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The wavelet characterization of the space Weak H^1

by

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Abstract. The space Weak H^1 was introduced and investigated by Fefferman and Soria. In this paper we characterize it in terms of wavelets. Equivalence of four conditions is proved.

1. Introduction. When we study the boundedness on $L^p(\mathbb{R}^n)$ for some of the basic operators in harmonic analysis, the case $p=1$ is often different from $p>1$. For example, if T is a Calderón–Zygmund singular integral operator, then T is bounded from $L^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$. So one finds a smaller space $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ such that T is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, which is well known. On the other hand, one can also find a space larger than $L^1(\mathbb{R}^n)$ and T is bounded from it to $WL^1(\mathbb{R}^n)$. This space is $WH^1(\mathbb{R}^n)$, introduced and investigated by Fefferman and Soria (see [3]).

We recall the definition of $WH^1(\mathbb{R}^n)$: Let f be a tempered distribution, and $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\int \varphi(x)\,dx = 1$. We define the maximal function

$$f^*(x) = \sup_{t>0} |f * \varphi_t(x)|\,.$$

Then we say that $f \in WH^1(\mathbb{R}^n)$ provided $f^* \in WL^1(\mathbb{R}^n)$, i.e.

$$|\{x \in \mathbb{R}^n : f^*(x) > \nu\}| \le C/\nu \quad \text{for all } \nu > 0\,.$$

The smallest C which makes the preceding estimate valid is called the "Weak H^1 norm" and denoted by $\|f\|_{WH^1}$. The choice of φ in the definition of $WH^1(\mathbb{R}^n)$ is of no importance. The space $WH^1(\mathbb{R}^n)$ is larger than $L^1(\mathbb{R}^n)$. In fact, the space of complex measures is continuously embedded as a subspace of $WH^1(\mathbb{R}^n)$. Another basic example is the distribution p.v. $\frac{1}{x}$ which belongs to $WH^1(\mathbb{R})$.

If we proceed to characterize the function spaces in terms of wavelets, we find ourselves in the same situation. Suppose $f = \sum a(\lambda)\psi_\lambda$ in the sense

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of distributions, where $a(\lambda) = \langle f, \psi_\lambda \rangle$ are the wavelet coefficients of f. Set

$$Wf(x) = \Big(\sum |a(\lambda)|^2 |\psi_\lambda(x)|^2\Big)^{1/2}.$$

Then $f \in L^p(\mathbb{R}^n)$ if and only if $Wf \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This is not true for $p = 1$ and we know that $f \in H^1(\mathbb{R}^n)$ if and only if $Wf \in L^1(\mathbb{R}^n)$ (see [1]). It is natural to ask whether $f \in WH^1(\mathbb{R}^n)$ if and only if $Wf \in WL^1(\mathbb{R}^n)$. The answer is affirmative and we will prove the equivalence of four conditions. It is well known that $WH^1(\mathbb{R}^n)$ is characterized by the condition that $S_\psi(f) \in WL^1(\mathbb{R}^n)$ where $S_\psi(f)$ denotes the area integral (with respect to a suitable nontrivial $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\int \psi = 0$) (see [3]). Moreover, a substitute for $S_\psi(f)$ is $(\sum |a(\lambda)|^2 |\psi_\lambda(x)|^2)^{1/2}$ when $\sum a(\lambda)\psi_\lambda(x)$ is the wavelet expansion of f. Then it is implicit that $f \in WH^1(\mathbb{R}^n)$ if and only if $(\sum |a(\lambda)|^2 |\psi_\lambda(x)|^2)^{1/2} \in WL^1(\mathbb{R}^n)$. The theorem we are going to prove is not new but we will give a new proof together with a slight sharpening.

2. Statement of results. The wavelets used in this paper are the compactly supported wavelets with r-regularity ($r \ge 1$) defined by I. Daubechies [2]. Let us recall it in more detail.

