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The Compact Approximation Property does not imply the Approximation Property

by

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Abstract. It is shown how to construct, given a Banach space which does not have the approximation property, another Banach space which does not have the approximation property but which does have the compact approximation property.

A Banach space, X, is said to have the *approximation property* if for every compact set $K \subseteq X$ and every $\varepsilon > 0$ there is a finite rank operator T on X such that $\|Tx - x\| < \varepsilon$ for every x in K. The approximation property is weaker than the notion of a Schauder basis. Classical Banach spaces have Schauder bases but it was shown by Enflo [E] that there are spaces which do not have the approximation property. A shorter construction of such spaces was given by Davie [D1], [D2] (see also [LT1], Theorem 2.d.3).

A Banach space, X, is said to have the *compact approximation property* if for every compact set $K \subseteq X$ and every $\varepsilon > 0$ there is a compact operator T on X such that $\|Tx - x\| < \varepsilon$ for every x in K. A finite rank operator is compact and so the compact approximation property is formally weaker than the approximation property. Known examples leave open the possibility that these two properties are equivalent. Indeed, many of the spaces which do not have the approximation property are known to also not have the compact approximation property (see [LT1], p. 94, and [LT2], Theorem 1.g.4). However, the following construction produces spaces which have the compact approximation property while not having the approximation property, thus showing that the two properties are not equivalent.

The construction is based on an argument due to Grothendieck [G] which shows that X has the approximation property if and only if for every Banach space Y, every compact operator $T : Y \to X$ and every $\varepsilon > 0$ there is a

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finite rank $F : Y \to X$ with $\|T - F\| < \varepsilon$. We shall follow the exposition given in [LT1], Theorem 1.e.4.

Let X be a Banach space which does not have the approximation property. Then there is a compact set $K \subseteq X$ such that the identity operator cannot be approximated on K by finite rank operators. By a theorem of Grothendieck (see [LT1], Proposition 1.e.2), it may be supposed that $K = \overline{\text{conv}}\{x_n\}_{n=1}^{\infty}$ where $\|x_n\| \le 1$ for all n and $\|x_n\|$ decreases to zero.

For each t between 0 and 1, put $U_t = \overline{\text{conv}}\{\pm x_n/\|x_n\|^t\}_{n=1}^{\infty}$. Then U_t is a compact, convex, symmetric subset of X. Let Y_t be the linear span of U_t and define a norm on Y_t by $|||x|||_t = \inf\{|\lambda| : \lambda^{-1}x \in U_t\}$, $x \in Y_t$. It may be checked that $(Y_t, |||\cdot|||_t)$ is a Banach space with unit ball U_t. If $s < t$, then $U_s \subseteq U_t$ and so $Y_s \subseteq Y_t$ and all spaces are contained in X. Denote the inclusion map of Y_t into X by L_t. Then L_t is compact and has norm at most one. It is shown in [LT1], Theorem 1.e.4, that the operator $L_{1/2} : Y_{1/2} \to X$ cannot be approximated by finite rank operators.

The space to be constructed will be a space of functions on $(0, 1)$ with values in X. For each $(s, t) \subseteq (0, 1)$ and y in Y_s, $y\,\chi_{(s,t)}$ is such a function, where $y\,\chi_{(s,t)}$ denotes the map

$$r \mapsto \begin{cases} y & \text{if } r \in (s, t), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{Z}$ be the linear span of $\{y\,\chi_{(s,t)} : 0 < s < t < 1,\ y \in Y_s\}$. Note that if f belongs to $\mathcal{Z}$, then $f(r)$ is in Y_r for all r. Hence we may define a norm on $\mathcal{Z}$ by

$$\|f\| = \int_0^1 |||f(r)|||_r \, dr \quad (f \in \mathcal{Z}).$$

Now let Z be the completion of $\mathcal{Z}$ with respect to this norm.

