \circ \widetilde{G}^{-1}(x)\,dx$$
$$\leq \sum_{n \in V} G(a^{2+n-m_{n+1}})/G(a^n) + \sum_{m=1}^{\infty} \sum_{n \in B_m \setminus V} G(a^{2+n-m_{n+1}})/G(a^n).$$

If $n \in V$, then either $n \notin A_{m_{n+1}-2}$, or $m_{n+1} = \infty$, and so

$$G(a^{2+n-m_{n+1}})/G(a^n) \leq a^{N+2-m_{n+1}}.$$

If $n \in B_m \setminus V$, then $m \leq m_{n+1} - 2$, and so

$$G(a^{2+n-m_{n+1}})/G(a^n) \leq G(a^{n-m})/G(a^n).$$

If we also know that $m > 1$, then $n \notin A_{m-1}$, and so

$$G(a^{2+n-m_{n+1}})/G(a^n) \leq G(a^{1+n-m})/G(a^n) \leq a^{N+1-m}.$$

Therefore,

$$\int_0^\infty G \circ f^* \circ \widetilde{G}^{-1}(x)\,dx$$
$$\leq \sum_{n \in V} a^{N+2-m_{n+1}} + \sum_{n \in B_1 \setminus V} G(a^{n-1})/G(a^n) + \sum_{m=2}^{\infty} \sum_{n \in B_m \setminus V} a^{N+1-m}$$

$$\leq a^{N-1} + \sum_{n \in B_1} G(a^{n-1})/G(a^n) + \sum_{m=2}^{\infty} c b^m a^{N+1-m},$$

which is a finite number whose value does not depend on f. However, by Lemma 5.2.1, G is dilatory, and hence by Lemma 5.1.1, $\|f\|_{1,G}$ is bounded by some number that does not depend on f. ∎

5.4. *Convexity and concavity conditions.* In this subsection, we give basic results about convexity and concavity, and their "almost" equivalents. First, we give a technical lemma whose proof is obvious.

Lemma 5.4.1. *Let G be a φ-function. Define a map $f : \mathbb{Z} \to \mathbb{Z}$ so that for all $n \in \mathbb{Z}$ we have*

$$a^{f(n)-1} \leq G(a^n) < a^{f(n)}.$$

(i) *If G is dilatory, then there is $L \in \mathbb{N}$ such that $f(n_1) \neq f(n_2)$ if $|n_1 - n_2| \geq L$.*

(ii) *If G satisfies the Δ_2-condition, then there is $M \in \mathbb{N}$ such that $|f(n_1) - f(n_2)| \leq M|n_1 - n_2|$ for all $n_1, n_2 \in \mathbb{Z}$.*

Next we give some results about convexity.

Lemma 5.4.2. *Let G_1 and G_2 be φ-functions. Consider the following statements.*

(i) *G_1 is equivalently more convex than G_2.*

(ii) *There is $c < \infty$ such that $G_1 \circ G_2^{-1}(st) \geq c^{-1} s G_1 \circ G_2^{-1}(t)$ for all $s \geq 1$ and $t \geq 0$.*

(iii) *There is $c < \infty$ such that $G_1(uv)/G_1(v) \geq c^{-1} G_2(uv)/G_2(v)$ for all $u \geq 1$ and $v > 0$.*

(iv) *There are $a > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ and $n \in \mathbb{Z}$,*

$$\frac{G_1(a^{n+m})}{G_1(a^n)} \geq a^{-N} \frac{G_2(a^{n+m})}{G_2(a^n)}.$$

Then we have (i)⇔(ii)⇔(iii)⇒(iv). If one of G_1 or G_2 satisfies the Δ_2-condition, then (iv)⇒(iii).

Proof. The implications (i)⇒(ii) and (iii)⇒(iv) are obvious. The implications (ii)⇔(iii) follow by setting $t = G_2(v)$ and $st = G_2(uv)$.

To show (ii)⇒(i), we let

$$H_0(t) = \inf_{s>1} \frac{G_1 \circ G_2^{-1}(st)}{s} \quad \text{and} \quad H(t) = \int_0^t \frac{H_0(s)}{s}\,ds.$$

Then it is easy to see that H is convex, and that H is equivalent to $G_1 \circ G_2^{-1}$.

Now suppose that G_1 satisfies the Δ_2-condition. We show (iv)⇒(iii). Let $L \in \mathbb{N}$ be such that $G_1(at) \leq a^L G_1(t)$ for all $t \geq 0$. Suppose that for some

$u > 1$ and $v > 0$ we have

$$\frac{G_1(uv)}{G_1(v)} < a^{-N-3L}\frac{G_2(uv)}{G_2(v)}.$$

Let m and n be such that $a^m \le u < a^{m+1}$ and $a^n \le v < a^{n+1}$. Then

$$a^{-3L}\frac{G_1(a^{m+n+2})}{G_1(a^n)} \le \frac{G_1(a^{m+n})}{G_1(a^{n+1})} < a^{-N-3L}\frac{G_2(a^{m+n+2})}{G_2(a^n)},$$

which is a contradiction.

