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Factors of ergodic group extensions of rotations

by

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Abstract. Diagonal metric subgroups of the metric centralizer $C(T_\varphi)$ of group extensions are investigated. Any diagonal compact subgroup Z of $C(T_\varphi)$ is determined by a compact subgroup Y of a given metric compact abelian group X, by a family $\{v_y : y \in Y\}$, of group automorphisms and by a measurable function $f : X \to G$ (G a metric compact abelian group). The group Z consists of the triples (y, F_y, v_y), $y \in Y$, where $F_y(x) = v_y(f(x)) - f(x+y), x \in X$.

Introduction. Lemańczyk and Mentzen [4] have proved that all factors of group extensions of dynamical systems with discrete spectra are completely determined by compact subgroups of their metric centralizers. If $T : X \to X$ is an automorphism with pure point spectrum and $\varphi : X \to G$ (G a metric compact abelian group) is a cocycle such that T_φ is ergodic, then the metric centralizer $C(T_\varphi)$ can be identified with a closed subset of $C(T) \times \mathcal{M} \times \mathrm{End}(G)$, where $\mathcal{M}$ is the set of all measurable functions $F : X \to G$, and $\mathrm{End}(G)$ is the set of all continuous epimorphisms of G. A triple $(S, F, v) \in C(T) \times \mathcal{M} \times \mathrm{End}(G)$ determines an element of $C(T_\varphi)$ if and only if

$$(*) \qquad F(Tx) - F(x) = \varphi(Sx) - v(\varphi(x)), \qquad x \in X.$$

One can distinguish a special class of compact subgroups of $C(T_\varphi)$, namely the so-called diagonal compact subgroups. We say that a compact subgroup C_0 of $C(T_\varphi)$ is *diagonal* if S runs over some compact subgroup Y of $C(T)$ and there is only one function $F = F_S$, $S \in Y$, satisfying $(*)$ (it is known [2] that if $(*)$ can be solved for some S, then v is determined univocally). In this way the diagonal compact subgroups of $C(T_\varphi)$ are of the form

$$C_0 = \{(S, F_S, v_S) : S \in Y\},$$

where Y is a compact subgroup of $C(T)$.

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The main result of this paper says that the families $\{F_S\}$ and $\{v_S\}$, $S \in Y$, are connected in the following way: there exists a measurable function $f : C(T) \to G$ satisfying the additional condition (see Theorem 1)

$$F_S(U) = v_S(f(U)) - f(U \circ S),$$

for $U \in C(T), S \in Y$. As a consequence we obtain a necessary and sufficient condition for the cocycle $\varphi : X \to G$ to be a coboundary. Namely, φ is a coboundary if and only if $\varphi^{(n)}$ (see Section 1) is close to 0 in measure whenever T^n is close to the identity in the weak topology of $C(T)$. This was proved by Veech [8] for X and G being the circle and T an ergodic rotation of X. Another proof of Veech's theorem was given by Rychlik [7]. L. Baggett [1] has given other criteria for circle-valued cocycles to be coboundaries.

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1. Notations and definitions. Let $T : (X, \mu) \to (X, \mu)$ be an ergodic automorphism with discrete spectrum, and let G be a metric compact abelian group with the normalized Haar measure m. Given a cocycle $\varphi : X \to G$ one can define an automorphism $T_\varphi : (X \times G, \mathcal{F}, \mu \times m) \to (X \times G, \mathcal{F}, \mu \times m)$,

$$T_\varphi(x, g) = (Tx, g + \varphi(x)),$$

where $\mathcal{F}$ is the product σ-algebra. The automorphism T_φ is ergodic [6] if and only if for every $\gamma \in \widehat{G}$ (the dual group of G), $\gamma \neq 1$, there is no measurable function $f : X \to K = \{z : |z| = 1\}$ such that

$$f(Tx)/f(x) = \gamma(\varphi(x)) \quad \text{for } \mu\text{-a.e. } x \in X.$$

By the *metric centralizer* $C(T_\varphi)$ of T_φ we mean the set of all measure preserving transformations $\widetilde{S} : X \times G \to X \times G$ commuting with T_φ. It is known [5] that if T_φ is ergodic then every $\widetilde{S}$ has the form

$$\widetilde{S}(x, g) = (Sx, F(x) + v(g)), \tag{1}$$

where $S \in C(T)$, $F : X \to G$ is a measurable function, and $v : G \to G$ is a continuous epimorphism satisfying

$$F(Tx) - F(x) = \varphi(Sx) - v(\varphi(x)) \quad \text{for } \mu\text{-a.e. } x \in X. \tag{2}$$