PROPOSITION 1. *Z does not have the approximation property.*

Proof. Define a map $R : Y_{1/2} \to Z$ by

(1) $$R(y) = 2y\,\chi_{(1/2,1)} \quad (y \in Y_{1/2}).$$

Then

(2) $$\begin{aligned} \|Rx_n\| &= 2\int_{1/2}^1 |||x_n|||_r \, dr \\ &\le 2\int_{1/2}^1 \|x_n\|^r \, dr, \quad \text{because } |||x_n|||_r \le \|x_n\|^r, \\ &< 2\|x_n\|^{1/2}/|\ln\|x_n\||. \end{aligned}$$

Since $\|x_n\| \to 0$ as $n \to \infty$, it follows that $\|R(x_n/\|x_n\|^{1/2})\| \to 0$ as $n \to \infty$, whence $RU_{1/2}$ is a totally bounded subset of Z. Since $U_{1/2}$ is the unit ball in $Y_{1/2}$ it follows that R is a compact operator.

Now define a map $J : Z \to X$ by

(3) $$J(f) = \int_0^1 f(r)\, dr \quad (f \in Z),$$

where the integral may be defined in the obvious way if f is in $\mathcal{Z}$ and this definition extends to all of Z by continuity. Then $JR = L_{1/2}$ and $L_{1/2}$ cannot be approximated by finite rank operators. It follows that R cannot be approximated by finite ranks. Therefore, by [LT1], Theorem 1.e.4, Z does not have the approximation property. ■

PROPOSITION 2. *Z does have the compact approximation property.*

Proof. For each r in $(0, 1)$ define the operator which shifts by r, S_r, on $\mathcal{Z}$ as follows: first, for $(s, t) \subseteq (0, 1)$ and y in Y_s, define $S_r(y\,\chi_{(s,t)}) = y\,\chi_{(s+r,t+r)}$; next extend S_r to $\mathcal{Z}$ by linearity; and then, since S_r is clearly a contraction mapping, extend to Z by continuity. As already mentioned, $\|S_r\| \le 1$ for each r. Furthermore, it may easily be checked that the function $r \mapsto S_r f$ is norm continuous, and $\|S_r f - f\| \to 0$ as $r \to 0$ for each f in Z. However, S_r is not a compact operator.

To obtain compact operators on Z which approximate the identity define, for each n, $T_n : Z \to Z$ by

(4) $$T_n f = n\int_0^{1/n} S_r f \, dr \quad (f \in Z).$$

The integral exists because $r \mapsto S_r f$ is norm continuous and $\|T_n\| \le 1$. Since the operators S_r approximate the identity as r approaches zero, it follows that $\|T_n f - f\| \to 0$ as $n \to \infty$ for each f in Z. Hence, since $\|T_n\| \le 1$ for every n, T_n approximates the identity operator on a given compact set when n is sufficiently large.

We shall see that T_n is compact for each n by showing that $T_n Z^{(1)}$ is totally bounded, where $Z^{(1)}$ denotes the unit ball in Z. First note that the functions of the form

$$\sum_{i=1}^{p} \lambda_i (t_i - s_i)^{-1} y_i\, \chi_{(s_i,t_i)}$$

(where: $s_1 < t_1 < s_2 < t_2 < \ldots < s_p < t_p$; $y_i \in Y_{s_i}$, $|||y_i|||_{s_i} \le 1$; and $\sum_{i=1}^{p} |\lambda_i| = 1$) are dense in $Z^{(1)}$. Hence, it will suffice to show that $T_n\{(t-s)^{-1} y\chi_{(s,t)} : s < t,\ y \in U_s\}$ is totally bounded. Next, since the unit ball in Y_s is $\overline{\text{conv}}\{\pm x_m/\|x_m\|^s\}_{m=1}^{\infty}$, it will suffice to show that

$$A_n = T_n\left\{(t-s)^{-1}\|x_m\|^{-s}x_m\chi_{(s,t)} : s < t,\ m = 1, 2, \ldots\right\}$$

is totally bounded.

For each s and t with $s < t$ we see that x_m belongs to Y_s for each m and

$$T_n(x_m\,\chi_{(s,t)}) = x_m\,h\,,$$

where $h = n\int_0^{1/n}\chi_{(s+r,t+r)}\,dr$.

