[4] S. Williams, *Toeplitz minimal flows which are not uniquely ergodic*, Z. Wahrsch. Verw. Gebiete 67 (1984), 95–107.

INSTITUTE OF MATHEMATICS
NICHOLAS COPERNICUS UNIVERSITY
CHOPINA 12/18
87-100 TORUŃ, POLAND

Received November 22, 1990
Revised version November 8, 1991 (2748)

STUDIA MATHEMATICA 103 (2) (1992)

Unbounded well-bounded operators, strongly continuous semigroups and the Laplace transform

by

RALPH DELAUBENFELS (Athens, O.)

Abstract. Suppose A is a (possibly unbounded) linear operator on a Banach space. We show that the following are equivalent.

(1) A is well-bounded on $[0,\infty)$.

(2) $-A$ generates a strongly continuous semigroup $\{e^{-sA}\}_{s\geq 0}$ such that $\{(1/s^2)e^{-sA}\}_{s>0}$ is the Laplace transform of a Lipschitz continuous family of operators that vanishes at 0.

(3) $-A$ generates a strongly continuous differentiable semigroup $\{e^{-sA}\}_{s\geq 0}$ and $\exists M < \infty$ such that

$$\|H_n(s)\| \equiv \left\| \left(\sum_{k=0}^{n} \frac{s^k A^k}{k!} \right) e^{-sA} \right\| \leq M, \quad \forall s>0,\ n\in\mathbb{N}\cup\{0\}.$$

(4) $-A$ generates a strongly continuous holomorphic semigroup $\{e^{-zA}\}_{\mathrm{Re}(z)>0}$ that is $O(|z|)$ in all half-planes $\mathrm{Re}(z) > a > 0$ and

$$K(t) \equiv \int_{1+i\mathbb{R}} e^{zt} e^{-zA} \frac{dz}{2\pi i z^3}$$

defines a differentiable function of t, with Lipschitz continuous derivative, with $K'(0)=0$.

We may then construct a decomposition of the identity, F, for A, from $K(t)$ or $H_n(s)$. For $\phi\in X^*$, $x\in X$,

$$(F(t)\phi)(x) = (d/dt)^2(\phi(K(t)x)) = \lim_{n\to\infty} \phi(H_n(n/t)x),$$

for almost all t.

I. Introduction. Scalar operators (see [5], [6]) with real spectrum are a generalization, to arbitrary Banach spaces, of self-adjoint operators on a Hilbert space. An early disappointment was the fact that most standard differential operators on an L^p space are scalar only when p equals 2. However, if one weakens the definition by requiring uniformly bounded spectral projections corresponding only to closed intervals, rather than arbitrary closed

1991 *Mathematics Subject Classification*: 47A60, 47B40.

sets, one is led to a more widely occurring class of operators, what are called *well-bounded* operators (see [5]).

For example, translation on $L^p(G)$, where G is a locally compact abelian group, is scalar if and only if p equals 2 or the operator has finite order, and is well-bounded whenever $1 < p < \infty$ (see [5]).

One may characterize both classes of operators purely in terms of functional calculi. A scalar operator is one with a functional calculus defined for all bounded Borel measurable functions; a well-bounded operator is one with a functional calculus defined for all absolutely continuous functions. This is the definition that we will use for an unbounded well-bounded operator (Definition 2.1).

The concept of an unbounded well-bounded operator appeared in [10], where they were called "well-boundable". The concept has also often appeared implicitly (see [2], [11]). The definition in [10] applies to the type B situation, but has no restriction on the spectrum, thus in one sense is wider and in another sense is more restrictive, than the definition we give in Section II. Our results in Sections II and III, and (b) of our application Theorem 4.1, are not covered by the work in [10], although there are close connections. We define an unbounded well-bounded operator in Section II and give some equivalences in Theorem 2.4.

We characterize well-bounded operators with nonnegative real spectrum as the generators of strongly continuous semigroups with certain regularity properties. It is necessary and sufficient that $-A$ generate a strongly continuous semigroup $\{e^{-sA}\}_{s\geq 0}$ such that $(1/s^2)e^{-sA}$ is the Laplace transform of a Lipschitz continuous family of bounded operators. This is also equivalent to e^{-sA} being holomorphic, with certain growth conditions on $A^k e^{-sA}$ similar to growth conditions that characterize e^{-sA} being holomorphic. These characterizations are in Section III.

Our characterization also enables us to construct a decomposition of the identity (these are the spectral projection corresponding to intervals, for A —see Definition 2.2), for a well-bounded operator, in terms of the semigroup generated by A.

These results are in the spirit of the Spectral Theorem, on a Hilbert space, which states that A being self-adjoint is equivalent to iA generating a bounded strongly continuous group. Of interest here are the relationships between spectral projections, functional calculus and generation of semigroups.

We apply our results to the Laplacian, on $L^p(\mathbb{R})$ $(1 \leq p < \infty)$ and $C_0(\mathbb{R})$, in Section IV.

All operators are linear, on a Banach space, X, with dual space X^*. We will denote by $L(X)$ the space of bounded operators from X into itself. We will write $\mathcal{D}(A)$ for the domain of the operator A, $\sigma(A)$ for the spectrum, $\varrho(A)$ for the resolvent set. Basic material on semigroups of operators may be found in [7] or [9]; for well-bounded operators, see [5].

We will denote by $AC[a,b]$ the Banach algebra of absolutely continuous functions on $[a,b]$, with norm $\|f\|_{AC[a,b]} \equiv |f(b)| + \int_a^b |f'(t)|\,dt$. The Banach algebra $AC[0,\infty)$ will consist of absolutely continuous functions on $[0,\infty)$ such that $\int_0^\infty |f'(t)|\,dt$ is finite, with norm $\|f\|_{AC[0,\infty)} \equiv |f(\infty)| + \int_0^\infty |f'(t)|\,dt$; note that $f(\infty) \equiv \lim_{x\to\infty} f(x)$ automatically exists.

