STUDIA MATHEMATICA 103 (2) (1992)

Strictly ergodic Toeplitz flows with positive entropies and trivial centralizers

by

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Abstract. A class of strictly ergodic Toeplitz flows with positive entropies and trivial topological centralizers is presented.

Introduction. D. Newton has asked in [2] if there is a coalescent ergodic dynamical system with positive metric entropy. This problem seems to be difficult and it has not been solved so far. The analogous problem in topological dynamics has been solved by P. Walters [3]. He has given an example of a coalescent topological flow with positive topological entropy. However, this example is not minimal. In this paper we construct a class of strictly ergodic topological Toeplitz flows with positive entropies and trivial topological centralizers. Of course, they are topologically coalescent.

We summarize some basic definitions and results. We shall use $\mathbb{Z}, \mathbb{N}$ to denote the integers and the positive integers respectively. By a *flow* we will mean a pair (X,T), where X is a compact metric space and T is a homeomorphism of X onto itself. A flow (X,T) is *minimal* if X has no proper closed T-invariant subsets. A flow (Y,S) is a *factor* of (X,T) if there is a continuous map Π of X onto Y with $\Pi\circ T = S\circ\Pi$. If Π is a homeomorphism, then (X,T) and (Y,S) are *isomorphic* as flows. Every minimal flow (X,T) has a maximal equicontinuous factor (G,g), $\Pi:(X,T)\to(G,g)$, where G is a compact metric monothetic group with a generator g. If $\Pi':(X,T)\to(G',g')$ is another such factor, then there is a factor map $\psi:(G,g)\to(G',g')$ such that $\psi\circ\Pi = \Pi'$.

By the *topological centralizer* of (X,T) we will mean the set of all continuous maps $U:X\to X$ which commute with T. We use $C(T)$ to denote the centralizer of T. $C(T)$ is automatically a semigroup. We say that (X,T) is *topologically coalescent* if every $U\in C(T)$ is a homeomorphism. In this case

1991 *Mathematics Subject Classification*: 28D20, 54H20.
Key words and phrases: Toeplitz flows, entropy.
Research supported by RP.I.10.

$C(T)$ is a group. A flow (X,T) *has trivial centralizer* if $C(T)=\{T^j : j\in\mathbb{Z}\}$. (X,T) is called *strictly ergodic* if it is minimal and if there exists a unique T-invariant Borel normalized measure on X.

Let P be a finite set. Let Ω be the space of all bisequences over P with its natural compact metric topology and let σ be the shift homeomorphism on Ω. In the special case when $P=\{0,1,\ldots,k-1\}$, $k\ge 2$, we will write Σ_k instead of Ω. If $\omega\in\Omega$, then $\omega[n]$ will denote the value of ω at $n\in\mathbb{Z}$ and $\mathrm{O}(\omega)$ will denote the orbit of ω. A finite sequence $B=(B[0]\ldots B[n-1])$, $B[i]\in P$, $n\ge 1$, is called a *block over* P or simply a *block*. The number n is called the *length* of B and is denoted by $|B|$. If $\omega\in\Omega$ and B is a block, then $\omega[i,k]$ $(i\le k)$ and $B[i,k]$ $(0\le i\le k\le n-1)$ denote the blocks $(\omega[i]\ldots\omega[k])$ and $(B[i]\ldots B[k])$ respectively. Let $C=(C[0]\ldots C[m-1])$ be another block. The *concatenation* of the blocks B and C is the block

$$BC=B[0]\ldots B[n-1]C[0]\ldots C[m-1].$$

In the same manner we can define the concatenation of more than two blocks. By $\mathrm{fr}(B,C)$, $|B|<|C|$, we mean the average relative frequency of B in C, i.e.

$$\mathrm{fr}(B,C)=\frac{1}{|C|-|B|}\cdot\mathrm{card}\{0\le i\le |C|-|B|-1 : C[i,i+|B|-1]=B\}.$$

Now, we are in a position to define a Toeplitz sequence over P. Let $\{p_t\}_{t=0}^{\infty}$ be a sequence of positive integers such that

(1) $\quad p_t$ divides p_{t+1} and $\lambda_{t+1}=p_{t+1}/p_t\ge 2,\ t\ge 0,\ \lambda_0=p_0\ge 2$.