The argument for G_2 satisfying the Δ_2-condition is similar. ∎

Now we start looking at the "almost" properties. First we relate almost convexity to almost concavity.

LEMMA 5.4.3. *Suppose that G is a φ-function.*

(i) *If G is almost convex and satisfies the Δ_2-condition, then G^{-1} is almost concave.*

(ii) *If G is almost concave, then G^{-1} is almost convex.*

This will follow from the next lemma.

LEMMA 5.4.4. *Suppose that G is a φ-function and that $a > 1$. Consider the following statements.*

(i) *There are $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do not have*

$$G(a^{n+m}) \ge a^{m-N}G(a^n)$$

is less than b^m.

(ii) *There are $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do not have*

$$G^{-1}(a^{n+m}) \le a^{m+N}G^{-1}(a^n)$$

is less than b^m.

Then (ii)⇒(i). *If, in addition, G satisfies the Δ_2-condition, then* (i)⇒(ii).

Proof. We will show that (i)⇒(ii) when G satisfies the Δ_2-condition. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined so that $a^{f(n)-1} \le G^{-1}(a^n) < a^{f(n)}$. Since G^{-1} is dilatory, by Lemma 5.4.1, we know that there is L such that $|f^{-1}(\{n\})| \le L$ for every $n \in \mathbb{Z}$. Let

$$A_m = \{n \in \mathbb{Z} : G^{-1}(a^{n+m}) > a^{m+1+N}G^{-1}(a^n)\}.$$

Then it can easily be shown that

$$f(A_m) \subseteq \{n \in \mathbb{Z} : G(a^{n+m+N}) < a^m G(a^n)\},$$

and hence $|A_m| \le Lb^{m+N} \le b_0^m$, where $b_0 = Lb^{N+1}$.

To show (ii)⇒(i) is similar. Let $g : \mathbb{Z} \to \mathbb{Z}$ be defined so that $a^{g(n)} \le G(a^n) < a^{g(n)+1}$. Since G is almost convex, it follows that G is dilatory. Now the proof proceeds as in (i)⇒(ii). ∎

Next, we deal with the composition of the "almost" properties. One of the main problems here is that given two φ-functions, each with an "almost" property, the a from the definition of the "almost" property for each φ-function could be different. Fortunately, we have already developed the tools to deal with this. First, for "almost convexity", the implication (iv)⇒(iii) in Lemma 5.3.1 tells us that the a may be any arbitrarily large number. If the φ-function is dilatory, then Lemma 5.4.4 also allows the a to be any arbitrarily large number for the "almost concavity" property. Finally, for other "almost" properties, Lemma 5.3.2 allows us to choose the a to be larger than any given number.

Thus we have the following result.

LEMMA 5.4.5. *Let G be a φ-function. If G is almost convex and almost concave, then G is almost linear.*

LEMMA 5.4.6. *Let G_1 and G_2 be φ-functions.*

(i) *If G_1 and G_2 are almost convex, then $G_1 \circ G_2$ is almost convex.*

(ii) *If G_1 and G_2 are almost concave, and if G_2 is dilatory, then $G_1 \circ G_2$ is almost concave.*

Proof. First we will prove part (ii). By the explanation given above, we may suppose that for one $a > 1$, there are $b_1 > 1$, $N_1 \in \mathbb{N}$, $b_2 > 1$ and $N_2 \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that

$$G_1(a^{n+m}) > a^{m+N_1}G_1(a^n)$$

is less than b_1^m, and the cardinality of the set of $n \in \mathbb{Z}$ such that

$$G_2(a^{n+m}) > a^{m+N_2}G_2(a^n)$$

is less than b_2^m. Define $f : \mathbb{Z} \to \mathbb{Z}$ so that $a^{f(n)-1} \le G_2(a^n) < a^{f(n)}$ for all $n \in \mathbb{Z}$. Then by Lemma 5.4.1, there is $L \in \mathbb{N}$ such that $|f^{-1}(\{n\})| \le L$ for all $n \in \mathbb{Z}$. Then we see that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that

$$G_1 \circ G_2(a^{n+m}) > a^{m+N_1+N_2+1}G_1 \circ G_2(a^n)$$

is less than $b_1^{m+N_1+N_2+1} + Lb_2^{m+N_2+1}$. For, if the above holds, and $G_2(a^{n+m}) \le a^{m+N_2}G_2(a^n)$, then

$$G_1(a^{f(n)-1+m+N_2+1}) > a^{m+N_2+1+N_1}G_1(a^{f(n)-1}).$$

The result follows.