For functions, f and g, in $L^1(0,1)$, let $f*g$ denote the usual convolution product of f and g restricted to $(0,1)$. Then $h = \chi_{(s,t)} * (n\,\chi_{(0,1/n)})$. It follows that for each f in $L^1(0,1)$, $T_n(x_m f) = x_m(f * (n\,\chi_{(0,1/n)}))$. Now it is well known, and may easily be checked, that the map $f \mapsto f*(n\,\chi_{(0,1/n)})$ is a compact operator on $L^1(0,1)$. Hence for each m the set $T_n\{(t-s)^{-1}\|x_m\|^{-s}\,x_m\,\chi_{(s,t)} : s < t\}$ is totally bounded. Furthermore,

$$\begin{aligned}(5)\qquad \|T_n(x_m\chi_{(s,t)})\| &= n\int_0^1 |||x_m|||_r|\chi_{(s,t)} * \chi_{(0,1/n)}(r)|\,dr\\ &\le n\int_s^{t+1/n}\|x_m\|^r(t-s)\,dr\\ &< n(t-s)\,\|x_m\|^s/|\ln\|x_m\|\,|\,.\end{aligned}$$

Hence, for each $\varepsilon > 0$, $\|T_n((t-s)^{-1}\|x_m\|^{-s}\,x_m\,\chi_{(s,t)})\| < \varepsilon$ whenever $\|x_m\| < e^{-n/\varepsilon}$. Since $\|x_m\| \to 0$ as $m \to \infty$, this is so for all m sufficiently large. It follows that A_n is totally bounded as required. Therefore $\{T_n\}_{n=1}^{\infty}$ is a sequence of compact operators on Z such that $\|T_n\,z - z\| \to 0$ as $n \to \infty$ for every z in Z. ∎

We have in fact shown by the above argument that the identity operator on Z can be approximated, in the topology of uniform convergence on compact sets, by compact operators of norm at most one, that is, that Z has the metric compact approximation property.

The existence of a space with the compact approximation property but not the approximation property raises questions as to whether results relating various stronger versions of the approximation property (see Section 1.e of [LT1]) have analogues for the compact approximation property. Some of these questions can be answered if there is a reflexive space with the compact approximation property but not the approximation property [C]. For instance, see [GW], Example 4.3, where the reflexive example constructed below is used to answer a question which arises in that paper and also in [S]. Now Z is not reflexive because the set $x_1L^1[0,1]$ is a closed subspace of Z which is isomorphic to $L^1[0,1]$. However, the construction of Z may be modified to produce a reflexive space as follows.

As before, let X be a Banach space which does not have the approximation property but now suppose that X is a closed subspace of ℓ^p for some $2 < p < \infty$. This value of p will remain fixed throughout the construction. Set $q = p/(p-1)$, so that q is the conjugate of p. See [LT1], Theorem 2.d.6 for a proof that ℓ^p has such subspaces.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that: $\|x_n\| \le 1$ for all n; $\|x_n\|$ decreases to zero; and the identity operator cannot be approximated on $\overline{\text{conv}}\{x_n\}_{n=1}^{\infty}$ by finite rank operators. Choose integers $n_1 < n_2 < \ldots$ such that $\|x_n\| < (1/2)^k$ whenever $n > n_k$ and define, for $k = 1, 2, \ldots$,

$$X_k = \text{span}\{x_n : n \le n_k\}\,.$$

Next define, for each t between 0 and 1,

$$V_t = \left\{\sum_{k=1}^{\infty}\alpha_k a_k/\|a_k\|^t : a_k \in X_k\,,\ \|a_k\| \le \left(\frac{1}{2}\right)^{k-1},\ \sum_{k=1}^{\infty}|\alpha_k|^p \le 1\right\}.$$

The properties of the sets V_t which we will need are given in the following

LEMMA 1. *Let X be a Banach space, $X_1, X_2, \ldots$ be finite-dimensional subspaces of X, and $r_1, r_2, \ldots$ be positive numbers such that $\sum_{n=1}^{\infty} r_n^q < \infty$. Define*

$$V = \left\{\sum_{k=1}^{\infty}\alpha_k x_k : x_k \in X_k,\ \|x_k\| \le r_k,\ \sum_{k=1}^{\infty}|\alpha_k|^p \le 1\right\}.$$

Then V is a compact, convex, symmetric subset of X.