II. Unbounded well-bounded operators on $[0,\infty)$. In this section, we define a (possibly unbounded) well-bounded operator on $[0,\infty)$ and characterize it in terms of resolvent, semigroup, and decomposition of the identity (Theorem 2.4).

We define a well-bounded operator as an operator with a functional calculus defined for absolutely continuous functions. For reflexive spaces, we show that our definition of well-bounded is a weak operator topology version of the usual direct definition of an unbounded scalar operator (see [6]), followed by an integration by parts ((e) of Theorem 2.4).

We show that a densely defined operator is well-bounded on $[0,\infty)$ if and only if it generates a strongly continuous semigroup $\{e^{-sA}\}_{s\geq 0}$ such that $(1/s)e^{-sA}$ is the Laplace transform of a decomposition of the identity ((d) of Theorem 2.4). Analogous results for scalar operators may be found in [3], [4] and [12].

We begin this section with preliminary material on well-bounded operators.

DEFINITION 2.1. A bounded operator, B, is *well-bounded on* $[a,b]$ if there exists a continuous algebra homomorphism, $f \mapsto f(B)$, from $AC[a,b]$ into $L(X)$, such that $f_0(B) = I$ and $f_1(B) = B$, where $f_0(x) \equiv 1$, $f_1(x) \equiv x$.

We will say that a (possibly unbounded) operator, A, is *well-bounded on* $[0,\infty)$ if $\sigma(A) \subseteq [0,\infty)$ and there exists a continuous algebra homomorphism, $f \mapsto f(A)$, from $AC[0,\infty)$ into $L(X)$, such that $f_0(A) = I$ and $g_z(A) = (z+A)^{-1}$, whenever z is outside $(-\infty,0]$, where $g_z(x) \equiv (z+x)^{-1}$.

Note that this algebra homomorphism, if it exists, is unique, since the unital algebra generated by $\{g_r \mid r > 0\}$ is dense in $AC[0,\infty)$.

It is shown in [5] that a bounded operator, A, that is well-bounded on $[a,b]$ has its spectrum contained in $[a,b]$, with $(z+A)^{-1}$ equal to $g_z(A)$, whenever $-z$ is not in $[a,b]$. Thus our definition for unbounded operators is consistent with the definition for bounded operators.

DEFINITION 2.2. A *decomposition of the identity for* X *on* $[a,b]$ is a family $\{E(t)\}_{t\in\mathbb{R}}$ of projections on X^* such that

(1) $E(s) = 0$, $\forall s < a$, and $E(s) = I$, $\forall s \geq b$.
(2) $E(s)E(t) = E(t)E(s) = E(s)$, $\forall s \leq t$.

(3) $\{\|E(s)\| \mid s \in \mathbb{R}\}$ is bounded.

(4) The map $s \mapsto (E(s)\phi)(x)$ is Lebesgue measurable, $\forall \phi \in X^*, x \in X$.

(5) If $x \in X$, $\phi \in X^*$, $a \le s < b$, and if the function

$$t \mapsto \int_a^t (E(r)\phi)(x)\, dr$$

is right differentiable at s, then the right derivative at s is $(E(s)\phi)(x)$.

(6) $\forall x \in X$, the map $\phi \mapsto (E(\cdot)\phi)(x)$, from X^* into $L^\infty(a,b)$, both with their weak*-topologies, is continuous.

A bounded operator, B, is well-bounded on $[a,b]$ if and only if there exists a decomposition of the identity, E, for X, on $[a,b]$ such that

$$\phi(Bx) = b\phi(x) - \int_a^b (E(t)\phi)(x)\, dt\,,$$

$\forall \phi \in X^*$, $x \in X$ (see [5]). For any $t \ge 0$, $E(t)$ is a spectral projection for B^* corresponding to $[a,t]$, that is, $E(t)X^*$ is invariant under B^*, and

$$\sigma(B^*|_{E(t)X^*}) \subseteq [a,t] \cap \sigma(B^*)\,.$$

When X is reflexive, then there exists a family of projections $\{F(t)\}$, on X, such that $E(t) = F(t)^*$. For each t, $F(t)$ is a spectral projection for B corresponding to $[a,t]$.

By a *decomposition of the identity for X on* $[0,\infty)$ we will mean a family $\{E(t)\}_{t\in\mathbb{R}}$ of projections on X^* satisfying (2) through (6) of the definition above, with $b = \infty$, $a = 0$, such that

(1′) $E(s) = 0$, $\forall s < 0$.

A consequence of Theorem 2.4 is that, for every operator, A, well-bounded on $[0,\infty)$, there will exist a decomposition of the identity, E, for X on $[0,\infty)$, such that $E(t)$ is a spectral projection for A^* corresponding to $[0,t]$. When X is reflexive, we will similarly obtain uniformly bounded spectral projections for A corresponding to $[0,t]$.

When A has countable spectrum and X is separable, then being scalar corresponds to the eigenvectors for A forming an unconditional basis for X, while being well-bounded corresponds to the eigenvectors forming a conditional basis.

DEFINITION 2.3. Suppose $0 < \theta \le \pi/2$. We say that the strongly continuous semigroup $\{T(t)\}_{t\ge 0}$ is a *bounded strongly continuous holomorphic semigroup of angle* θ if it extends to a holomorphic family of bounded operators in the sector $S_\theta \equiv \{z \mid |\arg(z)| < \theta\}$, that is bounded and strongly continuous on subsectors $\overline{S}_\psi$, $\forall \psi < \theta$.

The (possibly unbounded) operator A *generates* $\{T(t)\}_{t\ge 0}$ if

$$Ax = \lim_{t\to 0} \frac{1}{t}(T(t)x - x), \qquad \mathcal{D}(A) = \{x \mid \text{ limit exists }\}\,.$$

We then write $T(t)$ as e^{tA}, the *strongly continuous semigroup generated by* A.

THEOREM 2.4. *Suppose A is densely defined. Then the following are equivalent.*

(a) *A is well-bounded on* $[0,\infty)$.

(b) $(-1) \in \varrho(A)$ *and* $(1+A)^{-1}$ *is well-bounded on* $[0,1]$.