Assume that blocks A_t, $t\ge 0$, satisfy the following conditions:

(A) $|A_t|=p_t$,

(B) some places of A_t are occupied by elements of P (*filled places*) and part of them are not filled (*holes*),

(C) the block A_{t+1} is obtained as the concatenation of λ_{t+1} copies of A_t, $\underbrace{A_t\ldots A_t}_{\lambda_{t+1}}$, where some holes are filled by symbols in P,

(D) for every $i\in\mathbb{N}$ there exists an index t such that $A_t[i]\in P$ and $A_t[p_t-i]\in P$ (i and p_t-i are filled places in A_t).

We define bisequences ω_t, $t\ge 0$, over P and the symbol "$-$" (hole) as follows:

$$\omega_t[kp_t,(k+1)p_t-1]=A_t\quad\text{for every } k\in\mathbb{Z}\text{ and } t=0,1,\ldots$$

The sequences ω_t, $t\ge 0$, determine completely a bisequence ω such that

(2) $$\omega=\lim_t \omega_t.$$

The condition (D) implies that $\omega\in\Omega$ (all places are filled).

A sequence ω constructed as above is called a *Toeplitz sequence* if p_t is the smallest period of ω_t for every $t\ge 0$. The sequence $\{p_t\}$ is said to be the *period structure* of ω.

It follows from [4] that if p_t is a period structure of ω then ω is not periodic. We will assume additionally that for every $t\ge 0$ and every i, $0\le i\le p_t-1$, such that $A_t[i]=$"$-$" there exist two places j and j' in ω satisfying

(3) $$\omega[j]=\omega[j']\quad\text{and}\quad j\equiv j'\equiv i\ (\mathrm{mod}\,p_t).$$

A sequence ω is *regular* if

$$\lim_{t\to\infty} k_t/p_t=1,$$

where k_t is the number of all filled places in A_t. By a *t-symbol* of ω we mean every block of the form

$$\omega[kp_t,(k+1)p_t-1],\qquad k=0,\pm1,\pm2,\ldots$$

Every t-symbol coincides with A_t at the filled places. A *Toeplitz flow* is the pair $(\overline{\mathrm{O}(\omega)},\sigma)$, where $\overline{\mathrm{O}(\omega)}$ is the orbit closure of ω. It is known that $(\overline{\mathrm{O}(\omega)},\sigma)$ is minimal [4]. If ω is regular then $(\overline{\mathrm{O}(\omega)},\sigma)$ is strictly ergodic.

Let

$$G=\Big\{g=\sum_{t=0}^{\infty} g_t p_{t-1} :\ 0\le g_t\le\lambda_t-1,\ p_{-1}=1\Big\},$$

be the group of all p_t-adic integers and let T be the rotation of G by 1. It is proved in [4] that (G,T) is the minimal equicontinuous factor of $(\overline{\mathrm{O}(\omega)},\sigma)$. To define a corresponding homeomorphism Π from $(\overline{\mathrm{O}(\omega)},\sigma)$ onto (G,T) we construct a special partition $\{X_g\}$, $g\in G$, of $\overline{\mathrm{O}(\omega)}$. For fixed t, $t\ge 0$, and j, $0\le j\le p_t-1$, we set

$$X_j^t=\{x\in\overline{\mathrm{O}(\omega)} : x[-l_t+kp_t,-l_t+(k+1)p_t-1]\ \text{is a } t\text{-symbol for } k=0,\pm1,\ldots\},$$

where

$$l_t=\sum_{i=0}^{t} g_i p_{i-1}.$$

The sets X_j^t, $j=0,1,\ldots,p_t-1$, are pairwise disjoint, closed and open.