Let W be the linear subspace of X spanned by V and define $|||w||| = \inf\{\lambda > 0 : \lambda^{-1}w \in V\}$ $(w \in W)$. Then $(W, |||\cdot|||)$ is a Banach space. The map $Q : (\bigoplus_{k=1}^{\infty} X_k)_{\ell^p} \to W$ defined by

$$Q(x_1, x_2, \ldots) = \sum_{k=1}^{\infty} r_k x_k$$

is a quotient map.

Proof. It is clear that V is symmetric and it is compact because, as may be shown by a diagonal argument, every sequence in V has a convergent subsequence. Let $x = \sum_{k=1}^{\infty}\alpha_k x_k$ and $y = \sum_{k=1}^{\infty}\beta_k y_k$ be in V. Then

$$\frac{1}{2}(x+y) = \sum_{k=1}^{\infty}\frac{1}{2}(\alpha_k\,x_k + \beta_k\,y_k) = \sum_{k=1}^{\infty}\gamma_k z_k\,,$$

where $\gamma_k = \frac{1}{2}(|\alpha_k| + |\beta_k|)$, z_k belongs to X_k and $\|z_k\| \le r_k$. Since $\sum_{k=1}^{\infty}|\gamma_k|^p \le \sum_{k=1}^{\infty}\frac{1}{2}(|\alpha_k|^p + |\beta_k|^p) \le 1$, it follows that V is convex. These properties of V imply that $(W, |||\cdot|||)$ is a Banach space.

It is clear from the definition of V that Q maps the unit ball of $(\bigoplus_{k=1}^{\infty} X_k)_{\ell^p}$ onto V, which is the unit ball of W. Therefore Q is a quotient map. ∎

Let W_t denote the space spanned by V_t and normed so that V_t is its unit ball. Denote the norm on W_t by $|||\cdot|||_t$. Then $W_s \subset W_t$ if $s < t$. Denote by Q_t the quotient map from $(\bigoplus_{k=1}^{\infty} X_k)_{\ell^p}$ to W_t defined by

$$Q_t \mathbf{x} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{(1-t)(k-1)} x_k ,$$

where $\mathbf{x} = (x_1, x_2, \ldots)$. We will require the following.

LEMMA 2. *For each s in $(0,1)$ and $\mathbf{x}$ in $(\bigoplus_{k=1}^{\infty} X_k)_{\ell^p}$,*

$$\lim_{t\to s^+} |||Q_t \mathbf{x} - Q_s \mathbf{x}|||_t = 0 .$$

Proof. It is clear that the limit is zero if $\mathbf{x}$ is supported in only finitely many of the X_k's and that the set of finitely supported vectors is dense in $(\bigoplus_{k=1}^{\infty} X_k)_{\ell^p}$. The result follows because $\|Q_t\| = 1$ for each t. ∎

Let $\mathcal{Z}^\sharp$ be the linear span of $\{w\chi_{(s,t)} : 0 < s < t < 1,\ w \in W_s\}$. As in the previous construction, $\mathcal{Z}^\sharp$ is a space of functions, $f : (0,1) \to X$, such that $f(t)$ belongs to W_t for each t. Define a norm on $\mathcal{Z}^\sharp$ by

$$\|f\| = \Big(\int_0^1 |||f(r)|||_r^p \, dr \Big)^{1/p} \quad (f \in \mathcal{Z}^\sharp) .$$

Let $Z^\sharp$ denote the completion of $\mathcal{Z}^\sharp$ with respect to this norm.