Let $\mathcal{D}$ be the set of all dyadic cubes, i.e. $\mathcal{D} = \{Q_{j,k} : j \in \mathbb{Z},\ k = (k_1, \ldots, k_n) \in \mathbb{Z}^n\}$ where $Q_{j,k} = \{x \in \mathbb{R}^n : 2^j x - k \in [0,1)^n\}$. Set $E = \{0,1\}^n \setminus (0, \ldots, 0)$. Suppose φ and ψ are r-regular compactly supported functions obtained by multiresolution approximations in [1]. For any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E$ and $Q_{j,k} \in \mathcal{D}$, let

$$\psi^\varepsilon_{Q_{j,k}} = 2^{nj/2} \psi^{\varepsilon_1}(2^j x_1 - k_1) \ldots \psi^{\varepsilon_n}(2^j x_n - k_n),$$

where $\psi^0 = \varphi$ and $\psi^1 = \psi$. It is known that $\psi^\varepsilon_{Q_{j,k}}$ have the following properties:

(a) $\{\psi^\varepsilon_{Q_{j,k}}\}_{Q_{j,k} \in \mathcal{D}, \varepsilon \in E}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$;

(b) $\operatorname{supp} \psi^\varepsilon_{Q_{j,k}} \subset mQ_{j,k}$, $m \ge 1$, where mQ is the cube concentric with Q but with the side length m times that of Q (i.e. the "m-fold expansion");

(c) $\|(\partial^\alpha/\partial x^\alpha)\psi^\varepsilon_{Q_{j,k}}\|_\infty \le C 2^{nj/2 + |\alpha| j}$, $|\alpha| \le r$;

(d) $\int x^\alpha \psi^\varepsilon_{Q_{j,k}}(x)\,dx = 0$, $|\alpha| \le r$.

For notational convenience we shall write ψ_λ and $Q(\lambda)$ instead of $\psi^\varepsilon_{Q_{j,k}}$ and $Q_{j,k}$ respectively, where $\lambda = 2^{-j}k + 2^{-j-1}\varepsilon$. The set of all indices λ will be denoted by Λ.

Now we are in a position to state our results.

THEOREM. *Let $f = \sum_{\lambda \in \Lambda} a(\lambda)\psi_\lambda$ in the sense of distributions, where $a(\lambda) = \langle f, \psi_\lambda \rangle$ are the wavelet coefficients of f. Then the following conditions are equivalent:*

(A) $$Wf(x) = \Big(\sum_{\lambda \in \Lambda} |a(\lambda)|^2 |\psi_\lambda(x)|^2\Big)^{1/2} \in WL^1(\mathbb{R}^n);$$

(B) $$Sf(x) = \Big(\sum_{\lambda \in \Lambda} |a(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)}(x)\Big)^{1/2} \in WL^1(\mathbb{R}^n),$$

where $R(\lambda) \subset Q(\lambda)$ are measurable sets such that $|R(\lambda)| \ge \gamma |Q(\lambda)|$ for a fixed positive constant γ, and χ denotes the characteristic function;

(C) $$Gf(x) = \Big(\sum_{\lambda \in \Lambda} |a(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x)\Big)^{1/2} \in WL^1(\mathbb{R}^n);$$

(D) $$f \in WH^1(\mathbb{R}^n).$$

If T is a Calderón–Zygmund operator with $T^*(1) = 0$, then T is bounded on $WH^1(\mathbb{R}^n)$ (see [4]). Using this fact, we conclude that the theorem remains valid for any choice of the wavelet base $\{\psi_\lambda\}_{\lambda \in \Lambda}$.

3. Proof of Theorem. (A)⇒(B) and (C)⇒(B) are very easy. We shall prove (B)⇒(C), (B)⇒(D) and (D)⇒(A).

The letter C will denote a constant whose value may be different in different places.