Define

$$X_g = \bigcap_{t=0}^{\infty} X_{l_t}^t .$$

The sets X_g, $g \in G$, are closed and nonempty and they form a partition of $\overline{\mathrm{O}(\omega)}$. Moreover,

$$\sigma(X_g) = X_{g+1}, \qquad g \in G .$$

The factor map $\Pi : (\overline{\mathrm{O}(\omega)}, \sigma) \to (G, T)$ is defined by

$$\Pi(X_g) = g .$$

1. Toeplitz flows with trivial centralizers. In this part we define a special property of Toeplitz sequences which guarantees the trivial topological centralizers for the corresponding Toeplitz flows. We say that a Toeplitz sequence ω satisfies the *condition* $(*)$ if every subblock

$$A_{t+1}[kp_t, (k+1)p_t - 1], \qquad k = 0, 1, \ldots, \lambda_{t+1} - 1,$$

of A_{t+1} is either equal to A_t or is completely filled (such a filled fragment is a t-symbol).

THEOREM 1. *If ω is a nonperiodic Toeplitz sequence* (*regular or not*) *satisfying the condition* $(*)$, *then the topological centralizer $C(\sigma)$ of $(\overline{\mathrm{O}(\omega)}, \sigma)$ is trivial.*

Proof. Let $S \in C(\sigma)$. S induces a continuous map S' of G commuting with T because (G, T) is the maximal equicontinuous factor of $(\overline{\mathrm{O}(\omega)}, \sigma)$. It is known that S' is a translation of G by an element $h = \sum_{t=0}^{\infty} h_t p_{t-1}$, $0 \le h_t \le \lambda_t - 1$. This means that

$$S(X_g) = X_{g+h} ,$$

for any $g \in G$. In particular, $S(\omega) = y \in X_h$ because $\omega \in X_0$. At the same time the mapping S is determined by a code f having a length k, $k \ge 1$, i.e. $f : P^k \to P$ (P^k is the set of all blocks over P of length k) is a mapping such that

$$z[i] = f(u[i, i + k - 1])$$

for every $i \in \mathbb{Z}$, whenever $z = S(u)$, $z, u \in \overline{\mathrm{O}(\omega)}$. Let

$$m_t = \sum_{i=0}^{t} h_i p_{i-1}, \qquad t = 0, 1, \ldots$$

Consider the blocks

$$\mathrm{I}_l = \omega[lp_{t+1}, (l+1)p_{t+1} - 1] ,$$
$$\mathrm{II}_l = y[-m_{t+1} + lp_{t+1}, -m_{t+1} + (l+1)p_{t+1} - 1] ,$$

$l = 0, \pm 1, \pm 2, \ldots$ (see Fig. 1). Each of them is a $(t+1)$-symbol.

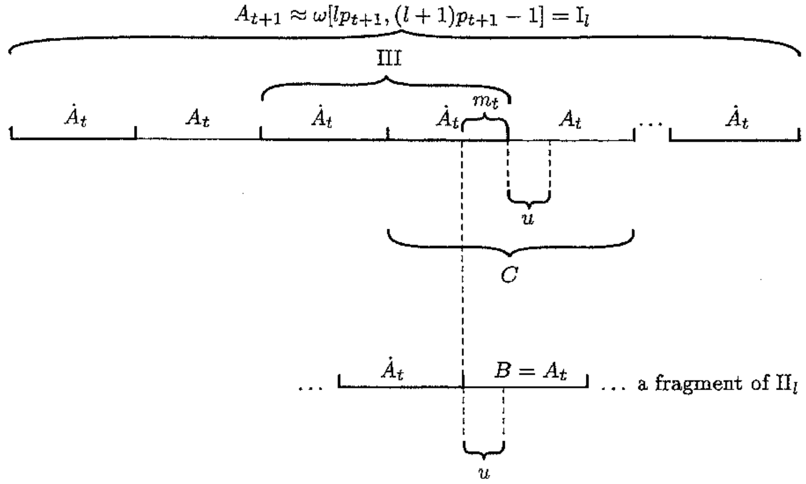

Fig. 1

The block A_{t+1} in the figure is a part of I_l. By $\dot{A}_t$ we denote those subblocks $A_{t+1}[sp_t, (s+1)p_t + 1]$, $0 \le s \le \lambda_{t+1} - 1$, that are completely filled (in Fig. 1 marked by the thick lines). The remaining such subblocks are equal to A_t. Suppose that there exists a series of successive fragments of A_{t+1} consisting of $\dot{A}_t$ (denoted by III in Fig. 1) such that the corresponding fragment B occurring in II_l (in fact in A_{t+1}) under the block $C = \dot{A}_t A_t$ is equal to A_t. Let u_t be the number of the hole in A_t nearest to the left end. The condition (D) implies $u_t \to \infty$. Then using a coding argument it is easy to see that $m_t \le k$ (for t large enough). In a similar way we prove that $m_t \ge p_t - k - 1$ if there exists a series of successive filled fragments of A_{t+1} such that the corresponding fragment of II_l appearing under the block $A_t \dot{A}_t = C$ (see Fig. 2) is equal to A_t.