We will show that $Z^\sharp$ has the required properties. In the proofs of these properties, $\mathcal{A}$ will denote the set of all functions of the form

$$\sum_{i=1}^{l} \lambda_i (t_i - s_i)^{-1/p} w_i \chi_{(s_i,t_i)} , \tag{6}$$

where: $s_1 < t_1 < s_2 < t_2 < \ldots < s_l < t_l$; $w_i \in W_{s_i}$, $|||w_i|||_{s_i} \le 1$; and $\sum_{i=1}^{l} |\lambda_i|^p = 1$. Clearly $\mathcal{A}$ is dense in the unit ball of $Z^\sharp$.

PROPOSITION 3. *$Z^\sharp$ is a quotient of a closed subspace of $L^p(0,1)$. In particular, $Z^\sharp$ is reflexive.*

Proof. Let X_k, $k = 1,2,\ldots$, be the subspaces of ℓ^p defined above. Then $(\bigoplus_{k=1}^{\infty} X_k)_{\ell^p}$ is a subspace of $(\bigoplus_{k=1}^{\infty} \ell^p)_{\ell^p}$. It follows that $L^p((0,1), (\bigoplus_{k=1}^{\infty} X_k)_{\ell^p})$ is isometric to a subspace of $L^p(0,1)$. We will show that $Z^\sharp$ is a quotient of this space.

The quotient map $Q : L^p((0,1), (\bigoplus_{k=1}^{\infty} X_k)_{\ell^p}) \to Z^\sharp$ will be defined first of all on simple functions from $(0,1)$ to $(\bigoplus_{k=1}^{\infty} X_k)_{\ell^p}$. Let f be such a function and put

$$(Qf)(t) = Q_t(f(t)) \quad (0 < t < 1) . \tag{7}$$

Then $(Qf)(t)$ belongs to W_t for each t and it follows from Lemma 2 that there are functions f_n, $n = 1,2,\ldots$, in $\mathrm{span}\{w\chi_{(s,t)} : 0 < s < t < 1;\ w \in W_s\}$ such that

$$\lim_{n\to\infty} \Big(\int_0^1 |||(Qf)(r) - f_n(r)|||_r^p \, dr \Big)^{1/p} = 0 .$$

It follows that: $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $Z^\sharp$; Qf may be identified with the limit of this sequence; and

$$\|Qf\| = \Big(\int_0^1 |||(Qf)(r)|||_r^p \, dr \Big)^{1/p} = \Big(\int_0^1 |||Q_r(f(r))|||_r^p \, dr \Big)^{1/p} .$$

Since $\|Q_r\| \le 1$ for each r, $\|Q\| \le 1$. Since the simple functions are dense, Q extends by continuity to be a norm one operator from $L^p((0,1), (\bigoplus_{k=1}^{\infty} X_k)_{\ell^p})$ to $Z^\sharp$. The equation (7) will hold for every f in $L^p((0,1), (\bigoplus_{k=1}^{\infty} X_k)_{\ell^p})$ and almost every t in $(0,1)$.

Now let

$$f = \sum_{i=1}^{l} \lambda_i (t_i - s_i)^{-1/p} w_i \chi_{(s_i,t_i)}$$

be in $\mathcal{A}$, where $w_i = \sum_{k=1}^{\infty} \alpha_k a_k / \|a_k\|^{s_i}$ belongs to V_{s_i} for each i. For each $s_i \le t \le t_i$, put

$$\mathbf{w}_i^{(t)} = (\alpha_1 a_1 / \|a_1\|^{s_i}, \ldots, \alpha_k (1/2)^{(t-1)(k-1)} a_k / \|a_k\|^{s_i}, \ldots)$$

in $(\bigoplus_{k=1}^{\infty} X_k)_{\ell^p}$. Then $\|\mathbf{w}_i^{(t)}\| \le \big(\sum_{k=1}^{\infty} |\alpha_k|^p (1/2)^{p(t-s_i)(k-1)} \big)^{1/p} \le 1$ and $Q_t(\mathbf{w}_i^{(t)}) = w_i$ for each $s_i \le t \le t_i$. Hence, if we define

$$\mathbf{f}(t) = \sum_{i=1}^{l} \lambda_i (t_i - s_i)^{-1/p} \mathbf{w}_i^{(t)} \chi_{(s_i,t_i)}(t) \quad (0 < t < 1) ,$$

then $\|\mathbf{f}\| \le 1$ and $Q\mathbf{f} = f$. Therefore Q maps the unit ball of $L^p((0,1), (\bigoplus_{k=1}^{\infty} X_k)_{\ell^p})$ onto the unit ball of $Z^\sharp$ and so Q is a quotient map. ∎

