

On generalized inverses in C^* -algebras

by

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Abstract. We investigate when a C^* -algebra element generates a closed ideal, and discuss Moore-Penrose and commuting generalized inverses.

0. Suppose A is a ring, with identity 1 and invertible group A^{-1} . Then we call an element $a \in A$ *regular* if $a \in aAa$, giving it a *generalized inverse* $b \in A$ for which

$$(0.1) \quad a = aba.$$

Generalized inverses breed *idempotents*: if (0.1) holds then $ba = p = p^2$ and $ab = q = q^2$ satisfy $aA = qA$ and $Aa = Ap$ with, for each $x \in A$,

$$ax = 0 \Leftrightarrow px = 0 \quad \text{and} \quad xa = 0 \Leftrightarrow xq = 0.$$

The generalized inverse $b \in A$ of (0.1) can be *normalized*: if $c = bab$ then

$$a = aca \quad \text{and} \quad c = cac.$$

The passage from b to c does not alter the projections p and q ; conversely, $c = bab$ is determined by p and q :

1. THEOREM. *If $a = aba$ then*

$$(1.1) \quad b'a = ba \quad \text{and} \quad ab' = ab \Rightarrow b'ab' = b'ab = bab,$$

and if also $e = e^2$ and $f = f^2$ with $Aa = Ae$ and $aA = fA$ then

$$c = ebf \Rightarrow a = aca \quad \text{with} \quad ca = e \quad \text{and} \quad ac = f.$$

Proof. Once stated, this is very routine. ■

Associated with $a \in A$ are the *multiplication operators* $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ from A to A ; when in particular A is a Hausdorff *topological ring* (addition and multiplication continuous) then L_a and R_a are both

continuous and each of the “kernel ideals”

$$a^{-1}(0) = L_a^{-1}(0) = \{x \in A : ax = 0\},$$

$$a_{-1}(0) = R_a^{-1}(0) = \{x \in A : xa = 0\}$$

closed. The same is not always the case for the “range ideals”:

2. THEOREM. *If $a \in A$ is regular then so are L_a and R_a , and the range ideals $aA = L_a(A)$ and $Aa = R_a(A)$ are both closed.*

Proof. If $a = aba$ then $L_a = L_a L_b L_a$ and $R_a = R_a R_b R_a$; hence also $aA = qA = (1 - q)^{-1}(0)$ and $Aa = Ap = (1 - p)_{-1}(0)$. ■

Much of this note is concerned with the converse of Theorem 2; we begin with an observation about “neighbouring” idempotents ([1], Proposition 19.1; [5], Theorem 26; [6], Proposition I.4.2):

3. THEOREM. *If $p = p^2$ and $q = q^2$ then*

$$p(p - q)^2 = (p - q)^2 p \quad \text{and} \quad q(p - q)^2 = (p - q)^2 q.$$

If in particular

$$1 - (p - q)^2 = r^{-1} \in A^{-1}$$

then $e = qpr$ is idempotent, with

$$eA = qA \quad \text{and} \quad e^{-1}(0) = p^{-1}(0),$$

$$Ae = Ap \quad \text{and} \quad e_{-1}(0) = q_{-1}(0).$$

Proof. This is easily checked, once the formula is given. ■

Kovarik [7] calls this the “poor man’s path” between p and q . The product pq of idempotents p and q need not be idempotent ([3], Theorem 2.5.4):

4. THEOREM. *If $p = p^2$ and $q = q^2$ in A satisfy*

$$(1 - q)(1 - p) = 0$$

then pq is idempotent, with

$$qA = p^{-1}(0) + pqA \quad \text{and} \quad p^{-1}(0) \cap pqA = \{0\},$$

$$Ap = q_{-1}(0) + Apq \quad \text{and} \quad q_{-1}(0) \cap Apq = \{0\}.$$

Proof. Again routine. ■

If the ring A has an *involution* $*$: $A \rightarrow A$ then we can introduce the concept of a “Moore–Penrose inverse”, in the sense of a (normalized) generalized inverse $b(=bab)$ for $a = aba \in A$ for which

$$ba = (ba)^* \quad \text{and} \quad ab = (ab)^*,$$

i.e., the induced projections are *self-adjoint*—strictly, this is implied when we call an idempotent a “projection”. For example the adjoint is a Moore–Penrose inverse for a *partial isometry*, $a \in A$ for which $a = aa^*a$.