(B)⇒(C). Set $E_k = \{x \in \mathbb{R}^n : Sf(x) > 2^k\}$. Then $|E_k| \le C 2^{-k}$. Take $0 < \beta < \gamma$ and let $\mathcal{D}_k$ be the set of dyadic cubes Q such that $|Q \cap E_k| \ge \beta |Q|$. Let $E_k^* = \bigcup_{Q \in \mathcal{D}_k} Q$. Then

$$|E_k^*| \le \frac{1}{\beta} |E_k| \le C 2^{-k} \quad \text{and} \quad |E_k \setminus E_k^*| = 0. \tag{1}$$

Q is called a *maximal dyadic cube in* $\mathcal{D}_k$ if $Q \in \mathcal{D}_k$ and $\widetilde{Q} \notin \mathcal{D}_k$ provided $\widetilde{Q} \supsetneq Q$ and $\widetilde{Q} \in \mathcal{D}$. Let $\{Q(k,i) : i \in F_k\}$ be the set of all maximal dyadic cubes in $\mathcal{D}_k$. Then E_k^* is a disjoint union of $Q(k,i)$, $i \in F_k$. Set $\Delta_k = \mathcal{D}_k \setminus \mathcal{D}_{k+1}$ and $\Delta(k,i) = \{Q \in \Delta_k : Q \subset Q(k,i)\}$. Then

$$\mathcal{D} = \bigcup_{k=-\infty}^{\infty} \bigcup_{i \in F_k} \Delta(k,i)$$

is a disjoint decomposition of $\mathcal{D}$. It is easy to get

$$\begin{aligned} \sum_{Q(\lambda) \in \Delta(k,i)} |a(\lambda)|^2 &\le \frac{1}{\gamma - \beta} \int_{Q(k,i) \setminus E_{k+1}} |Sf(x)|^2\,dx \\ &\le \frac{1}{\gamma - \beta} 4^{k+1} |Q(k,i)|. \end{aligned} \tag{2}$$

For any $\nu > 0$, take k_0 such that $2^{k_0} \le \nu < 2^{k_0+1}$. Set

$$G_1 f(x) = \Big(\sum_{k=-\infty}^{k_0} \sum_{i \in F_k} \sum_{Q(\lambda) \in \Delta(k,i)} |a(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \Big)^{1/2},$$

$$G_2 f(x) = \Big(\sum_{k=k_0+1}^{\infty} \sum_{i \in F_k} \sum_{Q(\lambda) \in \Delta(k,i)} |a(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \Big)^{1/2}.$$

From (1) and (2), we obtain

$$\text{(3)} \quad \|G_1 f\|_2^2 = \sum_{k=-\infty}^{k_0} \sum_{i \in F_k} \sum_{Q(\lambda) \in \Delta(k,i)} |a(\lambda)|^2$$
$$\le \sum_{k=-\infty}^{k_0} \sum_{i \in F_k} \frac{1}{\gamma - \beta} 4^{k+1} |Q(k,i)| \le C \sum_{k=-\infty}^{k_0} 4^k |E_k^*| \le C\nu.$$

Since $\operatorname{supp} G_2 f \subset \bigcup_{k=k_0+1}^{\infty} E_k^*$, we have

$$\text{(4)} \quad |\operatorname{supp} G_2 f| \le \sum_{k=-\infty}^{k_0} |E_k^*| \le C/\nu.$$

Therefore,

$$|\{x \in \mathbb{R}^n : Gf(x) > \nu\}| \le |\{x \in \mathbb{R}^n : G_1 f(x) > \nu\}| + |\operatorname{supp} G_2 f|$$
$$\le C(\|G_1 f\|_2/\nu)^2 + C/\nu \le C/\nu.$$

(B)$\Rightarrow$(D). We keep the above notations. Write

$$f_1 = \sum_{k=-\infty}^{k_0} \sum_{i \in F_k} \sum_{Q(\lambda) \in \Delta(k,i)} a(\lambda) \psi_\lambda,$$

$$f_2 = \sum_{k=k_0+1}^{\infty} \sum_{i \in F_k} \sum_{Q(\lambda) \in \Delta(k,i)} a(\lambda) \psi_\lambda.$$

As in (3), we obtain

$$\|f_1\|_2^2 = \sum_{k=-\infty}^{k_0} \sum_{i \in F_k} \sum_{Q(\lambda) \in \Delta(k,i)} |a(\lambda)|^2 \le C\nu.$$

