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Automorphisms and derivations of a Fréchet algebra of locally integrable functions

by

F. GHAHRAMANI† and J. P. McCLURE‡ (Winnipeg, Man.)

Abstract. We find representations for the automorphisms, derivations and multipliers of the Fréchet algebra L^1_{loc} of locally integrable functions on the half-line $\mathbb{R}^+$. We show, among other things, that every automorphism θ of L^1_{loc} is of the form $\theta = \varphi_a e^{\lambda X} e^D$, where D is a derivation, X is the operator of multiplication by coordinate, λ is a complex number, $a > 0$, and φ_a is the dilation operator $(\varphi_a f)(x) = af(ax)$ $(f \in L^1_{\mathrm{loc}}, x \in \mathbb{R}^+)$. It is also shown that the automorphism group is a topological group with the topology of uniform convergence on bounded sets and is the semidirect product of a connected subgroup and a discrete group which is isomorphic to the discrete group of real numbers.

1. Introduction. Weighted convolution algebras on the half-line $\mathbb{R}^+ = [0,\infty)$ were identified as examples of commutative Banach algebras early in the development of the subject [7]. More recently, in connection with the development of automatic continuity theory, *radical* weighted convolution algebras on $[0,\infty)$ have received a lot of attention [2]. These algebras are all subalgebras of the algebra of locally integrable functions on $[0,\infty)$, which is the main object of study in this paper. We denote this algebra by $L^1_{\mathrm{loc}}(\mathbb{R}^+)$, or just L^1_{loc}. As a vector space, L^1_{loc} consists of all (equivalence classes, with respect to equality almost everywhere, of) Lebesgue measurable, complex-valued functions f on $[0,\infty)$, for which

$$P_a(f) = \int_0^a |f(x)|\,dx \tag{1.1}$$

is finite, for all positive real numbers a; addition and scalar multiplication are defined pointwise. L^1_{loc} becomes a commutative, associative algebra, if

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we take the convolution product

$$(1.2)\qquad (f*g)(x)=\int_0^x f(x-y)g(y)\,dy \quad (x\in[0,\infty)).$$

In the next section, we shall see that the functionals P_a of (1.1) give L^1_{loc} a topology in which it becomes a Fréchet algebra. We shall show that the closed ideals of L^1_{loc} are just the standard ideals I_a: for $a>0$,

$$(1.3)\qquad I_a=\{f: f\in L^1_{\mathrm{loc}} \text{ and } f(x)=0 \text{ a.e. on } [0,a)\}$$

and we also put $I_0=L^1_{\mathrm{loc}}$, and $I_\infty=\{0\}$. We deduce that L^1_{loc} is a radical Fréchet algebra. We shall also give some preliminary remarks about the continuity of linear maps on L^1_{loc}, and about a topology on the algebra of continuous linear maps on L^1_{loc}, which will be relevant to the later sections of the paper. In those sections, we explore, in turn, the multipliers, the derivations, and the automorphisms of L^1_{loc}. It turns out that these maps are all continuous, and have representations analogous to the corresponding operators on Banach convolution algebras on $[0,\infty)$. In particular, we obtain a representation of automorphisms as products of dilation automorphisms and exponentials of derivations. This leads to a description of the automorphism group of L^1_{loc} as the semidirect product of a discrete subgroup isomorphic with $\mathbb{R}$, with a connected subgroup. We also consider the relationship between automorphisms of L^1_{loc} and automorphisms of its weighted (Banach) subalgebras. Automorphisms of Fréchet (and Banach) algebras of power series were studied in [13].

2. Preliminaries. It is easy to check, and well known, that each of the functionals P_a, defined in (1.1), is a seminorm on L^1_{loc}, and is submultiplicative with respect to the product (1.2). The null space of P_a is exactly the ideal I_a defined in (1.3). The quotient algebra L^1_{loc}/I_a is isomorphic with the algebra $L^1[0,a)$ of (equivalence classes of) Lebesgue integrable functions on $[0,a)$. $L^1[0,a)$ is a radical Banach algebra, normed by P_a and with product given by (1.2), restricted to x in $[0,a)$. As Banach algebras, the algebras $L^1[0,a)$ are all isomorphic with the Volterra algebra $V=L^1[0,1)$, which has been studied by a number of authors [10], [11], and [15]. We shall identify L^1_{loc}/I_a with $L^1(0,a)$; the quotient map is then represented by the restriction