To show (i), we note that as G_2 is almost convex, we already know that G_2 is dilatory. Now the argument follows as in part (ii). ∎

Now, we prove two lemmas that are "almost" analogues of Lemma 5.4.2.

LEMMA 5.4.7. *Let G_1 and G_2 be φ-functions such that G_2 is dilatory and satisfies the Δ_2-condition.*

(i) *$G_1 \circ G_2^{-1}$ is almost convex if and only if G_1 is almost more convex than G_2.*

(ii) *$G_1 \circ G_2^{-1}$ is almost concave if and only if G_1 is almost less concave than G_2.*

Proof. We will show that if $G_1 \circ G_2^{-1}$ is almost convex, then G_1 is almost more convex than G_2. All the other assertions follow similarly.

So, there are $a > 1$, $b > 2$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of

$$A_m = \{n \in \mathbb{Z} : G_1 \circ G_2^{-1}(a^{n+m}) < a^{m-N} G_1 \circ G_2^{-1}(a^n)\}$$

is less than b^m. Define $f : \mathbb{Z} \to \mathbb{Z}$ so that $a^{f(n)-1} \le G_2(a^n) < a^{f(n)}$ for all $n \in \mathbb{Z}$. Then, by Lemma 5.4.1, there are $L, M \in \mathbb{N}$ such that $f(n+L) > f(n)$ for all $n \in \mathbb{Z}$, and $f(m+n) - f(n) \le Mm$ for all $m \in \mathbb{N}$ and $n \in \mathbb{Z}$.

Now, for each $m \in \mathbb{N}$, consider the cardinality of the set

$$B_m = \left\{ n \in \mathbb{Z} : \frac{G_1(a^{L(n+m)})}{G_1(a^{Ln})} < a^{-N-2} \frac{G_2(a^{L(n+m)})}{G_2(a^{Ln})} \right\}.$$

If $n \in B_m$, let $n' = f(Ln)$ and $m' = f(L(m+n)) - n'$. Then

$$G_1 \circ G_2^{-1}(a^{n'+m'-1}) < a^{m'-1-N} G_1 \circ G_2^{-1}(a^{n'}).$$

Clearly, this is impossible if $m' \le 1$, and otherwise, this implies that $n' \in A_{m'-1}$. Since $m' \le Mm$, we see that $|B_m| < \sum_{m=1}^{Mm} b^{m'} \le b_0^m$, where $b_0 = b^{M+1}/(b-1)$. ∎

LEMMA 5.4.8. *Let G_1 and G_2 be φ-functions such that one of G_1 or G_2 is dilatory and one of G_1 or G_2 satisfies the Δ_2-condition. Then $G_1 \circ G_2^{-1}$ is almost linear if and only if G_1 is almost equivalent to G_2.*

Proof. We note that if one of G_1 or G_2 is dilatory, then both are, and if one of G_1 or G_2 satisfies the Δ_2-condition, then both do. Now the proof proceeds as in Lemma 5.4.7. ∎

5.5. *Condition (L) and condition (J).* In this subsection, we describe how the notions of satisfying condition (L) or condition (J) relate to the "almost" properties.

LEMMA 5.5.1. *Let G be a φ-function. Then the following are equivalent.*

(i) *G is almost constant.*
(ii) $\|1/\widetilde{G}\|_{G^{-1}} < \infty$.
(iii) *$\widetilde{G}$ satisfies condition (L).*

Proof. This proof is very similar to the proof of Lemma 5.3.1, and so we will omit many details. First we show that (i)⇒(ii). Following the same argument as the proof of (iv)⇒(i) in Lemma 5.3.1, we construct numbers $a > b > 1$, $c < \infty$ and $N \in \mathbb{N}$ and a sequence of sets B_m such that $|B_m| \le cb^m$, and such that if $n \in B_m$ for $m > 1$ then

$$G^{-1}(a^{-N} G(a^n)) \le a^{n-m+1}.$$

Hence,

$$\begin{aligned}\int_0^\infty G^{-1}(a^{-N}/\widetilde{G}(x))\,dx &= \sum_{n=-\infty}^{\infty} \int_{a^{-n}}^{a^{1-n}} G^{-1}(a^{-N}/\widetilde{G}(x))\,dx \\ &\le \sum_{m=1}^{\infty} \sum_{n \in B_m} a^{1-n} G^{-1}(a^{-N} G(a^n)) \\ &\le \sum_{n \in B_1} a^{1-n} G^{-1}(a^{-N} G(a^n)) \\ &\quad + \sum_{m=2}^{\infty} cb^m a^{1-n} a^{n-m+1},\end{aligned}$$

which is a finite number. By Lemma 5.2.1, G^{-1} is dilatory, and so the result follows by Lemma 5.1.1.