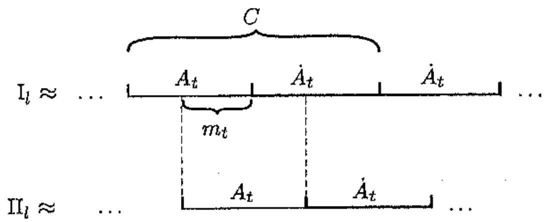

Fig. 2

It is not difficult to see that one of the cases described in Figures 1 and 2 must occur. Therefore, either $m_t \le k$ or $m_t \ge p_t - k - 1$ for t large enough. That means that h is a rational p_t-adic integer and $S = \sigma^h$. ∎

2. Strictly ergodic Toeplitz flows with positive entropy. We will need a result of Grillenberger [1].

THEOREM 2. *For every positive real number $\overline{h}$ and integer $k \geq 2$ such that $0 < \overline{h} < \log k$, there exists a sequence $x = x(\overline{h}, k) \in \Sigma_k$ such that the topological system $(\overline{\mathrm{O}(x)}, \sigma)$ is strictly ergodic and $\mathrm{h_{top}}(\sigma) = \overline{h}$.*

In the sequel we will use two properties of the sequence $x(\overline{h}, k)$.

For any block B over $\{0,1,\ldots,k-1\}$ and $\varepsilon > 0$ there exists a positive integer L_0 such that

$$\text{(E)} \qquad |\mathrm{fr}\,(B, x[n, n+L-1]) - \mu(B)| < \varepsilon$$

for every $L \geq L_0$ and $n \in \mathbb{Z}$, where μ is the unique σ-invariant measure on $\overline{\mathrm{O}(x)}$.

For any $\varepsilon > 0$ there exists n_0 such that

$$\text{(F)} \qquad \Theta(n) > \exp[n(h-\varepsilon)]$$

for $n \geq n_0$, where $\Theta(n)$ is the number of blocks of length n occurring in x.

The properties (E) and (F) follow from the strict ergodicity of $(\overline{\mathrm{O}(x)}, \sigma)$ and from the definition of the topological entropy.

First we construct blocks that will be t-symbols for a Toeplitz sequence.

Let $h > 0$. Let $\varepsilon_0, \varepsilon_1, \ldots$ be positive real numbers such that

$$\text{(4)} \qquad \sum_{i=0}^{\infty} \varepsilon_i < \frac{h}{2}.$$

We will construct positive integers $l_0, l_1, \ldots, l_t > 2$, $t \geq 0$, and blocks $C_1^{(t)}, \ldots, C_{l_{t+1}}^{(t)}$ over $\{0,1,\ldots,l_{t+1}-1\}$ such that

(5) $|C_1^{(t)}| = \ldots = |C_{l_{t+1}}^{(t)}| = \lambda_t, \quad \lambda_t > 2,$

(6) $C_1^{(t)}[0] = \ldots = C_{l_{t+1}}^{(t)}[0] = 0, \quad C_1^{(t)}[\lambda_t - 1] = \ldots = C_{l_{t+1}}^{(t)}[\lambda_t - 1] = 1,$

(7) $|\mathrm{fr}\,(i, C_j^{(t)}) - \mathrm{fr}(i, C_k^{(t)})| < \varepsilon_t/(4l_t)$
for every $i = 0, 1, \ldots, l_t - 1$ and every $j, k = 1, \ldots, l_{t+1}$ and $t \geq 0$,