$$(2.1)\qquad R_af=f|[0,a)\quad (f\in L^1_{\mathrm{loc}}).$$

When $0<a<b<\infty$, it will also be convenient to have a notation for the restriction (or quotient) map from $L^1[0,b)$ to $L^1[0,a)$; we put

$$(2.2)\qquad R_{a,b}f=f|[0,a)\quad (f\in L^1[0,b)).$$

Note that $R_{a,b}R_b=R_a$. In fact, the system of algebras $L^1[0,a)$, with the homomorphisms $R_{a,b}$ (when $0<a<b<\infty$) is a projective system of commutative (radical) Banach algebras and continuous homomorphisms, and L^1_{loc} is (isomorphic with) the projective limit of this system. The seminorms $\{P_a: a\in[0,\infty)\}$ clearly form a separating family on L^1_{loc}, and the countable subset $\{P_n: n\in\mathbb{N}\}$ is cofinal in $\{P_a\}$. Thus, the following result is an immediate consequence of the work of Michael [16].

PROPOSITION (2.3). *With the topology determined by the seminorms $\{P_a: 0<a<\infty\}$, L^1_{loc} is a commutative Fréchet algebra.*

Sometimes, it will be convenient to think of functions in $L^1[0,a)$ as defined on $[0,\infty)$. For $a>0$, and $f\in L^1[0,a)$, we define S_af in L^1_{loc} as follows:

$$(2.4)\qquad (S_af)(x)=\begin{cases} f(x) & \text{if } x<a,\\ 0 & \text{if } x\ge a.\end{cases}$$

It is easy to see that S_a is a linear map on $L^1[0,a)$ into L^1_{loc}. The image $S_a(L^1[0,a))$ is a closed subspace of L^1_{loc}, but *not* a subalgebra. Note that R_aS_a is the identity map on $L^1[0,a)$, while S_aR_a is the projection of L^1_{loc} onto its subspace of functions vanishing on $[a,\infty)$. It may be worth emphasizing that we are using the same symbol P_a for a *seminorm* on L^1_{loc} and for *the norm* on $L^1[0,a)$. Thus, for f in L^1_{loc}, we have $P_a(f)=P_a(R_af)$, while for f in $L^1[0,a)$, we have $P_a(f)=P_a(S_af)$.

Recall that, for f in L^1_{loc}, $\alpha(f)$ is the infimum of the support of f. Conventionally, we take $\alpha(0)=\infty$. With this notation, for $a\ge 0$, we have $I_a=\{f:\alpha(f)\ge a\}$, where I_a was defined in (1.3). As null spaces of the seminorms P_a defining the topology of L^1_{loc}, the ideals I_a are closed in L^1_{loc}. These are called *standard* ideals, and we now show that every closed ideal is standard.

PROPOSITION (2.5). (a) *Every closed ideal in L^1_{loc} is a standard ideal.*
(b) *L^1_{loc} is a radical algebra.*

Proof. (a) It is sufficient to show that every *principal* closed ideal is standard. We shall show that, for each f in L^1_{loc}, $(f*L^1_{\mathrm{loc}})^-=I_{\alpha(f)}$.

A function g belongs to $(f*L^1_{\mathrm{loc}})^-$ if and only if, for each $a>0$, and every $\varepsilon>0$, there is h in L^1_{loc} such that $P_a(g-f*h)<\varepsilon$. Since $P_a(g-f*h)=P_a(R_ag-(R_af)*(R_ah))$, it follows that g belongs to $(f*L^1_{\mathrm{loc}})^-$ if and only if, for each $a>0$, R_ag belongs to $(R_af*L^1[0,a))^-$. But, the closed ideals in the Volterra-type algebras $L^1[0,a)$ are known to be standard [6]; thus, g belongs to $(f*L^1_{\mathrm{loc}})^-$ if and only if, for each $a>0$, we have $\alpha(R_ag)\ge\alpha(R_af)$, and the latter condition is clearly equivalent to $\alpha(g)\ge\alpha(f)$. Thus $(f*L^1_{\mathrm{loc}})^-=I_{\alpha(f)}$.