Moreover, $\widetilde{S}$ is an automorphism if and only if so is v. We will write $\widetilde{S} = (S, F, v)$. The set $C(T_\varphi)$ is a topological semigroup with the weak topology. Lemańczyk and Liardet [3] have shown that the weak topology of $C(T_\varphi)$ coincides with the corresponding product topology, i.e. $\widetilde{S}_n = (S_n, F_n, v_n)$ converges to $\widetilde{S} = (S, F, v)$ if and only if $S_n \to S$ in $C(T)$, $F_n \to F$ in measure and $\|\gamma \circ v_n - \gamma \circ v\|_1 \to 0$ for every $\gamma \in \widehat{G}$.

We will assume that X is a monothetic group and let x_0 be an element of X such that the set $\{n \cdot x_0 : n \in \mathbb{Z}\}$ is dense in X. Put $T(x) = x + x_0$. For every $y \in X$ we define $S = S_y$, $S(x) = x + y$. It is known that $C(T) = \{S_y : y \in X\}$, so $C(T)$ can be identified with X. Then (1) and (2) can be rewritten as

$$\begin{cases} \widetilde{S}(x, g) = (x + y, F(x) + v(g)), \\ F(x + x_0) - F(x) = \varphi(x + y) - v(\varphi(x)). \end{cases} \tag{3}$$

We say that $y \in X$ *can be lifted to* $C(T_\varphi)$ if there exist a measurable function F and an epimorphism $v : G \to G$ satisfying (3). If $y \in X$ can be lifted, then there exists exactly one $v = v_y$ satisfying (3), and if $\overline{F}$ is another function satisfying (3) then, for some $h \in G$, $\overline{F}(x) = F(x) + h$ for a.e. $x \in X$ [2]. Of course, each element $y = n \cdot x_0$, $n = 0, \pm 1, \ldots$, lifts with $v = \mathrm{id}$ and the functions $\varphi^{(n)} + h$, where

$$\varphi^{(n)}(x) = \begin{cases} 0, & n = 0, \\ \varphi(x) + \ldots + \varphi(x + (n-1) \cdot x_0), & n \geq 1, \\ -\varphi(x - x_0) - \ldots - \varphi(x - n \cdot x_0) = \varphi^{(-n)}(T^n x), & n \leq -1. \end{cases}$$

In particular, $(0, h, \mathrm{id}) \in C(T_\varphi)$ for every $h \in G$. We will write σ_h instead of $(0, h, \mathrm{id})$ so that

$$\sigma_h(x, g) = (x, g + h).$$

The compact subgroups of $C(T_\varphi)$ play an important role. It is known [4] that there is one-to-one correspondence between such subgroups and the factors of T_φ. Given a compact subgroup $C_0 \subset C(T_\varphi)$ we define the corresponding factor $\mathcal{A} = \mathcal{A}(C_0)$ (a T_φ-invariant sub-σ-algebra of $\mathcal{F}$) as follows:

$$\mathcal{A} = \{A \in \mathcal{F} : \widetilde{S}^{-1}(A) = A \text{ for every } \widetilde{S} \in C_0\}.$$

Now, we want to describe all compact subgroups of $C(T_\varphi)$.

Let C_0 be such a subgroup and let Y be the projection of C_0 on X, i.e. $Y = \{y \in X : (y, F, v) \in C_0 \text{ for some } F, v\}$. It is clear that Y is a compact subgroup of X. It follows from [4] that $v = v_y$ is an automorphism of G for every $y \in Y$. Given a compact subgroup Y of $C(T)$ and a family $\{v_y\}$ of automorphisms of G we say that Y *and* $\{v_y\}$ *can be lifted to a compact subgroup* C_0 *of* $C(T_\varphi)$ if Y is the projection of C_0. Put

$$H = \{h \in G : \sigma_h \in C_0\}.$$

Then H is a closed subgroup of G. If $\widetilde{S} = (y, F, v_y) \in C_0$ and $\widetilde{U} = (y, \overline{F}, v_y) \in C_0$, then $\widetilde{S} \circ \widetilde{U}^{-1} \in C_0$. But

$$\widetilde{U}^{-1} = (-y, -v_y^{-1}(\overline{F}(x - y)), v_y^{-1})$$

and

$$\widetilde{U}^{-1} \circ \widetilde{S} = (0, v_y^{-1} \circ F - v_y^{-1} \circ \overline{F}, \mathrm{id}).$$

Therefore, there exists $h \in H$ satisfying

$$F = \overline{F} + h. \tag{4}$$

Moreover, if $h \in H$ then $(y, v_y(h) + F, v_y) = \widetilde{S} \circ \sigma_h \in C_0$, which gives $v_y(h) \in H$. Thus $v_y(H) = H$ and v_y induces an automorphism of G/H. Denote it by v_y again.