Hence

$$\text{(5)} \quad |\{x \in \mathbb{R}^n : f_1^*(x) > \nu\}| \le C(\|f_1\|_2/\nu)^2 \le C/\nu.$$

Set $\Omega = \bigcup_{k=k_0+1}^{\infty} \bigcup_{i \in F_k} 2mQ(k,i)$; then

$$\text{(6)} \quad |\Omega| \le C \sum_{k=-\infty}^{k_0} \sum_{i \in F_k} |Q(k,i)| \le C \sum_{k=-\infty}^{k_0} |E_k^*| \le C/\nu.$$

We shall prove that

$$\text{(7)} \quad |\{x \notin \Omega : f_2^*(x) > \nu\}| \le C/\nu.$$

Write

$$f_{k,i}(x) = \sum_{Q(\lambda) \in \Delta(k,i)} a(\lambda) \psi_\lambda(x).$$

Obviously, $\operatorname{supp} f_{k,i} \subset mQ(k,i)$. Let $x_{k,i}$ be the center of $Q(k,i)$. Suppose $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\int \varphi(x)\,dx = 1$. For $x \notin 2mQ(k,i)$,

$$|f_{k,i} * \varphi_t(x)| = \Big| \int_{mQ(k,i)} f_{k,i}(y) t^{-n} \Big(\varphi\Big(\frac{x-y}{t}\Big) - \varphi\Big(\frac{x - x_{k,i}}{t}\Big)\Big)\,dy \Big|$$
$$\le C|x - x_{k,i}|^{-n-1} \int_{mQ(k,i)} |f_{k,i}(y)||y - x_{k,i}|\,dy$$
$$\le C|x - x_{k,i}|^{-n-1} |Q(k,i)|^{1/n+1/2} \|f_{k,i}\|_2$$
$$= C|x - x_{k,i}|^{-n-1} |Q(k,i)|^{1/n+1/2} \Big(\sum_{Q(\lambda) \in \Delta(k,i)} |a(\lambda)|^2 \Big)^{1/2}$$
$$\le C 2^k |Q(k,i)|^{(n+1)/n} |x - x_{k,i}|^{-n-1}.$$

This implies

$$f_{k,i}^*(x) \le C 2^k |Q(k,i)|^{(n+1)/n} |x - x_{k,i}|^{-n-1}.$$

We shall use the superposition principle for weak type estimates which was found by Stein, Taibleson and Weiss [5]:

LEMMA 1. *Let g_k be a sequence of measurable functions and $0 < p < 1$. Assume that*

$$|\{x \in \mathbb{R}^n : |g_k(x)| > \nu\}| \le C/\nu^p,$$

where C is a constant not depending on k and ν. Then

$$\Big| \Big\{ x \in \mathbb{R}^n : \Big| \sum_k c_k g_k(x) \Big| > \nu \Big\} \Big| \le \frac{2-p}{1-p} \cdot \frac{C}{\nu^p} \sum_k |c_k|^p.$$

Set $c_{k,i} = C 2^k |Q(k,i)|^{(n+1)/n}$, $g_{k,i}(x) = |x - x_{k,i}|^{-n-1}$, and take $p = (n+1)/n$. We get

$$|\{x \notin \Omega : f_2^*(x) > \nu\}| \le \Big| \Big\{ x \notin \Omega : \sum_{k=-\infty}^{k_0} \sum_{i \in F_k} f_{k,i}^*(x) > \nu \Big\} \Big|$$
$$\le \Big| \Big\{ x \notin \mathbb{R}^n : \sum_{k=-\infty}^{k_0} \sum_{i \in F_k} c_{k,i} g_{k,i}(x) > \nu \Big\} \Big|$$

$$\leq \frac{C}{\nu^p}\sum_{k=-\infty}^{k_0}\sum_{i\in F_k} 2^{k(n+1)/n}|Q(k,i)|$$
$$\leq \frac{C}{\nu^p}\sum_{k=-\infty}^{k_0} 2^{k(n+1)/n}|E_k^*| \leq \frac{C}{\nu^p}2^{-(k_0+1)/(n+1)} \leq \frac{C}{\nu}.$$