(b) From (a), there is no character on L^1_{loc} (i.e. no nonzero, continuous, multiplicative linear functional). So, Corollary 5.5 of [16] implies that L^1_{loc} is a radical algebra.

Next, we recall some basic information about continuous linear maps on $L^1_{\rm loc}$ into itself. First, a linear map T on $L^1_{\rm loc}$ into itself is continuous if and only if for each $a > 0$, there are $b > 0$ and $K > 0$ such that

$$P_a(Tf) \le KP_b(f) \tag{2.6}$$

for all f in $L^1_{\rm loc}$. It follows from (2.6) that

$$T(I_b) \subseteq I_a\,, \tag{2.7}$$

and that there is a unique, bounded linear map $T_{a,b}$ in $B_{a,b} \equiv B(L^1[0,b), L^1[0,a))$ satisfying

$$R_aT = T_{a,b}R_b. \tag{2.8}$$

We remark in passing that, *for continuous* T, (2.7), and (2.8) *with bounded* $T_{a,b}$ are equivalent. The operator norm $\|T_{a,b}\|$ of $T_{a,b}$ in $B_{a,b}$ is exactly the infimum of the values of K for which (2.6) holds. Note that we also have, for any f in $L^1_{\rm loc}$,

$$(R_aT)(f) = (T_{a,b}R_b)(f) = (T_{a,b}R_bS_bR_b)(f) = (R_aT)[(S_bR_b)(f)]\,; \tag{2.9}$$

that is, Tf on $[0,a)$ is determined by f on $[0,b)$.

Write $B(L^1_{\rm loc})$ for the algebra of all continuous linear operators on $L^1_{\rm loc}$. The topology $\tau_{\rm b}$ of *uniform convergence on bounded sets* is defined on $B(L^1_{\rm loc})$ by the seminorms $P_{a,B}$, where, for $a > 0$ and B a bounded subset of $L^1_{\rm loc}$, $P_{a,B}(T) = \sup\{P_a(Tf) : f \in B\}$, for each T in $B(L^1_{\rm loc})$. This (or, an equivalent) definition is given in [17, p. 68] and in [19, p. 337]. More information about topologies on spaces of continuous operators may also be found in these sources; however, we shall need only the definition, and we shall use the topology only on certain subspaces of $B(L^1_{\rm loc})$, related to the automorphism group.

For each $r > 0$, we write B_r for the set of all linear operators on $L^1_{\rm loc}$ such that, for each $a > 0$, (2.6) holds with $b = ar$, for some K. It is easy to see that B_r is a linear subspace of $B(L^1_{\rm loc})$, and it is not the trivial subspace: the *dilation operator* φ_r defined by

$$(\varphi_rf)(x) = rf(rx) \tag{2.10}$$

for f in $L^1_{\rm loc}$, and x in $[0,\infty)$, is easily seen to belong to B_r. Another easy verification shows that each φ_r is, in fact, an automorphism of $L^1_{\rm loc}$; the inverse operator to φ_r is $\varphi_{1/r}$. Now, each T in B_r determines, and is determined by, a set of bounded linear maps $\{T_{a,ar} : a \in (0,\infty)\}$, where $T_{a,ar}$, in $B_{a,ar}$, satisfies (2.8) with $b = ar$. Of course, $T_{a,ar}$ depends linearly on T, so for each a in $(0,\infty)$, we can define a seminorm $q_{a,r}$ on B_r by

$$q_{a,r}(T) = \|T_{a,ar}\| \tag{2.11}$$

for T in B_r, where the norm symbol means the operator norm on $B_{a,ar}$. The following lemma will be used in our study of the automorphism group of $L^1_{\rm loc}$.

LEMMA (2.12). *If $\{a(n) : n \in \mathbb{N}\}$ is any sequence of real numbers tending to infinity, then the topology $\tau_{\rm b}$ on B_r is determined by the seminorms $q_{a(n),r}$. In particular, $\tau_{\rm b}$ is metrizable on B_r.*