Since $L^p(0,1)$ is uniformly convex, $L^p((0,1), (\bigoplus_{k=1}^{\infty} X_k)_{\ell^p})$ is uniformly convex and so it follows, by [Da], Theorem 5.5, that $Z^\sharp$ is also uniformly convex. This theorem of Day may also be deduced from the duality between uniform convexity and uniform smoothness (see [LT2], Proposition 1.e.2).

PROPOSITION 4. *$Z^\sharp$ has the metric compact approximation property but does not have the approximation property.*

Proof. The same argument as showed that Z does not have the approximation property also shows that $Z^\sharp$ does not have this property. It is immediate from the definitions of these sets that $U_{1/2} \subseteq V_{1/2}$ and so $Y_{1/2}$ is contained in $W_{1/2}$ and the embedding is a contraction. Hence maps $R^\sharp : Y_{1/2} \to Z^\sharp$ and $J^\sharp : Z^\sharp \to X$ are defined by equations (1) and (3) respectively and an estimate similar to (2) shows that $R^\sharp$ is compact. These new maps also satisfy $J^\sharp R^\sharp = L_{1/2}$ and so $Z^\sharp$ does not have the approximation property.

Shift operators $S_r^\sharp : Z^\sharp \to Z^\sharp$ may be defined for each r between 0 and 1 just as they were on Z and then used to define operators $T_n^\sharp : Z^\sharp \to Z^\sharp$ for $n = 1, 2, \ldots$ by an equation similar to (4). The same argument as used in Proposition 2 shows that these operators have norm at most one and that $\{T_n^\sharp\}_{n=1}^\infty$ converges to the identity operator uniformly on compact subsets of $Z^\sharp$. However, to show that $T_n^\sharp$ is compact for each n requires a slightly different argument to that used in Proposition 2. It suffices to show that $T_n^\sharp \mathcal{A}$ is totally bounded.

For each positive integer m define the subset, $V_s^{(m)}$, of V_s by

$$V_s^{(m)} = \left\{ \sum_{k=m}^{\infty} \alpha_k a_k / \|a_k\|^s \ : \ a_k \in X_k,\ \|a_k\| \le \left(\frac{1}{2}\right)^{k-1}, \sum_{k=m}^{\infty} |\alpha_k|^p \le 1 \right\}.$$

Let $w = \sum_{k=m}^\infty \alpha_k a_k / \|a_k\|^s$ be in $V_s^{(m)}$ and suppose that $|||w|||_s = 1$. Then, for $r > s$,

$$|||w|||_r = \left|\left|\left| \sum_{k=m}^{\infty} \alpha_k \|a_k\|^{r-s} a_k / \|a_k\|^r \right|\right|\right|_r \le \left(\sum_{k=m}^{\infty} |\alpha_k \|a_k\|^{r-s}|^p \right)^{1/p} \le \left(\frac{1}{2}\right)^{(m-1)(r-s)}$$

Corresponding to the estimate in (5) we now have

$$\|T_n^\sharp (w\,\chi_{(s,t)})\| \le n(t-s) \left(\int_s^{t+1/n} |||w|||_r^p \, dr \right)^{1/p} < n(t-s)(m-1)^{-1/p}. \tag{8}$$

Next, for each positive integer m denote by $\mathcal{B}^{(m)}$ the subset of $\mathcal{A}$ consisting of all functions of the form (6) where w_i belongs to $V_s^{(m)}$ for each i and by $\mathcal{C}^{(m)}$ the subset of $\mathcal{A}$ consisting of functions where w_i belongs to the finite-dimensional space $\operatorname{span}\{x_n : n \le n_{m-1}\} = X_{m-1}$. For a general w in V_s, $w = \sum_{k=1}^{m-1} \alpha_k a_k / \|a_k\|^s + w'$, where w' is in $V_s^{(m)}$ and $\sum_{k=1}^{m-1} \alpha_k a_k / \|a_k\|^s$ belongs to the unit ball of X_{m-1}. It follows that $\mathcal{A} \subseteq \mathcal{B}^{(m)} + \mathcal{C}^{(m)}$.