That (ii)⇒(iii) is straightforward. To show that (iii)⇒(i), choose $a > 2$, and note that for some $N, M \in \mathbb{N}$ we have

$$\int_0^\infty G^{-1}(a^{-N}/\widetilde{G}(x))\,dx \le a^M.$$

Then following a similar line of reasoning to that of the proof of (ii)⇒(iii) in Lemma 5.3.1, it is possible to show that there cannot be numbers $m \in \mathbb{N}$ and $n_1 < n_2 < \ldots < n_{a^{m+2+M}}$ such that $G(a^{n_i}) > a^N G(a^{n_i - m})$. ∎

LEMMA 5.5.2. *Let H be an N-function. Then the following are equivalent.*

(i) *H is almost concave.*
(ii) *H satisfies condition (J).*
(iii) *$\widetilde{H}^{*-1}$ satisfies condition (L).*

Proof. By Lemma 5.4.3, H is almost concave if and only if H^{-1} is almost convex. Clearly, H^{-1} is almost convex if and only if H^{*-1} is almost constant. Now the result follows by Lemma 5.5.1. ∎

5.6. *Condition (J) and the "almost" properties.* Now, we are ready to establish the relationship between being almost convex or almost concave, and being more or less convex than some N-function satisfying condition (J).

LEMMA 5.6.1. *Let G be a φ-function.*

(i) *G is almost convex if and only if there is an N-function H satisfying condition (J) such that $G \succ H^{-1}$.*

(ii) *If G is almost concave, then there is an N-function H satisfying condition (J) such that $G \prec H$.*

(iii) *If there is an N-function H satisfying condition (J) such that $G \prec H$, then G^{-1} is almost convex.*

Proof. We first note that if there is an N-function H satisfying condition (J) such that either $G \succ H^{-1}$ or $G^{-1} \prec H$, then by Lemma 5.4.2, G is almost convex. We will prove the other implication of part (i).

If G is almost convex, then there are $a > 1$, $b > 2$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that

$$G(a^{n+m}) < a^{m-N} G(a^n)$$

is less than b^m. Now we define a function $L : \{ a^n : n \in \mathbb{Z}\} \to (0, \infty)$ by

$$L(a^n) = \begin{cases} \inf\limits_{0=n_0<\ldots<n_K=n} \prod\limits_{k=1}^{K} a^N \min\left\{a^{n_k-n_{k-1}}, \dfrac{G(a^{n_k})}{G(a^{n_{k-1}})}\right\} & \text{if } n \geq 0, \\ \sup\limits_{0=n_0>\ldots>n_K=n} \prod\limits_{k=1}^{K} a^{-N} \max\left\{a^{n_k-n_{k-1}}, \dfrac{G(a^{n_k})}{G(a^{n_{k-1}})}\right\} & \text{if } n < 0. \end{cases}$$

We may extend the domain of L to $[0, \infty)$ "log-linearly", that is, by setting $L(0) = 0$, and

$$L(a^n t) = L(a^n) \exp\left(\frac{\log t}{\log a} \log\left(\frac{L(a^{n+1})}{L(a^n)}\right)\right),$$

for $n \in \mathbb{Z}$ and $1 \leq t < a$. We notice that $L(a^{n+1}) > L(a^n)$ for all $n \in \mathbb{Z}$, and hence L is a φ-function.

Now, we note that if $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, then

$$\frac{L(a^{n+m})}{L(a^n)} \leq \inf_{n=n_0<\ldots<n_K=n+m} \prod_{k=1}^{K} a^N \min\left\{a^{n_k-n_{k-1}}, \frac{G(a^{n_k})}{G(a^{n_{k-1}})}\right\}.$$

Thus, $L(a^{n+m}) \leq a^{m+N} L(a^n)$, and so, by Lemma 5.4.2, L^{-1} is equivalent to a convex function. We also have

$$\frac{L(a^{n+m})}{L(a^n)} \leq a^N \frac{G(a^{n+m})}{G(a^n)},$$

and therefore, by Lemma 5.4.2 and since L satisfies the Δ_2-condition, $G \succ L$.