Proof. Take any sequence $a(n)$ such that $a(n) \to \infty$ as $n \to \infty$. Then, for any a in $(0,\infty)$ and any bounded set B in $L^1_{\rm loc}$, we can pick and fix n so that $a(n) \ge a$, and we can take a constant K such that $P_{ra(n)}(f) \le K$ if f is in B. Let $B(n) = \{f \in L^1_{\rm loc} : P_{ra(n)}(f) \le K$ and f vanishes on $[ra(n),\infty)\}$. Then $B(n)$ is a bounded set in $L^1_{\rm loc}$, and $(S_{ra(n)}R_{ra(n)})(B) \subseteq B(n)$. Using (2.9), we have, for any T in B_r,

$$P_{a,B}(T) \le P_{a(n),B}(T) \le P_{a(n),B(n)}(T) \le K\|T_{a(n),ra(n)}\| = Kq_{a(n),r}(T)\,.$$

This calculation shows that, on B_r, each seminorm $P_{a,B}$ is dominated by a multiple of a seminorm $q_{a(n),r}$ and also that each $q_{a(n),r}$ *is* a $P_{a(n),B}$ (take $K = 1$ and $B = B(n)$). Thus, on B_r, $\tau_{\rm b}$ is determined by the countably many seminorms $q_{a(n),r}$.

It is an easy consequence of the definition of B_r that $B_r \subseteq B_s$ whenever $r \le s$; thus, when $r \le s$, the seminorms $q_{a,s}$ are defined on B_r, and can be used to determine $\tau_{\rm b}$ on B_r.

The subspace B_1 will be of particular interest. Note that an operator T in $B(L^1_{\rm loc})$ belongs to B_1 if and only if $T(I_a) \subseteq I_a$ for all a in $(0,\infty)$; it follows easily that B_1 is a subalgebra of $B(L^1_{\rm loc})$. For T in B_1 (and *only* for this case), we write T_a for the operator $T_{a,a}$ in $B_{a,a}$ determined by (2.8), for each a in $(0,\infty)$. Since each $B_{a,a}$ is a Banach algebra, we may define $\exp(T_a)$ in $B_{a,a}$. If $a \le b$, we have $R_aT = T_aR_a = T_aR_{ab}R_b$, and $R_aT = R_{ab}R_bT = R_{ab}T_bR_b$. It follows that $T_aR_{ab} = R_{ab}T_b$, and from that we get $\exp(T_a)R_{ab} = R_{ab}\exp(T_b)$. Therefore, the following definition is a good one.

DEFINITION (2.13). For T in B_1, we define $\exp(T) \equiv e^T$ on $L^1_{\rm loc}$ by requiring $R_a\exp(T) = \exp(T_a)R_a$ for all a in $(0,\infty)$.

For T in B_1, the operator e^T is continuous (since each $\exp(T_a)$ is), and belongs to B_1.

Finally in this preliminary section, we mention the algebra $M_{\rm loc}$ of all Radon measures on $[0,\infty)$. $M_{\rm loc}$ is a Fréchet algebra, the topology being determined by the seminorms

$$P_a(\mu) = |\mu|([0,a))$$

for $a > 0$, and the product being defined (indirectly) by

$$\int f(t)\,d(\mu*\nu)(t) = \int\int f(x+y)\,d\mu(x)\,d\nu(y)$$

(for f in C_c and μ,ν in $M_{\rm loc}$).

From the point of view of projective limits, we note that, by the Riesz representation theorem, the dual of $C_0[0,a)$ is the space $M[0,a)$ of bounded Borel measures on $[0,a)$. It is well known that, with the convolution product (restricted to $[0,a)$), $M[0,a)$ is (isomorphic with) the multiplier algebra of $L^1[0,a)$ [15, Remark 10]. If, for $b>a$, we again write $R_{a,b}$ for the restriction map from $M[0,b)$ to $M[0,a)$, then the maps $R_{a,b}$ are epimorphisms of convolution algebras, and with these epimorphisms, the algebras $M[0,a)$ form a projective system, whose projective limit is M_{loc}. As with $R_{a,b}$, we will use the same symbol R_a, already used for L^1_{loc}, to denote the restriction (or quotient) mapping from M_{loc} to $M[0,a)$, for each $a>0$.

The algebra L^1_{loc} can be identified with a closed ideal in M_{loc}, by identifying a locally integrable function f with the absolutely continuous measure $f(x)dx$. In fact, with this identification, L^1_{loc} becomes a closed ideal in M_{loc}, and thus we see that not every closed ideal in M_{loc} is (the analogue of) a standard ideal. We write δ_x for the point mass at x (a point of $[0,\infty)$). These point masses form a (multiplicative) semigroup in M_{loc}, and δ_0 is the multiplicative identity in M_{loc}.