In the case when $H = \{0\}$, C_0 has a special form. Namely, for every $y \in Y$ there is exactly one function $F_y : X \to G$ such that $C_0 = \{(y, F_y, v_y) : y \in Y\}$. This kind of compact subgroups of $C(T_\varphi)$ will be called *diagonal compact subgroups*. It turns out that to describe all compact subgroups of $C(T_\varphi)$ it is enough to examine the diagonal compact subgroups. In fact, the condition (4) means that the function $\widetilde{F} : X \to G/H$, $\widetilde{F}(x) = F(x) + H$, does not depend on the choice of F and we can denote it by $\widetilde{F}_y$. Thus

$$\widetilde{C}_0 = \{(y, \widetilde{F}_y, v_y) : y \in Y\}$$

is a compact subgroup of $C(T_{\widetilde{\varphi}})$, where $\widetilde{\varphi} : X \to G/H$ is induced by $\varphi : X \to G$. The subgroup $\widetilde{C}_0$ determines C_0 completely. It is evident that $\widetilde{C}_0$ is a diagonal compact subgroup of $C(T_{\widetilde{\varphi}})$. Therefore, in order to describe all compact subgroups of $C(T_\varphi)$ it suffices to describe all diagonal compact subgroups of $C(T_{\widetilde{\varphi}})$.

2. Diagonal compact subgroups of $C(T_\varphi)$. Now we are in a position to formulate the main result of the paper. Let Y be a compact subgroup of X and let $\{v_y : y \in Y\}$ be a family of automorphisms of G satisfying

(5) $v_{y+y'} = v_y \cdot v_{y'}$, $y, y' \in Y$, and the mapping $M(y) = v_y$, $M : Y \to \mathrm{Aut}(G)$, is continuous

($\mathrm{Aut}(G)$ is the group of all automorphisms of G). Denote by d a distance in X. We will assume that d is invariant under the rotations of X.

THEOREM 1. *The subgroup Y and the family $\{v_y : y \in Y\}$ can be lifted to a diagonal compact subgroup of $C(T_\varphi)$ if and only if there exists a measurable function $f : X \to G$ such that*

$$v_y[f(x + x_0) - f(x) + \varphi(x)] = f(x + x_0 + y) - f(x + y) + \varphi(x + y) \tag{6}$$

for a.e. $x \in X$ and every $y \in Y$.

Proof. *Sufficiency.* Let f satisfy (6). Put

$$F_y(x) = v_y(f(x)) - f(x + y), \qquad x \in X,\ y \in Y. \tag{7}$$

Using (5) it is not hard to check that

$$F_{y+y'}(x) = v_y(F_{y'}(x)) + F_y(x + y') \tag{8}$$

for a.e. $x \in X$ and all $y, y' \in Y$, and

(9) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y, y' \in Y)$

$$d(y', y) < \delta \Rightarrow \mu\{x \in X : d(F_{y'}(x), F_y(x)) > \varepsilon\} < \varepsilon.$$

Now define

$$C_0 = \{(y, F_y, v_y) : y \in Y\}.$$

The conditions (5), (8) and (9) guarantee that C_0 is a compact subgroup of $Y \times \mathcal{M} \times \mathrm{Aut}(G)$. In this case (6) implies (3) so that $(y, F_y, v_y) \in C(T_\varphi)$ and using Lemańczyk and Liardet's result [3] we see that C_0 is a compact subgroup of $C(T_\varphi)$. Of course, C_0 is a diagonal subgroup.