We also notice that, since

$$\min\left\{a^{n_2-n_0}, \frac{G(a^{n_2})}{G(a^{n_0})}\right\} \geq \min\left\{a^{n_2-n_1}, \frac{G(a^{n_2})}{G(a^{n_1})}\right\} \min\left\{a^{n_1-n_0}, \frac{G(a^{n_1})}{G(a^{n_0})}\right\}$$

for $n_0 \leq n_1 \leq n_2$, we have

$$\frac{L(a^{n+m})}{L(a^n)} \geq a^{-N} \inf_{n=n_0<\ldots<n_K=n+m} \prod_{k=1}^{K} a^N \min\left\{a^{n_k-n_{k-1}}, \frac{G(a^{n_k})}{G(a^{n_{k-1}})}\right\}.$$

Therefore, if $L(a^{n+m}) < a^{m-N} L(a^n)$, then for some $n \leq n' < n'+m' \leq n+m$, we have $G(a^{n'+m'}) < a^{m'-N} G(a^{n'})$. Therefore, the cardinality of the set of $n \in \mathbb{Z}$ satisfying $L(a^{n+m}) < a^{m-N} L(a^n)$ is less than $m(b + b^2 + \ldots + b^m)$, which is less than b_0^m for $b_0 = 2b^2/(b-1)$.

Therefore, L is almost convex. Now, we define the φ-function $H(t) = L^{-1}(t \operatorname{lm} t)$. It is clear that $L \succ H^{-1}$, and hence $G \succ H^{-1}$. Since $t \operatorname{lm} t$ is easily seen to be almost concave, it follows by Lemmas 5.4.3 and 5.4.6 that H is almost convex. Clearly H is an N-function, and so by Lemma 5.5.2, H satisfies condition (J).

The proof of part (ii) is similar. We know that there are $a > 1$, $b > 2$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that $G(a^{n+m}) > a^{m+N} G(a^n)$ is less than b^m. Now we define $L : \{ a^n : n \in \mathbb{Z}\} \to (0, \infty)$ by

$$L(a^n) = \begin{cases} \sup\limits_{0=n_0<\ldots<n_K=n} \prod\limits_{k=1}^{K} a^{-N} \max\left\{a^{n_k-n_{k-1}}, \dfrac{G(a^{n_k})}{G(a^{n_{k-1}})}\right\} & \text{if } n \geq 0, \\ \inf\limits_{0=n_0>\ldots>n_K=n} \prod\limits_{k=1}^{K} a^{N} \min\left\{a^{n_k-n_{k-1}}, \dfrac{G(a^{n_k})}{G(a^{n_{k-1}})}\right\} & \text{if } n < 0, \end{cases}$$

and extend L "log-linearly". By the same methods as in the proof of part (i), we see that L is convex, that $G \prec L$, and that L is almost concave. Finally, we set $H(t) = L(t) \operatorname{lm} L(t)$ to obtain the result. ∎

LEMMA 5.6.2. *Let G be a φ-function. Then the following are equivalent.*

(i) *G is almost linear.*

(ii) *There are N-functions H and K satisfying condition (J) such that $G = H \circ K^{-1}$.*

(iii) *There are N-functions H and K satisfying condition (J) such that $G = H^{-1} \circ K$.*

(iv) *There are N-functions H and K satisfying condition (J) and a number $c < \infty$ such that $c^{-1} G(t)/t \leq H(t)/K(t) \leq c G(t)/t$ for all $t > 0$.*

(v) *There are strictly monotone weight functions w_0 and w_1 satisfying condition (L) and a number $c < \infty$ such that $c^{-1} G(t)/t \leq w_0(t) w_1(t) \leq c G(t)/t$ for all $t \geq 0$.*

Before proving this result, we will require a couple of technical lemmas.

LEMMA 5.6.3. *If G_1 and G_2 are equivalent φ-functions, and if one of G_1 or G_2 satisfies the Δ_2-condition, then there is $c < \infty$ such that $c^{-1}G_1(t) \le G_2(t) \le cG_1(t)$ for all $t \ge 0$.*

LEMMA 5.6.4. *Suppose that $F : [0,\infty) \to [0,\infty)$ is a function such that for some $c_1 > 1$ and $c_2 > 1$ we have $F(c_1 t) \ge c_2 F(t)$ for all $t \ge 0$. Then there is a number $c < \infty$ and a dilatory φ-function G such that $G(c^{-1}t) \le F(t) \le G(ct)$ for all $t \ge 0$.*

Proof. We that $F(c_1^n t) \ge c_2^n F(t)$ for all $n \in \mathbb{N}$ and $t \ge 0$. Then it is clear that

$$G(t) = \sup_{s\ge 1} s^{-\log c_2/\log c_1} F(st)$$

satisfies the conclusion of the lemma. ■

Proof of Lemma 5.6.2. The implications (ii)⇒(i) and (iii)⇒(i) follow from Lemmas 5.4.5 and 5.4.6. The implications (iv)⇒(i) and (v)⇒(i) are obvious.