(c) $(-\infty,0) \subseteq \varrho(A)$ *and $\exists$ a decomposition of the identity, E, for X, on* $[0,\infty)$, *such that*

$$\phi((r+A)^{-1}x) = \int_0^\infty (r+t)^{-2}(E(t)\phi)(x)\, dt\,,$$

$\forall r > 0$, $x \in X$, $\phi \in X^*$.

(d) $-A$ *generates a strongly continuous holomorphic semigroup* $\{e^{-zA}\}_{\mathrm{Re}(z)>0}$ *of angle $\pi/2$, and $\exists$ a decomposition of the identity, E, for X, on* $[0,\infty)$, *such that*

$$\phi(e^{-zA}x) = \int_0^\infty z e^{-zt}(E(t)\phi)(x)\, dt\,,$$

when $\mathrm{Re}(z) > 0$, $x \in X$, $\phi \in X^*$.

If X is reflexive, then (a)–(d) *are also equivalent to*

(e) $\exists$ *projections $F(t)$, on X, such that $\{F(t)^*\}_{t\ge 0}$ is a decomposition of the identity for X, on* $[0,\infty)$, $\lim_{t\to\infty} \phi(F(t)x) = \phi(x)$, $\forall x \in X$, $\phi \in X^*$, *and*

$$\phi(Ax) = \lim_{N\to\infty}\Big[N\phi(x) - \int_0^N \phi(F(t)x)\, dt\Big], \qquad \forall \phi \in X^*,\ x \in \mathcal{D}(A)\,,$$

with $\mathcal{D}(A)$ equal to the set of all $x \in X$ for which the limit exists, $\forall \phi \in X^*$, *and defines a vector, Ax, in X.*

Proof. (a)⇔(b). This is merely composition of functions, in the appropriate algebra homomorphism. Let g_r be as in Definition 2.1. If A is well-bounded on $[0,\infty)$, then define a map from $AC[0,1]$ into $L(X)$ by $f((1+A)^{-1}) \equiv (f \circ g_1)(A)$. This is clearly the desired algebra homomorphism for $(1+A)^{-1}$. Conversely, if $(1+A)^{-1}$ is well-bounded on $[0,1]$, define a map from $AC[0,\infty)$ into $L(X)$ by $f(A) \equiv (f \circ g_1^{-1})((1+A)^{-1})$, where $(f \circ g_1^{-1})(0) \equiv f(\infty)$. We must check $g_z(A)$, when $z \notin (-\infty, 0]$. Since

$$(g_z \circ g_1^{-1})(x) = g_z\left(\frac{1}{x} - 1\right) = \frac{x}{(z-1)x+1},$$

it follows that $g_z(A) = (1+A)^{-1}[(z-1)(1+A)^{-1}+1]^{-1} = (z+A)^{-1}$, as desired.

(c)⇒(b). Define h_1, on $[0,\infty)$, by $h_1(s) \equiv 1 - 1/(1+s)$.

Let $G(s) \equiv (E(h_1^{-1}(s)))1_{[0,1)}(s) + 1_{[1,\infty)}(s)$. Then G is a decomposition of the identity for X on $[0,1]$; (5), of Definition 2.2, requires some computation.

For any $\phi \in X^*$, $x \in X$,

$$\phi((1-(1+A)^{-1})x) = \phi(x) - \int_0^\infty h_1'(t)(E(t)\phi)(x)\,dt$$
$$= \phi(x) - \int_0^1 (G(s)\phi)(x)\,ds\,.$$

Thus $1-(1+A)^{-1}$, and hence $(1+A)^{-1}$, is well-bounded on $[0,1]$.

(d)⇒(c). For $\phi \in X^*$, $x \in X$, $r > 0$,

$$\phi((r+A)^{-1}x) = \int_0^\infty e^{-rs}\phi(e^{-sA}x)\,ds$$
$$= \int_0^\infty \left(\int_0^\infty se^{-s(t+r)}\,ds\right)(E(t)\phi)(x)\,dt$$
$$= \int_0^\infty (r+t)^{-2}(E(t)\phi)(x)\,dt,$$

as desired.

(a)⇒(d). First, we will represent the functional calculus for A as an integral with respect to a decomposition of the identity.

By (a)⇔(b), $1-(1+A)^{-1}$ is well-bounded on $[0,1]$. This implies that ∃ a decomposition of the identity, G, for X, on $[0,1]$, such that the functional calculus for $1-(1+A)^{-1}$ is given by

$$\phi(f(1-(1+A)^{-1})x) = f(1)\phi(x) - \int_0^1 f'(s)(G(s)\phi)(x)\,ds\,,$$

$\forall f \in AC[0,1]$, $\phi \in X^*$, $x \in X$.

Let h_1 be as in (c)⇒(b) and let $E(t) \equiv G(h_1(t))$, for $t \ge 0$, $E(t) \equiv 0$, for $t < 0$. Then E is a decomposition of the identity on $[0,\infty)$; as in (c)⇒(b), (5), of Definition 2.2, requires some checking. As in the proof of (a)⇔(b), it may be shown that, for any $f \in AC[0,\infty)$, $f(A) = (f \circ h_1^{-1})(1-(1+A)^{-1})$, so that

$$\phi(f(A)x) = f(h_1^{-1}(1))\phi(x) - \int_0^1 (f \circ h_1^{-1})'(s)(G(s)\phi)(x)\,ds$$
$$= f(\infty)\phi(x) - \int_0^\infty f'(t)((G(h_1(t))\phi)(x)\,dt$$
$$= f(\infty)\phi(x) - \int_0^\infty f'(t)(E(t)\phi)(x)\,dt\,,$$

$\forall f \in AC[0,\infty)$, $\phi \in X^*$, $x \in X$.

For any $\pi > \theta > 0$, since $\{\|zg_z\|_{AC[0,\infty)} \mid |\arg(z)| < \theta\}$ is bounded and A is well-bounded on $[0,\infty)$, it follows that $\{\|z(z+A)^{-1}\| \mid |\arg(z)| < \theta\}$ is bounded. Since A is densely defined, this implies that $-A$ generates a bounded strongly continuous holomorphic semigroup of angle $\pi/2$ (see [7, Theorem 5.3] or [9, Theorem 5.2]).