Necessity. Suppose that $C_0 = \{(y, F_y, v_y) : y \in Y\}$ is a diagonal compact subgroup of $C(T_\varphi)$ over Y. It is obvious that (5), (8) and (9) are satisfied. We will show (Lemma 1 below) that $F_y(x)$ is measurable as a function of x, y. Let $Z = X/Y$ and let $L : X \to Z$ be the natural projection. There exists a measurable set $X_0 = \{x'_z\}_{z \in Z} \subset X$ such that $L(x'_z) = z$. The measure space (X, μ) can be identified with the product measure space $(Y \times Z, \mu_Y \times \mu_Z)$ by the mapping

$$l(y'', z) = y'' + x'_z, \qquad y'' \in Y.$$

Rewrite (8) as

$$F_{y+y'}(x'_z + y'') = v_y(F_{y'}(x'_z + y'')) + F_y(x'_z + y'' + y'), \tag{10}$$

$(y, y', y'', z) \in Y \times Y \times \underbrace{Y \times Z}_{X}$. If we set

$$V = \{(y, y', y'', z) \in Y \times Y \times Y \times Z : (10) \text{ is satisfied}\}$$

then $(\mu_Y \times \mu_Y \times \mu_Y \times \mu_Z)(V) = 1$. Further, let Y_0 be the set of all $y_0 \in Y$ for which

$$(\mu_Y \times \mu_Y \times \mu_Z)\underbrace{\{(y, y', z) : (y, y', y_0, z) \in V\}}_{V_{y_0}} = 1.$$

We have $\mu_Y(Y_0) = 1$. For μ_Y-a.e. $y'' \in Y$ we have

$$(\mu_Y \times \mu_Z)\overbrace{\{(y, z) : F_y(x'_z + y'') \text{ is defined}\}}^{U_{y''}} = 1,$$

so we can choose $y'' = y_0 \in Y_0$ such that

$$(\mu_Y \times \mu_Z)(U) = 1, \qquad U = U_{y''} = U_{y_0}.$$

Put $x_z = x'_z + y_0$, $z \in Z$ and let

$$X_1 = \{x_z + y : (y, z) \in U\}.$$

Then $\mu(X_1) = 1$. Now, we define a function f on X_1 by taking a measurable function $\bar{f} : Z \to G$ and putting

$$f(x_z + y) = v_y(f(x_z)) - F_y(x_z), \qquad (y, z) \in U,\ f(x_z) = \bar{f}(z). \tag{11}$$

Of course, f is measurable. We will show that f satisfies (7) for every $y \in Y$ and a.e. $x \in X$. Let

$$\begin{aligned} V_{y_0,y} &= \{(y', z) \in Y \times Z : (y, y', z) \in V_{y_0}\} \\ &= \{(y', z) : F_{y+y'}(x_z) = v_y(F_{y'}(x_z)) + F_y(x_z + y')\}. \end{aligned}$$

For μ_Y-a.e. $y \in Y$ we have

$$(\mu_Y \times \mu_Z)(V_{y_0,y}) = 1. \tag{12}$$

Fix y satisfying (12) and put $U' = \{(y', z) : (y + y', z) \in U\}$. Then $\{y + y' + x_z : (y', z) \in U\} = X_1 - y$ and $U' = l^{-1}(X_1 - y)$. That gives $(\mu_Y \times \mu_Z)(U') = 1$, which implies $(\mu_Y \times \mu_Z)(V_{y_0,y} \cap U \cap U') = 1$. For $(y', z) \in V_{y_0,y} \cap U \cap U'$ we have

(13) $F_y(x_z)$ and $F_{y+y'}(x_z)$ are defined and

$$F_{y+y'}(x_z) = v_y(F_{y'}(x_z)) + F_y(x_z + y').$$

Take $x = x_z + y'$. Using (11) we obtain

$$\begin{cases} f(x) = f(x_z + y') = v_{y'}(f(x_z)) - F_{y'}(x_z), \\ f(x + y) = f(x_z + y' + y) = v_{y+y'}(f(x_z)) - F_{y+y'}(x_z). \end{cases} \tag{14}$$

Now, (13) and (14) imply

$$\begin{aligned} v_y(f(x)) - f(x + y) &= v_y[v_{y'}(f(x_z)) - F_{y'}(x_z)] - v_{y+y'}(f(x_z)) + F_{y+y'}(x_z) \\ &= F_{y+y'}(x_z) - v_y(F_{y'}(x_z)) = F_y(x_z + y') = F_y(x). \end{aligned}$$