To show that (i)⇒(ii), we note that, since G is almost convex, by Lemma 5.6.1(i), there is an N-function K_0 satisfying condition (J) such that $G \succ K_0^{-1}$. If we let $K(t) = K_0(t \operatorname{lm} t)$, then $H = G \circ K$ is an N-function. Since G is almost concave, it follows by Lemma 5.4.6 that H satisfies condition (J). The implication (i)⇒(iii) is similar, using Lemma 5.6.1(ii).

To show (i)⇒(iv), we note that since G is almost concave, by Lemma 5.6.1(ii), there is an N-function H_0 satisfying condition (J) such that $G \prec H_0$. Then, from Lemmas 5.4.2 and 5.6.4, it follows that $tH_0(t)/G(t)$ is equivalent to a convex function K_0. Since G is almost convex, K_0 is almost concave. Now we let $H(t) = H_0(t)\operatorname{lm} t$ and $K(t) = K_0(t)\operatorname{lm} t$, and the result follows by Lemma 5.6.3.

To show that (iv)⇒(v), by Lemma 5.4.2, we may assume that H and K are convex. Thus, if we let $w_0(t) = (\operatorname{lm} t)H(t)/t$ and $w_1(t) = t/(\operatorname{lm} t)K(t)$, then $\widetilde{w}_0$ and $\widetilde{1/w_1}$ are both almost constant φ-functions. Then it follows from Lemma 5.5.1 that w_0 and w_1 satisfy condition (L). ■

5.7. *The proof of the results in Section* 4. Now we are ready to piece together all the lemmas we have just proved.

Proof of Theorem 4.2. First we will show that (ii)⇒(i). By Lemma 5.2.1, we know that $G_2 \circ G_1^{-1}$ is dilatory, and hence if G_1 is dilatory, then so is G_2. Therefore, we may assume that G_2 is dilatory.

By Lemma 5.3.1, there is $c_1 < \infty$ such that $\|f\|_{1,G_1\circ G_2^{-1}} \le c_1\|f\|_1$ for all measurable f. Since G_2 is dilatory, the result follows by Lemmas 5.1.3 and 5.1.2.

Now we show that (i)⇒(ii). By Lemma 5.2.5, G_2 satisfies the Δ_2-condition. Therefore, by Lemmas 5.1.2 and 5.1.3, there is $c_1 < \infty$ such that $\|f\|_{1,G_1\circ G_2^{-1}} \le c_1\|f\|_1$ for all measurable f. Now the result follows by Lemma 5.3.1.

The implication (ii)⇔(iv) follows from Lemma 5.4.7(i), and (ii)⇔(iii) follows from Lemma 5.6.1. ■

Proof of Theorem 4.3. First we show that (ii)⇒(i). If $G_1 \circ G_2^{-1}$ is almost linear, then by Lemma 5.2.1, $G_1 \circ G_2^{-1}$ satisfies the Δ_2-condition. Therefore, by Lemma 5.4.3, $G_2 \circ G_1^{-1}$ is almost convex. Hence, by Theorem 4.2, L_{F_1,G_1} and L_{F_1,G_2} are equivalent. Clearly, both G_1 and G_2 are dilatory, and by Proposition 4.1, L_{F_1,G_2} and L_{F_2,G_2} are equivalent. The result follows.

Next we show that (i)⇒(ii). First notice that, since $\widetilde{F}_1^{-1}(t) = \|\chi_{[0,t]}\|_{F_1,G_1}$ and $\widetilde{F}_2^{-1}(t) = \|\chi_{[0,t]}\|_{F_2,G_2}$, we have $F_1 \asymp F_2$. Now suppose without loss of generality that G_1 is dilatory. Then by Proposition 4.1, L_{F_1,G_1} and L_{F_2,G_1} are equivalent, and hence L_{F_2,G_1} and L_{F_2,G_2} are equivalent. Now, by Lemmas 5.2.3 and 5.2.5, both G_1 and G_2 are dilatory, and both satisfy the Δ_2-condition. Therefore, $G_2 \circ G_1^{-1}$ and $G_1 \circ G_2^{-1}$ satisfy the Δ_2-condition. By Theorem 4.2, both $G_1 \circ G_2^{-1}$ and $G_2 \circ G_1^{-1}$ are almost convex. Now the result follows by Lemmas 5.4.2 and 5.4.5.