When they exist, Moore–Penrose inverses are unique, and double commute with an element and its adjoint:

5. THEOREM. *If $b \in A$ and $b' \in A$ are Moore–Penrose inverses for $a \in A$ then $b' = b$. If b is the Moore–Penrose inverse of $a \in A$ then*

$$(5.1) \quad ca = ac \quad \text{and} \quad ca^* = a^*c \Rightarrow cb = bc.$$

Proof. With $p = ba$ and $p' = b'a$, by (0.1),

$$p'p = b'aba = b'a = p' \quad \text{and} \quad pp' = bab'a = ba = p.$$

Taking adjoints gives $p' = (p')^* = p^*(p')^* = pp' = p$, and similarly $q' = q$; now use (1.1). Towards (5.1) we claim that if b is the Moore–Penrose inverse for a in A and $p = ba$ the induced idempotent then

$$(5.2) \quad ca = ac \Rightarrow pc = pcp,$$

$$(5.3) \quad ca^* = a^*c \Rightarrow pcp = cp.$$

For (5.2) argue

$$(5.4) \quad pc = bac = bca = bcaba = bacba = pcp,$$

and similarly for (5.3); alternatively take adjoints in (5.2). Thus if c is in the double commutant of a and a^* then c commutes with $p = ba$, and similarly, or by “reversal of products”, with $q = ab$. But now also $bc = babc = bcab = bacb = cbab = cb$, giving (5.1). ■

We shall write a^+ for the Moore–Penrose inverse of $a \in A$. For example if $a \in A$ has Moore–Penrose inverse $b = a^+$ then also

$$(5.5) \quad \exists (a^*)^+ = (a^+)^*$$

and hence the Moore–Penrose inverse of a self-adjoint element will also be self-adjoint. In general (notice for example $a^* = a^*aa^+$)

$$(a^+)^{-1}(0) = (a^*)^{-1}(0) \quad \text{and} \quad a^+A = a^*A.$$

The Moore–Penrose inverse exists for all regular elements when A is a C^* -algebra [1], [3], [4]:

6. THEOREM. *If a is regular in a C^* -algebra A then it has a Moore–Penrose inverse $a^+ \in A$.*

Proof. We use the construction of Theorem 3 and apply Theorem 1: if $p = p^2 \in A$ then

$$1 - (p - p^*)^2 = 1 + (p - p^*)(p - p^*)$$

is invertible [1], [3], [4] by the B^* -condition, so that Theorem 3 applies with

$$e = [p] = p^*p(1 - (p - p^*)^2)^{-1},$$

and also, after “reversal of products”, with

$$f = [q^*] = qq^*(1 - (q - q^*)^2)^{-1},$$

where $a = aba$ with $ba = p$ and $ab = q$. We may now apply Theorem 1: evidently $a^+ = [p]b[q^*]$ is a Moore–Penrose inverse for a . ■

It is rather clear, if A has an involution, that a regular $\Leftrightarrow a^*$ regular; in a C^* -algebra we have more:

7. THEOREM. If $a \in A$ for a C^* -algebra A then

$$(7.1) \quad a \text{ regular} \Leftrightarrow a^*a \text{ regular} \Leftrightarrow aa^* \text{ regular.}$$

In particular,

$$(7.2) \quad (a^*a)^+ = a^+(a^+)^* \quad \text{and} \quad (aa^+)^+ = (a^+)^*a^+.$$

Proof. If more generally A has an involution and $a = aba \in A$ with $ab = (ab)^*$ then

$$a^*abb^*a^*a = a^*ab(ab)^*a = a^*ababa = a^*a,$$

so that $a^*a \in A$ is also regular; thus if in particular a has a Moore–Penrose inverse then both aa^* and a^*a must be regular. By Theorem 6 this applies to all regular $a \in A$ when A is a C^* -algebra. Conversely, if for example a^*a is regular then there is $c \in A$ for which $a^*a = a^*aca^*$, which gives

$$(a - aca^*)^*(a - aca^*) = 0 \in A.$$

Now if the B^* -condition holds in A it follows that $a = aca^*$; this with a similar argument for aa^* finishes the proof of (7.1). For the first part of (7.2) observe that if $a = aba$ with self-adjoint ba and ab then

$$bb^*a^*a = b(ab)^*a = baba = (ba)^*ba \text{ self-adjoint.} \blacksquare$$

Our main result is a converse for Theorem 2:

8. THEOREM. If $a \in A$ for a C^* -algebra A then

$$(8.1) \quad aA \text{ closed} \Rightarrow a \text{ regular,}$$

and hence also

$$aA = \text{cl } aA \Leftrightarrow Aa = \text{cl } Aa.$$

Proof. Begin with the special case of a *positive* element, which always has a “square root” [1], [3], [4]:

$$0 \leq a \in A \Rightarrow \exists a^{1/2} = (a^{1/2})^* \in \text{cl } aA.$$

If aA is closed it follows that there is $c \in A$ for which $a^{1/2} = ac$; but now

$$acc^*a = ac(ac)^* = (a^{1/2})^2 = a.$$

For general $a \in A$ the product a^*a is positive, and satisfies

$$\|ax\| = \|(a^*a)^{1/2}x\| \quad \text{for each } x \in A,$$

so that, since also $\text{dist}(x, a^{-1}(0)) = \text{dist}(x, (a^*a)^{-1/2}(0))$,

$$aA \text{ closed} \Leftrightarrow (a^*a)^{1/2}A \text{ closed.}$$

Now (8.1) follows from two applications of Theorem 7: if aA is closed then so is cA for the positive element $c = (a^*a)^{1/2}$, which by the first part of this argument is also regular; but now $a^*a = c^*c$ is regular, and hence also a . ■