Next we give a description of the multipliers of L^1_{loc}. Recall that a linear map T on L^1_{loc} is a multiplier if $T(f*g)=Tf*g$ for any f and g in L^1_{loc}.

We shall employ the Titchmarsh convolution theorem, a proof of which can be found for example in [2].

THEOREM (2.14). (a) *For each μ in M_{loc}, the mapping $T_\mu f=\mu*f$ is a continuous multiplier on L^1_{loc}.*

(b) *If T is any multiplier on L^1_{loc}, then there is a measure μ in M_{loc} such that $Tf=\mu*f$ for all f in L^1_{loc}. In particular, every multiplier on L^1_{loc} is continuous.*

Proof. (a) That T_μ is a multiplier is implicit in the fact, mentioned in the previous section, that L^1_{loc} is an ideal in M_{loc}. The continuity of T_μ is shown by the calculation

$$P_a(\mu*f)=P_a(R_a\mu*R_af)\le P_a(R_a\mu)P_a(R_af)=P_a(R_a\mu)P_a(f). \tag{2.15}$$

(b) Let g be any element of L^1_{loc} with $\alpha(g)=0$. Then, for any f in L^1_{loc}, $Tf*g=f*Tg$, so the Titchmarsh convolution theorem implies $\alpha(Tf*g)=\alpha(Tf)+\alpha(g)=\alpha(f)+\alpha(Tg)\ge\alpha(f)$. It follows that $T(I_a)\subseteq I_a$ for all $a>0$, and thus that there are unique maps T_a on $L^1[0,a)$ satisfying $T_aR_a=R_aT$ for all a. It is routine to check that T_a is a multiplier on $L^1[0,a)$; thus, there is a measure μ_a in $M[0,a)$ such that $T_af=\mu_a*f$ for all f in $L^1[0,a)$ [15, Remark 10]. Now suppose $b>a$. Then, for f in $L^1[0,a)$, we have

$$\begin{aligned}(R_{a,b}\mu_b)*f&=(R_{a,b}\mu_b)*(R_aS_af)=(R_{a,b}\mu_b)*(R_{a,b}R_bS_af)\\&=R_{a,b}(\mu_b*R_bS_af)=R_{a,b}T_bR_bS_af=R_{a,b}R_bTS_af\\&=R_aTS_af=T_aR_aS_af=T_af=\mu_a*f.\end{aligned}$$

Thus $R_{ab}\mu_b=\mu_a$ whenever $b>a$, and it follows that there is a unique measure μ in M_{loc} satisfying $R_a\mu=\mu_a$ for all $a>0$. But then, for any $a>0$ and any f, we have

$$(R_aT)(f)=(T_aR_a)(f)=\mu_a*R_a(f)=R_a(\mu)*R_a(f)=R_a(\mu*f).$$

Thus, $T(f)=\mu*f$ for all f, as required.

We remark that, since each multiplier T is continuous and satisfies $P_a(Tf)\le KP_a(f)$ for all a in $(0,\infty)$ and all f (see (2.15)), T is in B_1, so the exponential e^T exists and is a continuous linear map on L^1_{loc}. It is straightforward to check that e^T is a multiplier. Of course, if $T=T_\mu$ then $e^T=T_{\exp(\mu)}$, where, for μ in M_{loc}, e^μ exists, by functional calculus for Fréchet algebras, since the exponential function is an entire function.

We also remark that, given a measure μ in M_{loc}, we can construct a continuous, radical weight $w(x)$ on $[0,\infty)$ such that μ belongs to $M(w)$, i.e. such that $\int w\,d|\mu|<\infty$. The weight $w(x)$ can even be taken to be star-shaped ([1], [18]) or to satisfy other desirable regularity properties. We omit the details of this particular construction; a more difficult one will be given in the next section. The point of the remark at this stage is to observe that every multiplier on L^1_{loc} is, in fact, the extension of a multiplier on a (Banach) weighted convolution subalgebra $L^1(w)$.