(8) $\mathrm{fr}\,(i, C_j^{(t)}) > 0, \quad t \geq 1,$

(9) $l_{t+1} \geq q \exp[\lambda_t(h_t - \varepsilon_t/2)]$, where $h_0 = h$, $h_t = \log l_t - \eta_t$, $t \geq 1$,

where $\eta_0, \eta_1, \ldots$ are positive numbers with

$$\text{(10)} \qquad \eta_t < \varepsilon_t/2.$$

S t a r t. We choose a positive integer $l_0 \geq 2$ such that

$$\log(l_0 - 1) \leq h \leq \log l_0$$

and a sequence $x_0 \in \Sigma_{l_0}$ satisfying the conclusion of Theorem 2 with $\overline{h} = h$ and $k = l_0$. Let μ_0 be the unique σ-invariant measure on $\mathrm{O}(x_0)$. Then applying (E) and (F) we choose q_0 such that

$$\text{(11)} \qquad \frac{2}{q_0 + 2} < \frac{\varepsilon_0}{16 l_0} \quad \text{and} \quad |\mathrm{fr}\,(i, B) - \mu_0(i)| < \frac{\varepsilon_0}{16 l_0},$$

where B is any block occurring in x_0 with length q_0. At the same time we require $l_1 = \Theta(q_0)$ to satisfy

$$\text{(12)} \qquad l_1 \geq \exp[(q_0 + 2)(h - \varepsilon_0/2)], \qquad l_1 > 2.$$

Let $B_1^{(0)}, B_2^{(0)}, \ldots, B_{l_1}^{(0)}$ be all blocks of length q_0 appearing in x_0. Now, we define

$$C_j^{(0)} = 0 B_{j+1}^{(0)} 1, \qquad j = 0, 1, \ldots, l_1 - 1.$$

It follows from (11) and (12) that the numbers l_0, l_1 and the blocks $C_1^{(0)}, \ldots, C_{l_1}^{(0)}$ satisfy (5)–(10) with $\lambda_0 = q_0 + 2$.

I n d u c t i o n s t e p. Suppose we have defined positive integers $l_0, \ldots, l_t$ and blocks $C_1^{(t-1)}, \ldots, C_{l_t}^{(t-1)}$ satisfying (5)–(10). Applying Theorem 2 we choose a sequence Σ_{l_t} such that $(\overline{\mathrm{O}(x_t)}, \sigma)$ is strictly ergodic and

$$\mathrm{h_{top}}(\overline{\mathrm{O}(x_t)}, \sigma) = h_t = \log l_t - \eta_t,$$

where

$$\eta_t < \varepsilon_t/2 \quad \text{and} \quad \eta_t < \log l_t - \log(l_t - 1).$$

Let μ_t denote the unique σ-invariant measure on $\overline{\mathrm{O}(x_t)}$. Then $\mathrm{fr}(i, x_t) = \mu_t(i) > 0$ for every $i = 0, 1, \ldots, l_t - 1$, because in the opposite case we would have $\mathrm{h_{top}}(\overline{\mathrm{O}(x_t)}, \sigma) \leq \log(l_t - 1)$. Applying again (E) and (F) we choose a positive integer q_t such that

$$\text{(13)} \qquad \frac{2}{q_t + 2} < \frac{\varepsilon_t}{16 l_t}, \qquad |\mathrm{fr}\,(i, B) - \mu_t(i)| < \frac{\varepsilon_t}{16 l_t},$$

$$\text{(14)} \qquad \mathrm{fr}(i, B) > 0,$$

for $i = 0, 1, \ldots, l_t - 1$ and for any block B, $|B| = q_t$, appearing in x_t. Moreover,

$$\text{(15)} \qquad \Theta(q_t) = l_{t+1} > \exp[(q_t + 2)(h_t - \varepsilon_t/2)].$$

Of course, we can require that $l_{t+1} > 2$.

Define

$$C_j^{(t)} = 0 B_{j+1}^{(t)} 1, \qquad j = 0, 1, \ldots, l_{t+1} - 1,$$

where $B_1^{(t)}, \ldots, B_{l_{t+1}}^{(t)}$ are all blocks of length q_t occurring in x_t. The definition of $C_j^{(t)}$, $0 \leq j \leq l_{t+1} - 1$, and (13)–(15) imply (5)–(10) for $\lambda_t = q_t + 2$.