With the help of the square root we can also see that the Moore–Penrose inverse of a positive element will always be positive:

$$0 \leq a \Rightarrow a^+ = a^+aa^+ = (a^{1/2}a^+)^*a^{1/2}a^+.$$

We shall call an element $a \in A$ *decomposably regular* if $a \in aA^{-1}a$, i.e., it has an *invertible* generalized inverse, and *simply polar* ([2], Definition 3.1; [3], Definition 7.3.5) if $a \in a \text{ comm}(a)a$, i.e., it has a *commuting* generalized inverse. This is a very strong condition to impose:

9. THEOREM. A normalized commuting generalized inverse is unique. If $a \in A$ has a commuting generalized inverse then it is decomposably regular, and

$$(9.1) \quad A = aA + a^{-1}(0) \quad \text{with } aA \cap a^{-1}(0) = \{0\},$$

$$(9.2) \quad A = Aa + a_{-1}(0) \quad \text{with } Aa \cap a_{-1}(0) = \{0\}.$$

Proof. If $a = aba$ with $ba = ab$ and also $a = ab'a$ with $b'a = ab'$ then

$$ab' = abab' = bab'a = ba,$$

so that the projection $p = ba = ab$ is uniquely determined, and hence also by Theorem 1 the (normalized) commuting generalized inverse $b = bab$. We can—at the cost of normalization—convert b to an invertible generalized inverse:

$$c = b + (1 - p), \quad c' = a + (1 - p) \Rightarrow a = aca \text{ and } c'c = 1 = cc'.$$

The decomposition (9.1) is accomplished by writing $x = px + (1 - p)x$ for each $x \in A$, where $p = ab = ba$, and similarly for (9.2). ■

The conditions (9.1) and (9.2) are not together sufficient for $a \in aAa$ to be simply polar: for example ([2], Example 4.7; [3], (7.3.6.8)) take

$$a = T = \begin{pmatrix} W & -I \\ 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} c_{00} \\ c_{00} \end{pmatrix} \rightarrow \begin{pmatrix} c_{00} \\ c_{00} \end{pmatrix}$$

with $(Wx)_n = (1/n)x_n$ ($n \in \mathbb{N}$) for each $x \in c_{00}$, the terminating sequences.

When $a \in A$ is simply polar then [2], [3] its “Drazin inverse” $a^\times = b$ and its *support projection* $p = ab = ba$ lie in the *double commutant* $\text{comm}^2(a)$ of a in A , and all the powers a^n of a are regular: $a^n = a^n b^n a^n$ for each $n \in \mathbb{N}$.

A normal element in a C^* -algebra with a generalized inverse has all these properties:

10. THEOREM. *The Moore-Penrose inverse a^+ of a regular normal element a in a C^* -algebra A is also normal and commutes with a :*

$$(10.1) \quad a \in aAa \text{ and } aa^* = a^*a \Rightarrow a^+a^{**} = a^{**}a^+ \text{ and } aa^+ = a^+a.$$

In particular, normal regular elements are simply polar.

PROOF. If $a \in A$ is normal then $c = a$ lies in the double commutant of a and a^* , and therefore by (5.1) commutes with the Moore-Penrose inverse a^+ . This gives the second part of (10.1); for the normality recall (5.5) and (7.2):

$$(a^*a)^+ = a^+a^{**} = a^+a^{**} \quad \text{and} \quad (aa^*)^+ = a^{**}a^+ = a^{**}a^+. \quad \blacksquare$$

We are indebted to the referee for the argument (5.4), which enables us to deduce (5.1) without using the B^* -condition.

The special case of Theorem 8, in which $A = BL(X, X)$ is the algebra of all bounded operators on a Hilbert space X , tells us that an operator T has closed range in X if and only if the multiplication L_T has closed range in $BL(X, X)$. This can be seen directly: if, more generally, X, Y and $Z \neq \{0\}$ are normed spaces and $T \in BL(X, Y)$ then with the help of "rank one" operators $h \odot y : z \rightarrow h(z)y$ there is implication

$$L_T BL(Z, X) \text{ closed in } BL(Z, Y) \Rightarrow T(X) \text{ closed in } Y,$$

for if $y = \lim Tx_n$ and if $z \in Z$ and $h \in Z^\dagger$ satisfy $h(z) = 1$ then there must be $U \in BL(Z, X)$ for which

$$L_T(h \odot x_n) = h \odot Tx_n \rightarrow h \odot y = L_T(U) = TU,$$

giving $y = T(Uz) \in T(X)$. Conversely, if the spaces X and Y are complete then $T(X)$ is closed if and only if $T^\wedge : X/T^{-1}(0) \rightarrow Y$ is bounded below, and the factorization $T = T^\wedge \circ \pi$ gives inclusion

$$(10.2) \quad \text{Range } L_T \subseteq \text{Range } L_{T^\wedge}.$$

If also the null space $T^{-1}(0)$ is complemented then there exists $\omega : X/T^{-1}(0) \rightarrow X$ such that $T^\wedge = T \circ \omega$, giving equality in (10.2). The same sort of argument shows that the right multiplication R_T has closed range if and only if the dual operator $T^\dagger : Y^\dagger \rightarrow X^\dagger$ has closed range, provided the closure of the range of T is complemented.

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