The implications (ii)⇔(v) follow from Lemma 5.4.8, the implication (v)⇒(viii) follows from Lemmas 5.6.4 and 5.6.3, and the implication (viii)⇒(v) is obvious. Finally, (viii)⇔(iii)⇔(iv)⇔(vi) ⇔(vii) all follow from Lemma 5.6.2. ■

Proof of Theorem 4.4. The implication (ii)⇒(i) is obvious, so we show that (i)⇒(ii). As in the proof of Theorem 4.2, we may suppose that G_2 satisfies the Δ_2-condition. By Lemma 5.1.2, we may assume without loss of generality that $F(t) = t$. Now the result follows by Lemmas 5.1.3 and 5.3.1, in the same manner as in the proof of Theorem 4.2. ■

Proof of Theorem 4.5. The implications (iii)⇒(i) and (iii)⇒(ii) are obvious. We show that (i)⇒(ii). First notice that $\chi_{[0,t)} \in \mathcal{T}_{F_1}$, and so as $\widetilde{F}_1^{-1}(t) = \|\chi_{[0,t)}\|_{F_1,G_1}$ and $\widetilde{F}_2^{-1}(t) = \|\chi_{[0,t)}\|_{F_2,G_2}$, we see that F_1 and F_2 are equivalent. Then it is clear that there is $c < \infty$ such that $f^* \in \mathcal{T}_{F_1}$ if and only if there is $g^* \in \mathcal{T}_{F_2}$ such that $c^{-1}f^* \le g^* \le cf^*$. Similarly, (ii)⇒(i).

Now we show that (i) and (ii)⇒(iii). Suppose, without loss of generality, that G_1 is dilatory. Then by Proposition 4.1, L_{F_1,G_1} and L_{F_2,G_1} are equivalent, and hence L_{F_2,G_1} and L_{F_2,G_2} are equivalent. Now, by Lemma 5.2.3, it follows that if one of G_1 or G_2 is dilatory, then both are. Then the result follows by Theorem 4.4. ■

Proof of Theorem 4.6. We show that (i)⇔(iii). By Proposition 4.1, if F_1 and F_2 are equivalent, then $L_{F_1,\infty}$ and $L_{F_2,\infty}$ are equivalent. Also, if $L_{F_1,G}$ and $L_{F_2,\infty}$ are equivalent, then since $\widetilde{F}_1^{-1}(t) = \|\chi_{[0,t]}\|_{F_1,G}$ and

$\widetilde{F}_2^{-1}(t) = \|\chi_{[0,t]}\|_{F_2,\infty}$, we see that F_1 and F_2 are equivalent. Thus, without loss of generality, we may assume that $F_1 = F_2 = F$.

Now we note that we always have $\|f\|_{F,\infty} \le \|f\|_{F,G}$. This follows because $f^* \ge f^*(x)\chi_{[0,x]}$ for all $x \ge 0$, and hence

$$\|f\|_{F,G} \ge \|f^*(x)\chi_{[0,x]}\|_{F,G} \ge f^*(x)\widetilde{F}^{-1}(x).$$

Now we show that $L_{F,G}$ is equivalent to $L_{F,\infty}$ if and only if $\|1/\widetilde{F}^{-1}\|_{F,G} < \infty$. That the first statement implies the second is obvious, because $\|1/\widetilde{F}^{-1}\|_{F,\infty} = 1$. To show the converse, note that if $\|f\|_{F,\infty} \le 1$, then $f^*(x) \le 1/\widetilde{F}^{-1}$.

But $\|1/\widetilde{F}^{-1}\|_{F,G} = \|1/\widetilde{G}^{-1}\|_G$, and the result follows. The other implications follow by Lemma 5.5.2. ∎

6. Is every r.i. space equivalent to an Orlicz–Lorentz space? We can answer the question in the negative easily, as follows. It is well known that $L_{1,\infty}$ is not separable. Then it is not hard to see that $L^0_{1,\infty}$, the closure of the simple functions in $L_{1,\infty}$, is not an Orlicz–Lorentz space.

However, the reader may consider this cheating. So to avoid all this "infinite-dimensional nonsense", we might ask the following question. Is there a rearrangement invariant space X such that for all Orlicz–Lorentz spaces $L_{F,G}$, the norms $\|\cdot\|_X$ and $\|\cdot\|_{F,G}$ are inequivalent on $X \cap L_{F,G}$? We answer this question in the positive by the following example.

THEOREM 6.1. *There is a rearrangement invariant Banach space X, where the measure space is $[0,\infty)$ with Lebesgue measure, such that for every Orlicz–Lorentz space $L_{F,G}$, the norms $\|\cdot\|_X$ and $\|\cdot\|_{F,G}$ are inequivalent on the vector space of simple functions.*

Proof. We define the following norm for measurable functions f:

$$\|f\|_X = \sup\{\|f^*g\|_1/\|g\|_2 : g \in \mathcal{T}_{t^2}\}.$$

We let X be the vector space of all measurable functions f such that $\|f\|_X < \infty$, modulo functions that are zero almost everywhere. Then it is an easy matter to see that X is a rearrangement invariant space such that $\|g\|_X = \|g\|_2$ for all $g \in \mathcal{T}_{t^2}$. Thus, if for some φ-functions F and G we have

$$c_1^{-1}\|f\|_X \le \|f\|_{F,G} \le c_1\|f\|_X$$

for all simple functions f, then by Theorem 4.5, there is a constant $c_2 < \infty$ such that

$$c_2^{-1}\|f\|_X \le \|f\|_2 \le c_2\|f\|_X$$

for all simple functions f. We will show that this cannot happen.