Construction of a Toeplitz sequence. To construct t-symbols we use an operation over blocks. For a block B, $|B| = k$, over a set of symbols $\overline{S} = \{s_1, \ldots, s_n\}$ and blocks $A_{s_1}, \ldots, A_{s_n}$ over another set of symbols $\overline{\overline{S}}$, define a block

$$\{A_{s_1}, \ldots, A_{s_n}\} * B$$

over $\overline{\overline{S}}$ to be the concatenation

$$A_{B[0]}A_{B[1]} \ldots A_{B[k-1]}\,.$$

We will define t-symbols $A_0^{(t)}, \ldots, A_{l_{t+1}-1}^{(t)}$, $t \geq 0$, over the symbols $P = \Sigma_{l_0}$ such that

$$|A_j^{(t)}| = p_t, \quad j = 0, 1, \ldots, l_{t+1} - 1\,,$$

where $p_t = \lambda_0 \ldots \lambda_t$, $t \geq 0$. Let

(16) $$A_j^{(0)} = C_j^{(0)}, \quad j = 0, \ldots, l_1 - 1\,,$$

(17) $$A_j^{(t)} = \{A_0^{(t-1)}, \ldots, A_{l_t-1}^{(t-1)}\} * C_j^{(t)}, \quad j = 0, \ldots, l_{t+1} - 1,\ t \geq 1\,.$$

Using the blocks $A_j^{(t)}$, $0 \leq j \leq l_{t+1} - 1$, we now define blocks A_t, $t \geq 0$, over P and over the symbol "$-$" satisfying (A), (B), (C), (D) and $|A_t| = p_t$. Set

(18) $$A_0 = \overbrace{0\underbrace{- - \ldots -}_{q_0 \text{ times}}1}^{p_0}, \quad p_0 = q_0 + 2\,,$$

(19) $$A_{t+1} = A_0^{(t)}\underbrace{A_t \ldots A_t}_{q_{t+1} \text{ times}}A_1^{(t)}, \quad t \geq 0\,.$$

THEOREM 3. *Let ω be the Toeplitz sequence determined by the sequence of blocks* (18) *and* (19). *Then the Toeplitz flow $(\overline{\mathrm{O}(\omega)}, \sigma)$ is strictly ergodic and $\mathrm{h_{top}}(\overline{\mathrm{O}(\omega)}, \sigma) > 0$.*

Proof. We start with an estimation of the topological entropy. We have

$$\mathrm{h_{top}}(\overline{\mathrm{O}(\omega)}, \sigma) = \lim_{t\to\infty} \Theta_t/p_t\,,$$

where Θ_t is the number of blocks of length p_t appearing in ω. It follows from (13), (14) and (17) that each t-symbol $A_j^{(t)}$, $0 \leq j \leq l_{t+1} - 1$, contains all $(t-1)$-symbols as subblocks. Then (18) and (19) imply that ω contains all t-symbols for $t = 0, 1, \ldots$ Therefore $\Theta_t \geq l_{t+1}$ and

(20) $$\mathrm{h_{top}}(\overline{\mathrm{O}(\omega)}, \sigma) \geq \limsup_{t\to\infty} \frac{\log l_{t+1}}{p_t}\,.$$

Then (9) and (10) give

$$\frac{\log l_{t+1}}{p_t} \geq \frac{\lambda_t[h_t - \varepsilon_t/2]}{p_t} = \frac{h_t - \varepsilon_t/2}{p_{t-1}} \geq \frac{\log l_t}{p_{t-1}} - \varepsilon_t\,.$$

Repeatedly using (12) and the above inequality we obtain

$$\frac{\log l_{t+1}}{p_t} \geq h - \varepsilon_0 - \ldots - \varepsilon_t\,.$$

Now (4) and (20) give

$$\mathrm{h_{top}}(\overline{\mathrm{O}(\omega)}, \sigma) \geq h/2\,.$$

It remains to show the unique ergodicity of $(\overline{\mathrm{O}(\omega)}, \sigma)$. Take any block B over P and $\varepsilon > 0$. Choose t_0 such that