We have shown (7) for μ_Y-a.e. $y \in Y$. In particular, the set of such y's is dense. Now, take an arbitrary $y \in Y$. There exists a sequence $y_n, n \ge 1$, such that $y = \lim_n y_n$ and each y_n satisfies (7). Then $f(x + y_n) \to f(x + y)$ in measure and (9) implies $F_{y_n} \to F_y$ in measure. We will show in Lemma 2 below that $v_{y_n} \to v_y$ uniformly. Hence $v_{y_n}(f(x)) \to v_y(f(x))$ for a.e. $x \in X$. The above properties imply (7) for every $y \in Y$. In this way we have proved the theorem because (3) and (7) imply (6). ∎

Let $\xi = (D_1, \ldots, D_k)$ be a finite measurable partition of Y. By the *diameter* of ξ we mean the number $\operatorname{diam}(\xi) = \max_{1 \le i \le k} \sup_{y,y' \in D_i} d(y, y')$.

LEMMA 1. *If a family of measurable functions $F_y : X \to G$, $y \in Y$, satisfies* (9), *then $F_y(x)$ is a measurable function of two variables x and y, $x \in X$, $y \in Y$.*

Proof. Take a sequence of finite measurable partitions ξ_n of Y such that $\xi_n \prec \xi_{n+1}$ and $\operatorname{diam}(\xi_n) \to 0$. Let $\xi_n = (D_1^{(n)}, \ldots, D_{k_n}^{(n)})$ and choose $y_i^{(n)} \in D_i^{(n)}$, $i = 1, \ldots, k_n$. Define

$$F_n(x, y) = F_{y_i^{(n)}}(x) \quad \text{if } y \in D_i^{(n)}.$$

The functions $F_n : X \times Y \to G$ are measurable and the sequence F_n satisfies the Cauchy condition for convergence in measure. Hence $F_n(x, y) \to F(x, y)$ in $\mu \times \mu_Y$-measure for a measurable function $F(x, y)$. Taking a subsequence we can assume that F_n converges to F for $\mu \times \mu_Y$-a.e. (x, y). Thus for μ-a.e. $x \in X$ and μ_Y-a.e. $y \in Y$

$$\lim_n F_n(x, y) = F(x, y).$$

At the same time $F_n(\cdot, y) \to F_y$ in measure for a.e. $y \in Y$. Consequently, $F_y(x)$ is a measurable function of x and y. ∎

LEMMA 2. *Let v_n, v be automorphisms of G such that $\gamma \circ v_n \xrightarrow{L_1} \gamma \circ v$ for every $\gamma \in \widehat{G}$. Then $v_n \to v$ uniformly.*

Proof. For every $n \ge 1$, $\gamma \circ v_n \in \widehat{G}$ and $\gamma \circ v \in \widehat{G}$. Since

$$\int_G |(\gamma \circ v_n)(g) - (\gamma \circ v)(g)|\, m(dg) \to 0,$$

we have

$$\int_G (\gamma \circ v_n)(g) \cdot \overline{(\gamma \circ v)}(g)\, m(dg) \ne 0$$

for sufficiently large n. Thus for every $\gamma \in \widehat{G}$ there exists a positive integer n_γ such that

$$\gamma \circ v_n = \gamma \circ v \quad \text{if } n \ge n_\gamma. \tag{15}$$

Consider the following distance d_G in G:

$$d_G(g, h) = \sum_{\gamma \in \widehat{G}} a_\gamma |\gamma(g) - \gamma(h)|,$$

where $a_\gamma > 0$ and $\sum_{\gamma \in \widehat{G}} a_\gamma = 1$. Then, for each $\varepsilon > 0$, (15) implies $d_G(v_n(g), v(g)) < \varepsilon$ for n large enough and every $g \in G$. Therefore $v_n \to v$ uniformly. ∎

THEOREM 2. *Suppose that $C_0 = \{(y, F_y, \mathrm{id}) : y \in Y\}$ and $C_1 = \{(y, \overline{F}_y, \mathrm{id}) : y \in Y\}$ are diagonal compact subgroups of $C(T_\varphi)$ over Y. Then there exists a continuous homomorphism $a : Y \to G$ such that*

$$\overline{F}_y(x) = F_y(x) + a(y) \tag{16}$$

for a.e. $x \in X$. Conversely, if C_0 as above is a diagonal compact subgroup of $C(T_\varphi)$ over Y and $\overline{F}_y$ satisfies (16) *then C_1 defined as above is a diagonal compact subgroup of $C(T_\varphi)$ over Y.*