Define $\mathcal{T}'$ to be the set of functions $h : [0,\infty) \to [0,\infty)$ such that for some integers $k_1 < \ldots < k_n$, and setting $k_0 = -\infty$, we have

$$h(x) = \begin{cases} 2^{-k_i} & \text{if } 4^{k_{i-1}} \le x < 4^{k_i} \text{ and } 1 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is an easy matter to see that if $g \in \mathcal{T}_{t^2}$, then there is $h \in \mathcal{T}'$ such that $h(4x)/2 \le g(x) \le 2h(x/4)$. Therefore,

$$\tfrac{1}{4}\|f\|_X \le \sup\{\|f^*h\|_1/\|h\|_2 : h \in \mathcal{T}'\} \le 4\|f\|_X.$$

Now, for each $N \in \mathbb{N}$, let f_N be the simple function

$$f_N(x) = \begin{cases} 1 & \text{if } 0 \le t < 4, \\ k^{-1/2}2^{-k} & \text{if } 4^k \le t < 4^{k+1} \text{ and } 1 \le k \le N, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that $\|f_N\|_2 \to \infty$ as $N \to \infty$. However, a simple, but laborious, calculation shows that there is $c < \infty$ such that $\|f_N^*h\|_1/\|h\|_2 \le c$ for all $h \in \mathcal{T}'$, and hence $\|f_N\|_X \le 4c$. ∎

7. The definition of Torchinsky and Raynaud. The definition of the Orlicz–Lorentz spaces presented here is not the only possible definition. In fact, given any weight function w and any φ-functions H and G, one can form the functional

$$\|f\|_{w,H,G} = \|w \cdot (f^* \circ H)\|_G.$$

We have investigated the case when $w(x) = 1$. However, Torchinsky [T] gave the following definition for the Orlicz–Lorentz functional. If F and G are φ-functions, then we define

$$\|f\|^{T}_{F,G} = \|\widetilde{F}^{-1}(e^x)f^*(e^x)\|_G = \inf\left\{c : \int_0^\infty G(\widetilde{F}^{-1}(x)f^*(x)/c)\,\frac{dx}{x} \le 1\right\},$$

and call the corresponding space $L^T_{F,G}$ (my notation).

These spaces were investigated by Raynaud [R]. He showed that if F is dilatory and satisfies the Δ_2-condition, and if G is dilatory, then $\|\chi_A\|^T_{F,G} \approx \widetilde{F}^{-1}(\mu(A))$ for all measurable A. Thus, $L^T_{F,G}$ and $L_{F,G}$ are equivalent if $G(t) = t^p$.

The comparison results for these spaces are much more straightforward. Raynaud [R] showed that if F_1 and F_2 are dilatory and satisfy the Δ_2-condition, and if G_1 and G_2 are dilatory, then $L^T_{F_1,G_1}$ and $L^T_{F_2,G_2}$ are equivalent if F_1 and F_2 are equivalent, and the *sequence* spaces l_{G_1} and l_{G_2} are equivalent. The converse result is also easy to show.

We also comment that the Boyd indices of these spaces are much easier to compute. This will be dealt with more fully in [Mo2].

Also, unlike the Orlicz–Lorentz spaces we have used here, $L^T_{F,F}$ is not always equivalent to the Orlicz space L_F. For example, if $F(t) = t\,\mathrm{lm}\,t$, then L_F is equivalent to $L^T_{F,1}$ by Theorem 4.3, and since l_F and l_1 are not equivalent, this is not equivalent to $L^T_{F,F}$.

We finally add that we may define the spaces $L_{F,X}$, where X is a rearrangement invariant quasi-Banach space on $\mathbb{R}$ satisfying certain mild restrictions. Corresponding to the definition used in this paper, we may define

$$\|f\|_{F,X} = \|f^* \circ \widetilde{F} \circ \phi_X\|_X ,$$

where $\phi_X(t) = \|\chi_{[0,t]}\|_X$ is the fundamental function of X. Corresponding to the definition used by Torchinsky and Raynaud, we may define

$$\|f\|^T_{F,X} = \|\widetilde{F}^{-1}(e^x) f^*(e^x)\|_X .$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI
COLUMBIA, MISSOURI 65211
U.S.A.
E-mail: MATHSMS@UMCVMB.BITNET
or STEPHEN@MONT.CS.MISSOURI.EDU

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