To show that e^{-zA} is given, as one might expect, by the functional calculus, $e^{-zA} = k_z(A)$, $k_z(t) \equiv e^{-zt}$, we must first prove the following claim, where $f_1(x) \equiv x$:

$(*)$ $\quad (f_1f)(A) = Af(A)$, when both f and f_1f are in $AC[0,\infty)$.

Suppose $f(x) = (r+x)^{-n}$, for some $r > 0$, $n \in \mathbb{N}$. Then $Af(A) = (r+A-r)(r+A)^{-n} = (r+A)^{-(n-1)} - r(r+A)^{-n} = (f_1f)(A)$. Let $\mathcal{F}$ be the algebra generated by all functions of the form $x \to (r+x)^{-1}$, for some $r > 0$. For any f as in $(*)$, $\exists\langle h_n\rangle \subseteq \mathcal{F}$ such that $h_n \to f$ and $f_1h_n \to f_1f$, both in $AC[0,\infty)$, as $n \to \infty$; this may be seen by choosing $\langle k_n\rangle$ in the span of $\mathcal{F}$ and the constant functions, converging to $(1+f_1)f$ in $AC[0,\infty)$, and letting $h_n(x) \equiv k_n(x)/(1+x)$. Since A is closed, this implies that $\mathrm{Im}(f(A)) \subseteq \mathcal{D}(A)$, with $Af(A) = \lim_{n\to\infty}(f_1h_n)(A) = (f_1f)(A)$, proving $(*)$.

A calculation shows that $(1/h)(k_{z+h} - k_z)$ converges to $-f_1k_z$ in $AC[0,\infty)$, as $h \to 0$, when $\mathrm{Re}(z) > 0$. Thus $(d/dz)k_z(A) = (-f_1k_z)(A) = -Ak_z(A)$, which implies that

$$\phi(e^{-zA}x) = \phi(k_z(A)x) = \int_0^\infty ze^{-zt}(E(t)\phi)(x)\,dt\,,$$

$\forall \phi \in X^*$, $x \in X$, as desired.

(b)⇒(e). Since X is reflexive, ∃ a family of projections $\{H(t)\}_{t\ge0}$, on X, such that $\{H(t)^*\}_{t\ge0}$ is a decomposition of the identity for X on $[0,1]$, $H(t^+)x \equiv \lim_{s\to t^+} H(s)x$ and $H(t^-)x \equiv \lim_{s\to t^-} H(s)x$ both exist, $\forall x \in X$,

$t \in [0,1]$, and

$$\phi((1-(1+A)^{-1})x) = \phi(x) - \int_0^1 \phi(H(t)x)\,dt,$$

$\forall x \in X$, $\phi \in X^*$ (see [5, Theorem 17.17]).

Define F, on $[0,\infty)$, by $F(t) \equiv H(1-1/(1+t))$, analogously to (a)$\Rightarrow$(d). Since $(1+A)^{-1}$ is injective, $H(1^-)x = x$, $\forall x \in X$ (see [5, Theorem 17.15(iii)]). Thus $\lim_{t\to\infty} F(t)x = x$, $\forall x \in X$.

The proofs of the previous equivalences show that

$$\phi((1+A)^{-1}y) = \int_0^\infty \phi(F(s)y)\,\frac{ds}{(1+s)^2},$$

$\forall y \in X, \phi \in X^*$.

Define an operator B, as in (e), as follows:

$$\phi(Bx) = \lim_{N\to\infty}\Big[N\phi(x) - \int_0^N \phi(F(t)x)\,dt\Big], \quad \forall \phi \in X^*,\ x \in \mathcal{D}(B),$$

with $\mathcal{D}(B)$ equal to the set of all $x \in X$ for which the limit exists, $\forall \phi \in X^*$, and defines a vector, Bx, in X.

We will show that $A \subseteq B$.

Suppose $x \in \mathcal{D}(A)$. Then $\exists y \in X$ such that $x = (1+A)^{-1}y$, so that, for any $N > 0$, $\phi \in X^*$,

$$\begin{aligned}
&N\phi(x) - \int_0^N \phi(F(t)x)\,dt \\
&\quad = N\phi((1+A)^{-1}y) - \int_0^N \phi(F(t)(1+A)^{-1}y)\,dt \\
&\quad = N\int_0^\infty \phi(F(s)y)\,\frac{ds}{(1+s)^2} - \int_0^N \int_0^\infty \phi(F(t)F(s)y)\,\frac{ds}{(1+s)^2}\,dt \\
&\quad = N\int_0^\infty \phi(F(s)y)\,\frac{ds}{(1+s)^2} \\
&\qquad - \int_0^N \Big[\int_0^t \phi(F(s)y)\,\frac{ds}{(1+s)^2} + \int_t^\infty \phi(F(t)y)\,\frac{ds}{(1+s)^2}\Big]\,dt \\
&\quad = N\int_0^\infty \phi(F(s)y)\,\frac{ds}{(1+s)^2}
\end{aligned}$$

$$\begin{aligned}
&\qquad - \int_0^N \int_s^N \phi(F(s)y)\,\frac{dt}{(1+s)^2}\,ds - \int_0^N \phi(F(t)y)\,\frac{dt}{1+t} \\
&\quad = N\int_0^\infty \phi(F(s)y)\,\frac{ds}{(1+s)^2} - \int_0^N (N-s)\phi(F(s)y)\,\frac{ds}{(1+s)^2} \\
&\qquad - \int_0^N \phi(F(t)y)\,\frac{dt}{1+t} \\
&\quad = N\int_N^\infty \phi(F(s)y)\,\frac{ds}{(1+s)^2} - \int_0^N \phi(F(s)y)\,\frac{ds}{(1+s)^2} \\
&\quad = (N+1)\int_N^\infty \phi(F(s)y)\,\frac{ds}{(1+s)^2} - \int_0^\infty \phi(F(s)y)\,\frac{ds}{(1+s)^2};
\end{aligned}$$

the convergence of $F(s)y$ to y, as $s \to \infty$, implies that the first term converges to $\phi(y)$ as $N \to \infty$.