Proof. We have $\overline{F}_y(x) = F_y(x) + a(y)$ because

$$F_y(x+x_0) - F_y(x) = \varphi(x+y) - \varphi(x) = \overline{F}_y(x+x_0) - \overline{F}_y(x)$$

and $T(x) = x + x_0$ is an ergodic transformation of (X,μ). Using (8) for $F_{y+y'}$ and $\overline{F}_{y+y'}$ and (16) we obtain

$$\begin{aligned} &F_{y+y'}(x) + a(y+y') \\ &\quad = \overline{F}_{y+y'}(x) = \overline{F}_{y'}(x) + F_y(x+y') \\ &\quad = F_{y'}(x) + F_y(x+y') + a(y) + a(y') = F_{y+y'}(x) + a(y') + a(y). \end{aligned}$$

Hence $a(y+y') = a(y) + a(y')$. It follows from (9) and (16) that $a(y) \to 0$ whenever $y \to 0$. We have proved the first part of the theorem. The second part is evident. ∎

THEOREM 3. *Let $\varphi : X \to G$ be a cocycle. Then φ is coboundary if and only if*

(17) $(\forall \varepsilon > 0)(\exists \delta > 0)$

$$d(n\cdot x_0, 0) < \delta \Rightarrow \mu\{x \in X : d_G(\varphi^{(n)}(x), 0) > \varepsilon\} < \varepsilon.$$

Proof. *Necessity.* Suppose that φ is a coboundary. Then there exists a measurable function $F : X \to G$ satisfying

$$F(x+x_0) - F(x) = \varphi(x) \quad \text{ for a.e. } x \in X.$$

This implies

$$F(x + n\cdot x_0) - F(x) = \varphi^{(n)}(x)$$

for every $n \geq 1$. It follows from Lusin's theorem that given $\varepsilon > 0$ there exists $\delta > 0$ such that $d(n\cdot x_0, 0) < \delta$ implies $\mu\{x : d_G(F(x+n\cdot x_0), f(x)) > \varepsilon\} < \varepsilon$. In this manner (17) is proved for $n \geq 1$. If $n \leq -1$ then (17) follows from the equality $\varphi^{(n)}(x) = -\varphi^{(-n)}(T^n x)$.

Sufficiency. Assuming (17) we will construct a family of measurable functions $F_y : X \to G$, $y \in X$. Put

$$F_{x_0} = \varphi, \qquad F_{n\cdot x_0} = \varphi^{(n)}, \qquad n = 0, \pm 1, \ldots$$

Take $y \in X$. There exists a sequence of positive integers $\{n_m\}$ such that $d(n_m \cdot x_0, y) \to 0$. The sequence $F_{n_m\cdot x_0}$ satisfies the Cauchy condition for convergence in measure. In fact, for $s > l$ we have

$$\varphi^{(s)}(x) = \varphi^{(l)}(x) + \varphi^{(s-l)}(x + l\cdot x_0).$$

Taking $s = n_m$, $l = n_k$, $m > k$, and using (17) we obtain for sufficiently large k and arbitrary $m > k$

$$\mu\{x : d_G(F_{n_m\cdot x_0}(x), F_{n_k\cdot x_0}(x)) > \varepsilon\} < \varepsilon.$$

We have shown $F_{n_m\cdot x_0} \to F_y$ in measure. The condition (17) implies that F_y does not depend on the choice of $\{n_m\}$. Since

$$F_{n\cdot x_0}(x+x_0) - F_{n\cdot x_0}(x) = \varphi(x + n\cdot x_0) - \varphi(x)$$

for every $n = 0, \pm 1, \ldots$, taking $n = n_m$ and letting $m \to \infty$ we obtain

$$F_y(x+x_0) - F_y(x) = \varphi(x+y) - \varphi(x) \quad \text{ for } \mu\text{-a.e. } x \in X,$$

because $\varphi(x + n_m\cdot x_0) \to \varphi(x+y)$ in measure. It is evident that the family $\{F_y\}, y \in X$, satisfies (8) and (9) with $v_y = \mathrm{id}$. It follows from Theorem 1 that there exists a measurable function $f : X \to G$ such that

$$F_y(x) = f(x) - f(x+y) \quad \text{ for } \mu\text{-a.e. } x \in X$$

and for every $y \in X$. Taking $y = x_0$ we get $f(x) - f(x+x_0) = F_{x_0}(x) = \varphi(x)$, i.e. φ is a coboundary. ∎

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