This proves (7). Now (5)–(7) give us

$$|\{x\in\mathbb{R}^n : f^*(x) > 2\nu\}|$$
$$\leq |\{x\in\mathbb{R}^n : f_1^*(x) > \nu\}| + |\Omega| + |\{x\notin\Omega : f_2^*(x) > \nu\}| \leq C/\nu.$$

(D)⇒(A). The following atomic decomposition of $WH^1(\mathbb{R}^n)$ is due to Fefferman and Soria [3]:

LEMMA 2. *Given $f\in WH^1(\mathbb{R}^n)$, there exists a sequence of functions $\{f_k\}_{k=-\infty}^{\infty}$ with the following properties:*

(a) *$f=\sum_{k=-\infty}^{\infty} f_k$ in the sense of distributions.*

(b) *$f_k=\sum_{i=1}^{\infty} h_{k,i}$ in L^1, where $h_{k,i}$ satisfy:*

1) *$h_{k,i}$ is supported in a cube $B_{k,i}$ with $\{B_{k,i}\}_{i=1}^{\infty}$ having bounded overlap for each k;*
2) *$\int h_{k,i}(x)\,dx=0$;*
3) *$\|h_{k,i}\|_\infty \leq C2^k$ and $\sum_{i=1}^{\infty}\|B_{k,i}\| \leq C2^{-k}$.*

Write

$$F_1=\sum_{k=-\infty}^{k_0} f_k, \quad F_2=\sum_{k=k_0+1}^{\infty} f_k.$$

Their wavelet series are respectively

$$F_1=\sum_{\lambda\in\Lambda} a_1(\lambda)\psi_\lambda, \quad F_2=\sum_{\lambda\in\Lambda} a_2(\lambda)\psi_\lambda.$$

We have

$$\|WF_1\|_2^2 = \Big\|\Big(\sum_{\lambda\in\Lambda}|a_1(\lambda)|^2|\psi_\lambda|^2\Big)^{1/2}\Big\|_2^2 = \sum_{\lambda\in\Lambda}|a_1(\lambda)|^2 = \|F_1\|_2^2$$
$$\leq \Big(\sum_{k=-\infty}^{k_0}\|f_k\|_2\Big)^2 \leq C\Big(\sum_{k=-\infty}^{k_0}2^{k/2}\Big)^2 \leq C\nu.$$

Therefore,

$$|\{x\in\mathbb{R}^n : WF_1(x) > \nu\}| \leq (\|WF_1\|_2/\nu)^2 \leq C/\nu. \tag{8}$$

Let $\widetilde{B}_{k,i}$ denote the "expansion" of the cube $B_{k,i}$ by the factor $C_1(3/2)^{(k-k_0)/n}$, where C_1 is a large constant depending on m and determined later. Set $A=\bigcup_{k=k_0+1}^{\infty}\bigcup_{i=1}^{\infty}\widetilde{B}_{k,i}$. It follows that

$$|A| \leq \sum_{k=k_0+1}^{\infty}\sum_{i=1}^{\infty}|\widetilde{B}_{k,i}| \leq C\sum_{k=k_0+1}^{\infty}\sum_{i=1}^{\infty}\Big(\frac{3}{2}\Big)^{k-k_0}|B_{k,i}| \tag{9}$$
$$\leq C\sum_{k=k_0+1}^{\infty}\Big(\frac{3}{4}\Big)^{k-k_0}2^{-k_0} \leq \frac{C}{\nu}.$$

We shall prove

$$\int_{A^c}|WF_2(x)|\,dx \leq C. \tag{10}$$

Write

$$h_{k,i}(x)=\sum_{\lambda\in\Lambda} a_{k,i}(\lambda)\psi_\lambda(x),$$

where

$$a_{k,i}(\lambda)=\int_{B_{k,i}} h_{k,i}(y)\overline{\psi}_\lambda(y)\,dy$$

are the wavelet coefficients of $h_{k,i}$. Set $A_s=2^{s+1}\widetilde{B}_{k,i}\setminus 2^s\widetilde{B}_{k,i}$. Then

$$\int_{A^c}\Big(\sum_{\lambda\in\Lambda}|a_{k,i}(\lambda)|^2|\psi_\lambda(x)|^2\Big)^{1/2}dx \leq \sum_{s=0}^{\infty}\int_{A_s}\Big(\sum_{\lambda\in\Lambda}|a_{k,i}(\lambda)|^2|\psi_\lambda(x)|^2\Big)^{1/2}dx.$$