Thus the limit, as $N \to \infty$, exists, so that $x \in \mathcal{D}(B)$, with

$$\phi(Bx) = \phi(y) - \int_0^\infty \phi(F(s)y)\,\frac{ds}{(1+s)^2} = \phi(y) - \phi((1+A)^{-1}y) = \phi(Ax).$$

Thus $1 + A \subseteq 1 + B$. We will now show that $1+B$ is injective. For any N, define $B_N \in L(X)$ by

$$\phi(B_N x) \equiv N\phi(x) - \int_0^N \phi(F(t)x)\,dt.$$

Suppose $0 = (1+B)x$. Then $\forall \phi \in X^*$, $0 = \lim_{N\to\infty} \phi((1+B_N)x)$. It is well-known that $((1+A)^{-1})^*$ commutes with any decomposition of the identity associated with $(1+A)^{-1}$. Thus, $\forall \phi \in X^*$,

$$0 = \lim_{N\to\infty} ((1+A)^{-1})^*\phi((1+B_N)x) = \lim_{N\to\infty} \phi((1+B_N)(1+A)^{-1}x),$$

which we have previously shown to be $\phi(x)$.

Thus $1+B$ is injective. Since it contains $1+A$, which is surjective, $1+A$ must equal $1+B$, so that $A = B$, as desired.

(e)$\Rightarrow$(b). Define $G(s) \equiv I - F((1/s-1)^-)$, for $0 < s < 1$, $G(s) = I$, for $s \ge 1$, $G(s) = 0$, for $s \le 0$. Then $\{G(t)^*\}$ is the decomposition of the identity for a well-bounded operator, R, on X, that is,

$$\phi(Rx) = \phi(x) - \int_0^1 \phi(G(t)x)\,dt,$$

$\forall x \in X$, $\phi \in X^*$. After a change of variables,

$$\phi(Rx) = \int_0^\infty \phi(F(s)x)\,\frac{ds}{(1+s)^2}\,.$$

The same computations as in (b)⇒(e) now show that R maps X into $\mathcal{D}(A)$, $(1+A)Rx = x$, $\forall x \in X$ and $R(1+A)x = x$, $\forall x \in \mathcal{D}(A)$. This establishes (b). ∎

Comments 2.5. Somewhat more than (e) of Theorem 2.4 can be said when X is reflexive; $(1+A)^{-1}$ is then a well-bounded operator *of type* (B) (see [5, Definition 16.8]). This implies that $(1+A)^{-1}$, and hence A, has a functional calculus defined for functions of bounded variation, and this functional calculus may be represented as a vector-valued Riemann–Stieltjes integral. To go into this in detail would take us too far afield, hence we will merely refer the interested reader to [5, Chapters 16 and 17].

Note that a consequence of the proof of Theorem 2.4 is that, for any Banach space X, when A is well-bounded on $[0,\infty)$, the algebra homomorphism for A is given by

$$\phi(f(A)x) = f(\infty)\phi(x) - \int_0^\infty f'(t)(E(t)\phi)(x)\,dt\,,$$

for $\phi \in X^*$, $x \in X$.

III. Well-bounded operators, semigroups and the Laplace transform. The relationship between the resolvent of an operator and the semigroup it generates is very similar to the relationship between the semigroup generated by a well-bounded operator and its decomposition of the identity. Informally, the resolvent family $\{(r+A)^{-1}\}_{r>0}$ is the Laplace transform of $\{e^{-sA}\}_{s\geq 0}$ and $\{(1/s)e^{-sA}\}_{s>0}$ is the Laplace transform of $\{E(t)\}_{t\geq 0}$, a decomposition of the identity.

Important analytic differences appear when one writes down precisely what is meant by being a Laplace transform. When we say that the resolvent is the Laplace transform of the semigroup, we mean, in the strong operator topology, the Laplace transform of a continuous function, that is, for all x in X, $(r+A)^{-1}x$ is the Laplace transform of the continuous function $e^{-sA}x$. When, as in Theorem 2.4(d), we say that $(1/s)e^{-sA}$ is the Laplace transform of $E(t)$, we mean, in the weak operator topology, the Laplace transform of a measurable function, that is, for all x in X and ϕ in X^*, $\phi((1/s)e^{-sA}x)$ is the Laplace transform of the measurable function $(E(t)\phi)(x)$.

Recently, some nice results on vector-valued Laplace transforms have appeared (see [1] and [8] and Lemmas 3.3 and 3.4). Because we are not concerned with strong continuity of $E(t)$, it is actually easier to apply Lemma 3.3 to decompositions of the identity than to strongly continuous semigroups. One merely takes the Laplace transform of Theorem 2.4(d), and applies integration by parts, to obtain $\{(1/s^2)e^{-sA}\}_{s>0}$ as the Laplace transform of a Lipschitz continuous family of operators. We may now use Lemma 3.3 to obtain results in terms of the operator norm, rather than the weak operator topology.

Once we have established the semigroup generated by $-A$ as the Laplace transform of a decomposition of the identity, we may then use results about inversion of the Laplace transform to construct the decomposition of the identity from the semigroup generated by $-A$. Lemma 3.4, from [8], allows us to obtain results again in terms of the operator norm, rather than the weak operator topology, as would follow from classical Laplace transform theory.