For each ξ in $\mathcal{B}^{(m)}$ we have, by (6) and (8),

$$\|T_n^\sharp \xi\| \le n(m-1)^{-1/p} \sum_{i=1}^{l} |\lambda_i|\, |t_i - s_i|^{1-1/p} \le n(m-1)^{-1/p},$$

because $\sum_{i=1}^l |\lambda_i|^p = 1$ and $\sum_{i=1}^l |t_i - s_i| \le 1$. If, given $\varepsilon > 0$, we choose $m > 1 + (n/\varepsilon)^p$, then it follows that $\|T_n^\sharp f\| < \varepsilon$ for every f in $\mathcal{B}^{(m)}$. Also, since the map $f \mapsto nf * \chi_{(0,1/n)}$ is a compact operator on $L^p(0,1)$, $T_n^\sharp \mathcal{C}^{(m)}$ is totally bounded for each m. Therefore $T_n^\sharp \mathcal{A}$ is totally bounded and so $T_n^\sharp$ is a compact operator. ■

It may also be shown that, for $1 < p < 2$, there are quotients of subspaces of $L^p(0,1)$ which have the metric compact approximation property but not the approximation property. This may be shown by choosing a subspace, X, of ℓ^p which does not have the approximation property (such spaces have been shown to exist by Szankowski, see [Sz] or [LT2], Theorem 1.g.4), and then repeating the above construction. Alternatively, the dual of the above example has the required properties. I am grateful to Professor T. Figiel for this remark and also for some other suggestions which shortened some proofs and improved Proposition 3.

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The wavelet characterization of the space Weak H^1

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Abstract. The space Weak H^1 was introduced and investigated by Fefferman and Soria. In this paper we characterize it in terms of wavelets. Equivalence of four conditions is proved.

1. Introduction. When we study the boundedness on $L^p(\mathbb{R}^n)$ for some of the basic operators in harmonic analysis, the case $p = 1$ is often different from $p > 1$. For example, if T is a Calderón–Zygmund singular integral operator, then T is bounded from $L^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$. So one finds a smaller space $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ such that T is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, which is well known. On the other hand, one can also find a space larger than $L^1(\mathbb{R}^n)$ and T is bounded from it to $WL^1(\mathbb{R}^n)$. This space is $WH^1(\mathbb{R}^n)$, introduced and investigated by Fefferman and Soria (see [3]).

We recall the definition of $WH^1(\mathbb{R}^n)$: Let f be a tempered distribution, and $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\int \varphi(x)\,dx = 1$. We define the maximal function

$$f^*(x) = \sup_{t>0} |f * \varphi_t(x)|\,.$$

Then we say that $f \in WH^1(\mathbb{R}^n)$ provided $f^* \in WL^1(\mathbb{R}^n)$, i.e.

$$|\{x \in \mathbb{R}^n : f^*(x) > \nu\}| \le C/\nu \quad \text{for all } \nu > 0\,.$$

The smallest C which makes the preceding estimate valid is called the "Weak H^1 norm" and denoted by $\|f\|_{WH^1}$. The choice of φ in the definition of $WH^1(\mathbb{R}^n)$ is of no importance. The space $WH^1(\mathbb{R}^n)$ is larger than $L^1(\mathbb{R}^n)$. In fact, the space of complex measures is continuously embedded as a subspace of $WH^1(\mathbb{R}^n)$. Another basic example is the distribution p.v. $\frac{1}{x}$ which belongs to $WH^1(\mathbb{R})$.

If we proceed to characterize the function spaces in terms of wavelets, we find ourselves in the same situation. Suppose $f = \sum a(\lambda)\psi_\lambda$ in the sense

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