If $mQ(\lambda)\cap B_{k,i}=\emptyset$, $a_{k,i}(\lambda)=0$. When $mQ(\lambda)\cap A_s=\emptyset$, $\psi_\lambda(x)\chi_{A_s}(x)=0$. So we assume that $mQ(\lambda)\cap B_{k,i}\neq\emptyset$ and $mQ(\lambda)\cap A_s\neq\emptyset$. This implies that

$$2^{-j} \geq 2^s(3/2)^{(k-k_0)/n}|B_{k,i}|^{1/n} \tag{11}$$

when C_1 is large enough. For fixed j, the number of $Q(\lambda)$ satisfying the conditions given above has a universal upper bound. We denote by j_0 the largest integer satisfying (11). Let $b_{k,i}$ be the center of $B_{k,i}$. Then

$$|a_{k,i}(\lambda)| = \Big|\int_{B_{k,i}} h_{k,i}(y)(\overline{\psi}_\lambda(y)-\overline{\psi}_\lambda(b_{k,i}))\,dy\Big|$$
$$\leq \int_{B_{k,i}}|h_{k,i}(y)|\,|y-b_{k,i}|\,dy\,\|\nabla\psi_\lambda\|_\infty$$
$$\leq C2^k2^{(n/2+1)j}|B_{k,i}|^{(n+1)/n}.$$

For $x \in A_s$, we have

$$\Big(\sum_{\lambda\in\Lambda}|a_{k,i}(\lambda)|^2|\psi_\lambda(x)|^2\Big)^{1/2} \le C\Big(\sum_{j=-\infty}^{j_0} 2^{2k}2^{2(n+1)j}|B_{k,i}|^{2(n+1)/n}\Big)^{1/2}$$
$$\le C2^k2^{(n+1)j_0}|B_{k,i}|^{(n+1)/n}$$
$$\le C2^k2^{-s(n+1)}\left(\frac{3}{2}\right)^{-(k-k_0)(n+1)/n}.$$

Therefore,

$$\sum_{s=0}^{\infty}\int_{A_s}\Big(\sum_{\lambda\in\Lambda}|a_{k,i}(\lambda)|^2|\psi_\lambda(x)|^2\Big)^{1/2}dx \le C\sum_{s=0}^{\infty}2^{-s}2^k\left(\frac{3}{2}\right)^{-(k-k_0)/n}|B_{k,i}|$$
$$\le C2^k\left(\frac{3}{2}\right)^{-(k-k_0)/n}|B_{k,i}|.$$

It follows that

$$\int_{A^c}|WF_2(x)|\,dx \le \sum_{k=k_0+1}^{\infty}\sum_{i=1}^{\infty}\int_{A^c}\Big(\sum_{\lambda\in\Lambda}|a_{k,i}(\lambda)|^2|\psi_\lambda(x)|^2\Big)^{1/2}dx$$
$$\le C\sum_{k=k_0+1}^{\infty}\sum_{i=1}^{\infty}2^k\left(\frac{3}{2}\right)^{-(k-k_0)/n}|B_{k,i}|$$
$$\le C\sum_{k=k_0+1}^{\infty}\left(\frac{3}{2}\right)^{-(k-k_0)/n} \le C.$$

This proves (10). From (8)–(10), we obtain

$$|\{x\in\mathbb{R}^n : Wf(x) > 2\nu\}|$$
$$\le |\{x\in\mathbb{R}^n : WF_1(x) > \nu\}| + |A| + |\{x\notin A : WF_2(x) > \nu\}| \le C/\nu.$$

The proof of the Theorem is complete.

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