THEOREM 3.1. *Suppose A is densely defined. Then the following are equivalent.*

(a) *A is well-bounded on $[0,\infty)$.*

(b) *$-A$ generates a strongly continuous semigroup $\{e^{-sA}\}_{s\geq 0}$ such that $\{(1/s^2)e^{-sA}\}_{s>0}$ is the Laplace transform of a Lipschitz continuous function, from $[0,\infty)$ into $L(X)$, that vanishes at 0.*

(c) *$-A$ generates a strongly continuous differentiable semigroup $\{e^{-zA}\}_{s\geq 0}$ and $\exists M<\infty$ such that*

$$\|H_n(s)\| \equiv \left\|\left(\sum_{k=0}^{n}\frac{s^kA^k}{k!}\right)e^{-sA}\right\| \leq M, \qquad \forall s>0,\ n\in\mathbb{N}\cup\{0\}\,.$$

(d) *$-A$ generates a strongly continuous holomorphic semigroup $\{e^{-zA}\}_{\mathrm{Re}(z)>0}$ that is $O(|z|)$ in all half-planes $\mathrm{Re}(z)>a>0$ and*

$$G(t) \equiv \lim_{N\to\infty}\int_{a-iN}^{a+iN} e^{zt}e^{-zA}\,\frac{dz}{2\pi i z^2}$$

exists, independently of a, and defines a Lipschitz continuous function of t, $G(0)=0$ and for any $a, R>0$, the limit is uniform in $t\in[0,R]$.

(e) *$-A$ generates a strongly continuous holomorphic semigroup $\{e^{-zA}\}_{\mathrm{Re}(z)>0}$ that is $O(|z|)$ in all half-planes $\mathrm{Re}(z)>a>0$ and*

$$K(t) \equiv \int_{1+i\mathbb{R}} e^{zt}e^{-zA}\,\frac{dz}{2\pi i z^3}$$

is a differentiable function from $[0,\infty)$ into $L(X)$, with $K'(t)$ Lipschitz continuous and $K'(0)=0$.

A decomposition of the identity for A is then given by

$$(E(t)\phi)(x) = \frac{d}{dt}\phi(G(t)x) = \left(\frac{d}{dt}\right)^2 \phi(K(t)x) = \lim_{n\to\infty} \phi\left(H_n\left(\frac{n}{t}\right)x\right) \quad \text{a.e.},$$

for $\phi \in X^*$, $x \in X$.

Comments 3.2. H_n involves boundary values of a Taylor series expansion for e^{-sA}. It may be shown that, under the conclusions of the theorem, $\lim_{n\to\infty} H_n(s)x = x$, $\forall x \in X$, $s > 0$.

H_0 being uniformly bounded is saying that $\{e^{-sA}\}_{s\ge 0}$ is uniformly bounded, while H_1 being uniformly bounded is equivalent to $\{e^{-sA}\}$ extending to a bounded holomorphic strongly continuous semigroup (see [7] or [9]).

It is interesting that $G(t)$ is in $L(X)$, $\forall t \ge 0$, although, in general, there exists no $F(t) \in L(X)$ such that $E(t) = F(t)^*$, where E is any decomposition of the identity for A (see Definition 2.2); note that $\phi(G(t)x) = \int_0^t (E(s)\phi)(x)\,ds$, $\forall \phi \in X^*$, $x \in X$, for a decomposition of the identity E.

The following indicates just how "close" to being well-bounded a generator of a bounded strongly continuous holomorphic semigroup of angle $\pi/2$ is. Compare this Proposition with (c) of Theorem 3.1.

PROPOSITION. *Suppose $-A$ generates a bounded strongly continuous holomorphic semigroup of angle $\pi/2$. Then $\forall r > 1$, $\exists M_r < \infty$ such that*

$$\|H_n(s)\| \le M_r r^n, \quad \forall n \in \mathbb{N},\ s > 0.$$

Proof. Fix $r > 1$. For any $s > 0$, let Γ_s be the circle, in the complex plane, of radius s/r, centered at s (on the real line). Some calculation, using the Cauchy integral formula, shows that

$$H_n(s) = \int_{\Gamma_s} e^{-zA}\left(1 - \left(\frac{s}{s-z}\right)^{n+1}\right)\frac{dz}{2\pi i z}.$$

There exists $\phi_r < \pi/2$ such that $\Gamma_s \subseteq S_{\phi_r} \equiv \{z \mid |\arg(z)| < \phi_r\}$, $\forall s > 0$. This implies that $\exists K_r < \infty$ such that $\|e^{-zA}\| < K_r$, $\forall z \in \Gamma_s$, $s > 0$ (see Definition 2.3). After the change of variables $z \equiv s + (s/r)e^{i\theta}$, the integral above, after some calculation, implies that

$$\|H_n(s)\| \le K_r(1 + r^{n+1})/(r-1).$$

Letting $M_r \equiv 2rK_r/(r-1)$ now concludes the proof. ∎

LEMMA 3.3 (Theorem 1.1, from [1]). *Suppose that $f : (0,\infty) \to X$ and $M < \infty$. Then the following are equivalent.*

(a) *f is infinitely differentiable, and*

$$\left\|\frac{s^{n+1}}{n!} f^{(n)}(s)\right\| \le M, \quad \forall s > 0,\ n+1 \in \mathbb{N}.$$

(b) *$\exists G : [0,\infty) \to X$ such that $G(0) = 0$,*

$$\|G(t) - G(s)\| \le M|t-s|, \quad \forall s,t \ge 0,$$

and

$$f(s) = s\int_0^\infty e^{-st} G(t)\,dt, \quad \forall s > 0.$$

LEMMA 3.4 (Theorem 3.2 from [8]). *Suppose* (b) *of Lemma* 3.3 *holds. Then,* $\forall t, a > 0$,

$$G(t) = \lim_{N\to\infty} \int_{a-iN}^{a+iN} e^{zt} f(z)\,\frac{dz}{2\pi i z},$$

where the limit is uniform in $t \in [0,R]$, $\forall R > 0$.

Proof of Theorem 3.1. To consider the Laplace transform, we need derivatives of the function $s \to (1/s)e^{-sA}$. For any $s > 0$, using the product rule $(fg)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(n-k)} g^{(k)}$, when $-A$ generates a strongly continuous differentiable semigroup, it is straightforward to calculate that, $\forall s > 0$,

$$(*) \qquad \frac{s^{n+1}}{n!}\left(\frac{d}{ds}\right)^n \left(\frac{1}{s}e^{-sA}\right) = (-1)^n H_n(s).$$

(a)⇒(b). Theorem 2.4(a)⇒(d) implies that $f(s) \equiv (1/s)e^{-sA}$ satisfies (a) of Lemma 3.3, thus Lemma 3.3 implies (b).