(21) $$|B|/p_t < \varepsilon_t/4 \quad \text{for } t \geq t_0.$$

Define

$$a_{ij}^{(t)} = \mathrm{fr}(i, C_j^{(t)}), \quad i = 0, 1, \ldots, l_t - 1 \text{ and } j = 0, \ldots, l_{t+1} - 1\,.$$

It follows from (17) that

$$\mathrm{fr}(B, A_j^{(t)}) = \sum_{i=0}^{l_t-1} \mathrm{fr}(B, A_i^{(t-1)}) \circ a_{ij}^{(t)} + \delta_t\,,$$

where $\delta_t < |B|/p_{t-1}$, $t > t_0$. The inequalities (7) and (21) imply

(22) $$|\mathrm{fr}\,(B, A_j^{(t)}) - \mathrm{fr}(B, A_k^{(t)})| \leq \sum_{i=0}^{l_t-1} \mathrm{fr}(B, A_i^{(t-1)}) \cdot |a_{ij}^{(t)} - a_{ik}^{(t)}| \cdot 2\delta_t < \varepsilon_t/4 + 2\delta_t < \varepsilon_t$$

for $j, k = 0, 1, \ldots, l_{t+1} - 1$. Combining (22) with the fact that ω is an infinite concatenation of t-symbols for every $t \geq 0$, we conclude that for given $\varepsilon > 0$ there exists a positive integer L such that

$$|\mathrm{fr}\,(B, \omega[n, n + L - 1]) - \mathrm{fr}(B, \omega[m, m + L - 1])| < \varepsilon$$

for all $n, m \in \mathbb{Z}$. This means the unique ergodicity of the Toeplitz flow $(\overline{\mathrm{O}(\omega)}, \sigma)$. Theorem 3 is proved. ∎

The Toeplitz sequence determined by the blocks (18) and (19) satisfies the condition (8). Hence Theorem 1 implies that the topological centralizer $C(\sigma)$ is trivial.

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Received November 22, 1990 (2748)
Revised version November 8, 1991

STUDIA MATHEMATICA 103 (2) (1992)

Unbounded well-bounded operators, strongly continuous semigroups and the Laplace transform

by

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Abstract. Suppose A is a (possibly unbounded) linear operator on a Banach space. We show that the following are equivalent.

(1) A is well-bounded on $[0,\infty)$.

(2) $-A$ generates a strongly continuous semigroup $\{e^{-sA}\}_{s\geq 0}$ such that $\{(1/s^2)e^{-sA}\}_{s>0}$ is the Laplace transform of a Lipschitz continuous family of operators that vanishes at 0.

(3) $-A$ generates a strongly continuous differentiable semigroup $\{e^{-sA}\}_{s\geq 0}$ and $\exists M<\infty$ such that

$$\|H_n(s)\| \equiv \left\|\left(\sum_{k=0}^{n}\frac{s^kA^k}{k!}\right)e^{-sA}\right\| \leq M, \quad \forall s>0,\ n\in\mathbb{N}\cup\{0\}.$$

(4) $-A$ generates a strongly continuous holomorphic semigroup $\{e^{-zA}\}_{\mathrm{Re}(z)>0}$ that is $O(|z|)$ in all half-planes $\mathrm{Re}(z)>a>0$ and

$$K(t) \equiv \int_{1+i\mathbb{R}} e^{zt}e^{-zA}\,\frac{dz}{2\pi i z^3}$$

defines a differentiable function of t, with Lipschitz continuous derivative, with $K'(0)=0$.

We may then construct a decomposition of the identity, F, for A, from $K(t)$ or $H_n(s)$. For $\phi\in X^*$, $x\in X$,

$$(F(t)\phi)(x) = (d/dt)^2(\phi(K(t)x)) = \lim_{n\to\infty}\phi(H_n(n/t)x),$$

for almost all t.

I. Introduction. Scalar operators (see [5], [6]) with real spectrum are a generalization, to arbitrary Banach spaces, of self-adjoint operators on a Hilbert space. An early disappointment was the fact that most standard differential operators on an L^p space are scalar only when p equals 2. However, if one weakens the definition by requiring uniformly bounded spectral projections corresponding only to closed intervals, rather than arbitrary closed

1991 *Mathematics Subject Classification*: 47A60, 47B40.