(b)⇒(a). Let G be the Lipschitz continuous family of operators of (b), with Lipschitz constant M. Since $G(0) = 0$, $\{e^{-sA}\}_{s\ge 0}$ is bounded, thus $-1 \in \varrho(A)$.

For fixed $\phi \in X^*$, $x \in X$, let $G_{\phi,x}(t) \equiv \phi(G(t)x)$. Note that $G_{\phi,x}$ is differentiable a.e., with $|G'_{\phi,x}(t)| \le M\|\phi\|\|x\|$, for almost all $t \ge 0$.

For any $n \in \mathbb{N}$,

$$\begin{aligned}\phi((1+A)^{-n}x) &= \frac{1}{(n-1)!}\int_0^\infty s^{n-1}e^{-s}\phi(e^{-sA}x)\,ds \\ &= \frac{1}{(n-1)!}\int_0^\infty \Big(\int_0^\infty s^{n+1}e^{-s}e^{-ts}\,ds\Big) G_{\phi,x}(t)\,dt \\ &= n(n+1)\int_0^\infty \left(\frac{1}{1+t}\right)^{n+2} G_{\phi,x}(t)\,dt \\ &= n\int_0^\infty \left(\frac{1}{1+t}\right)^{n+1} G'_{\phi,x}(t)\,dt \\ &= n\int_0^1 w^{n-1} G'_{\phi,x}\left(\frac{1}{w}-1\right)dw.\end{aligned}$$

This implies that, if p is a polynomial, then

$$\phi(p((1+A)^{-1})x) = p(0)\phi(x) + \int_0^1 p'(w)G'_{\phi,x}\left(\frac{1}{w}-1\right)dw\,,$$

so that

$$|\phi(p((1+A)^{-1})x)| \le \|\phi\|\|x\|\left(|p(0)| + M\int_0^1 |p'(w)|\,dw\right),$$

which implies that

$$\|p((1+A)^{-1})\| \le 2M\|p\|_{AC[0,1]}\,.$$

Since the polynomials are dense in $AC[0,1]$, this implies that $(1+A)^{-1}$ is well-bounded on $[0,1]$, hence, by Theorem 2.4(b)⇒(a), A is well-bounded on $[0,\infty)$.

(b)⇔(c). By (b)⇒(a) and Theorem 2.4(a)⇒(d), (b) implies that $-A$ generates a strongly continuous holomorphic semigroup. Lemma 3.3 and (*) now imply that (b) and (c) are equivalent.

(b)⇒(d). As in (b) ⇒(c), $-A$ generates a strongly continuous holomorphic semigroup. If $\{(1/s^2)e^{-sA}\}_{s>0}$ is the Laplace transform of a Lipschitz continuous function, G, vanishing at 0, with Lipschitz constant M, then a simple calculation shows that

$$\left\|\frac{1}{z}e^{-zA}\right\| \le \frac{M}{\mathrm{Re}(z)}\,,$$

whenever $\mathrm{Re}(z) > 0$. This proves the growth condition. The integral representation of G follows from Lemma 3.4.

(d)⇒(b). For $0 < a < s$, $M > 0$, the uniform convergence of the integrals on $t \in [0, M]$ allows us to calculate as follows:

$$\begin{aligned}2\pi i\int_0^M e^{-st}G(t)\,dt &= \lim_{N\to\infty}\int_0^M e^{-st}\int_{a-iN}^{a+iN} e^{zt}e^{-zA}\frac{dz}{z^2}\,dt\\ &= \lim_{N\to\infty}\int_{a-iN}^{a+iN}\left(\int_0^M e^{t(z-s)}\,dt\right)e^{-zA}\frac{dz}{z^2}\\ &= \lim_{N\to\infty}\int_{a-iN}^{a+iN}\frac{1}{s-z}(1-e^{M(z-s)})e^{-zA}\frac{dz}{z^2}\\ &= \int_{a+i\mathbb{R}}\frac{1}{s-z}(1-e^{M(z-s)})e^{-zA}\frac{dz}{z^2}\,,\end{aligned}$$

so that dominated convergence implies that the limit, as $M\to\infty$, exists,

with

$$\begin{aligned}\int_0^\infty e^{-st}G(t)\,dt &= \lim_{M\to\infty}\int_0^M e^{-st}G(t)\,dt\\ &= \int_{a+i\mathbb{R}}\frac{1}{s-z}e^{-zA}\frac{dz}{2\pi i z^2} = \frac{1}{s^2}e^{-sA}\,,\end{aligned}$$

where the final equality follows from a calculus of residues argument.

(d)⇒(e) is clear, by letting $K(t) \equiv \int_0^t G(s)\,ds$ and using dominated convergence to obtain the integral representation of K.

(e)⇒(b). A calculation as in (d)⇒(b) implies that $(1/s^3)e^{-sA}$ is the Laplace transform of K, thus (b) follows after an integration by parts, using the fact that $K'(0) = 0$.

The uniqueness of the determining function in the Laplace transform and Theorem 2.4 imply that

$$\frac{d}{dt}\phi(G(t)x) = \left(\frac{d}{dt}\right)^2\phi(K(t)x)\,,$$

for $\phi \in X^*$, $x \in X$, defines a decomposition of the identity.

The fact that $(E(t)\phi)(x) = \lim_{n\to\infty}\phi(H_n(n/t)x)$ a.e. follows from (*) and complex-valued Laplace inversion theorems (see [14]). ■

IV. An application. We will write Δ for the Laplacian, the generator of the strongly continuous holomorphic semigroup

$$(e^{z\Delta}f)(x) \equiv \frac{1}{\sqrt{4\pi z}}\int_{\mathbb{R}} e^{-(x-y)^2/(4z)}f(y)\,dy\,,$$

on $L^p(\mathbb{R})$ $(1 \le p < \infty)$ or $C_0(\mathbb{R})$.

In [10], and, in effect, in [11], it is shown that $-\Delta$, on $L^p(\mathbb{R})$, is well-bounded if $1 < p < \infty$. We will use Theorem 3.1 to show that $-\Delta$, on $L^1(\mathbb{R})$ or $C_0(\mathbb{R})$, fails to be well-bounded.

THEOREM 4.1. (a) *$-\Delta$, on $L^p(\mathbb{R})$, is well-bounded on $[0,\infty)$ if $1 < p < \infty$.*

(b) *$-\Delta$, on $C_0(\mathbb{R})$ or $L^1(\mathbb{R})$, is not well-bounded on $[0,\infty)$.*

Proof. As we mentioned before stating the theorem, (a) is in [10] and [11]. We will use Theorem 3.1(a)⇔(e) to show (b). It is well-known that $\|e^{z\Delta}\| \le |z/\mathrm{Re}\,(z)|$, whenever $\mathrm{Re}(z) > 0$, on any of the indicated spaces.

Define

$$P_t(x) \equiv \frac{1}{\pi x}\sin(x\sqrt{t})\,.$$

We will first consider Δ on $C_0(\mathbb{R})$. As in Theorem 3.1(e), define

$$K(t)f \equiv \int_{1+i\mathbb{R}} e^{zt}(e^{z\Delta}f)\,\frac{dz}{2\pi i z^3}.$$

We claim that, if f has compact support, then $\forall x \in \mathbb{R}$, $(K(t)f)(x)$ is a twice differentiable function of t, with

$$(*) \qquad (d/dt)^2(K(t)f)(x) = (P_t * f)(x).$$

To prove $(*)$, note that, if $k_z(x) \equiv e^{-zx^2}$, then

$$e^{z\Delta}f = (\mathcal{F}^{-1}(k_z)) * f,$$

where $\mathcal{F}$ is the Fourier transform, thus,

$$(K(t)f)(x) = \int_{\mathbb{R}}\Big[\int_{\mathbb{R}} e^{is(x-y)}\Big[\int_{1+i\mathbb{R}} e^{z(t-s^2)}\,\frac{dz}{2\pi i z^3}\Big]\frac{ds}{2\pi}\Big] f(y)\,dy.$$

A calculus of residues argument shows that the innermost integral equals $\frac{1}{2}(t-s^2)^2 1_{(-\sqrt{t},\sqrt{t})}(s)$; explicit calculation of the inverse Fourier transform now gives us

$$\Big(\frac{d}{dt}\Big)^2\Big[\int_{\mathbb{R}} e^{isx}\Big[\int_{1+i\mathbb{R}} e^{z(t-s^2)}\,\frac{dz}{2\pi i z^3}\Big]\frac{ds}{2\pi}\Big] = P_t(x),$$

proving $(*)$.

Now suppose, for the sake of contradiction, that $-\Delta$, on $C_0(\mathbb{R})$, is well-bounded on $[0,\infty)$. By Theorem 3.1(a)$\Leftrightarrow$(e), the map $t \mapsto K'(t)$, from $[0,\infty)$ into $L(X)$, is Lipschitz continuous. Let M be the Lipschitz constant. By $(*)$, for any compactly supported f,

$$|(P_t * f)(x)| \le M\|f\|_\infty,$$

for all $t \ge 0$, $x \in \mathbb{R}$. This implies that $\|P_t\|_1 \le M$, $\forall t \ge 0$, which is false; in fact, $\|P_t\|_1$ is infinite for all $t > 0$.

Thus $-\Delta$, on $C_0(\mathbb{R})$, is not well-bounded on $[0,\infty)$. We may show the same result on $L^1(\mathbb{R})$ with a duality argument, as follows. If $-\Delta$, on $L^1(\mathbb{R})$, were well-bounded on $[0,\infty)$, let $K(t)$ be as in Theorem 3.1(e). Then arguments similar to those above show that $\exists$ a constant M such that, for any compactly supported $g \in L^1(\mathbb{R})$, f continuous,

$$\Big|\int_{\mathbb{R}} (P_t * g)(x) f(x)\,dx\Big| \le M\|g\|_1\|f\|_\infty,$$

which implies that

$$\|P_t * g\|_1 \le M\|g\|_1,$$

for any compactly supported $g \in L^1(\mathbb{R})$, which implies that $\|P_t\|_1$ is finite, a contradiction. ■

References

[1] W. Arendt, *Vector valued Laplace transforms and Cauchy problems*, Israel J. Math. 59 (1987), 327–352.

[2] H. Benzinger, E. Berkson and T. A. Gillespie, *Spectral families of projections, semigroups, and differential operators*, Trans. Amer. Math. Soc. 275 (1983), 431–475.

[3] E. Berkson, *Semigroups of scalar type operators and a theorem of Stone*, Illinois J. Math. 10 (1966), 345–352.

[4] R. deLaubenfels, *Scalar-type spectral operators and holomorphic semigroups*, Semigroup Forum 33 (1986), 257–263.

[5] H. R. Dowson, *Spectral Theory of Linear Operators*, Academic Press, 1978.

[6] N. Dunford and J. T. Schwartz, *Linear Operators*, Part III, Interscience, New York 1971.

[7] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford University Press, 1985.

[8] F. Neubrander and B. Hennig, *On representations, inversions and approximations of Laplace transforms in Banach spaces*, Resultate Math., to appear.

[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York 1983.

[10] D. J. Ralph, *Semigroups of well-bounded operators and multipliers*, Thesis, Univ. of Edinburgh, 1977.

[11] W. Ricker, *Spectral properties of the Laplace operator in $L^p(\mathbb{R})$*, Osaka J. Math. 25 (1988), 399–410.

[12] A. R. Sourour, *Semigroups of scalar type operators on Banach spaces*, Trans. Amer. Math. Soc. 200 (1974), 207–232.

[13] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton 1970.

[14] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton 1946.

DEPARTMENT OF MATHEMATICS
OHIO UNIVERSITY
ATHENS, OHIO 45701
U.S.A.

Received January 16, 1991
Revised version January 22, 1992
(2768)