3. Derivations. In this section, we describe the derivations on L^1_{loc}. Let D be such a derivation. Since M_{loc} is the multiplier algebra of L^1_{loc}, we can extend D to a derivation on M_{loc}, as follows. Given μ in M_{loc}, we define a mapping T on L^1_{loc} by $T(f)=D(\mu*f)-\mu*D(f)$. It is easy to check that T is a multiplier on L^1_{loc}. Thus, by Theorem (2.14), there is a measure in M_{loc}, which we denote by $\Delta(\mu)$, which satisfies $D(\mu*f)=\Delta(\mu)*f+\mu*D(f)$ for all f in L^1_{loc}. Further routine calculations show that the map Δ thus defined on M_{loc} is a derivation, and that $\Delta|L^1_{\text{loc}}=D$.

Now let X denote the mapping on M_{loc} (or L^1_{loc}) of multiplication by the coordinate function: for ν in M_{loc}, $d(X\nu)(t)=td\nu(t)$ (or, for f in L^1_{loc}, $(Xf)(t)=tf(t)$). Then it is a result of H. G. Diamond [4], [5] that Δ is a derivation on M_{loc} if and only if there is a measure μ in M_{loc} such that $\Delta(\nu)=(X\nu)*\mu$ for all ν in M_{loc}. In particular, since Δ extends D, we have $Df=(Xf)*\mu$ for f in L^1_{loc}. Thus, we have the following result.

THEOREM (3.1). *D is a derivation on L^1_{loc} if and only if there is a measure μ in M_{loc} such that $Df=Xf*\mu$ for all f in L^1_{loc}. In particular, every derivation D is continuous.*

Proof. If μ is given, it is easy to check that $Df=Xf*\mu$ defines a derivation. On the other hand, if D is a given derivation, the existence of μ satisfying $Df=Xf*\mu$ was established in the discussion preceding the statement of the theorem. The continuity follows from the inequality

$$P_a(xf * \mu) \le aP_a(\mu)P_a(f) \tag{3.2}$$

so the theorem is proved.

In a later section, it will be convenient to know, for a given derivation D on L^1_{loc}, whether there is a weight $w(x)$ such that D restricts to a derivation on $L^1(w)$. It is shown in [8] that, for a given weight $w(x)$ and measure μ, $Df = Xf * \mu$ defines a derivation on $L^1(w)$ if and only if

$$\sup\left\{\frac{x}{w(x)}\int w(x+y)\,d|\mu|(y) : x > 0\right\} < \infty. \tag{3.3}$$

We first note that if $\mu(\{0\}) \ne 0$, then (3.3) cannot hold for any weight $w(x)$, since, for all x in $(0,\infty)$,

$$\frac{x}{w(x)}\int w(x+y)\,d|\mu|(y) \ge x|\mu|(\{0\}).$$

It turns out that this is the only barrier to the existence of a suitable weight, as shown in the following theorem. Recall that a weight $w(x)$ is called *star-shaped* if the function $\eta(x) = -\log w(x)$ has the property that $\eta(x)/x$ is a non-decreasing function of x in $[0,\infty)$.

THEOREM (3.4). *Let μ be a Radon measure such that $\mu(\{0\}) = 0$. Then there is a continuous, star-shaped weight $w(x)$ such that $Df = Xf*\mu$ defines a derivation on $L^1(w)$.*

Proof. We are going to obtain weights $w(x)$ by first constructing a function $\eta(x)$, and then putting $w(x) = e^{-\eta(x)}$. To construct $\eta(x)$, we shall first choose a sequence $\{A_n : n = 0,1,\ldots\}$ such that $\{A_n\}$ is positive and increasing, and then we shall take sequences $\{a_n\}$ and $\{b_n\}$ so that, by putting

$$\eta(x) = a_n x + b_n \quad \text{for } x \text{ in } [n, n+1),\ n = 0,1,\ldots, \tag{3.5}$$

we obtain $\eta(x)$ continuous on $[0,\infty)$, and satisfying

$$\eta(n) = nA_n \quad \text{for } n = 0,1,\ldots \tag{3.6}$$

In fact, these conditions determine $\{a_n\}$ and $\{b_n\}$, once $\{A_n\}$ has been chosen. For, we must have $nA_n = na_n + b_n$, and $(n+1)a_n + b_n = (n+1)A_{n+1}$, whence

$$a_n = (n+1)A_{n+1} - nA_n. \tag{3.7}$$

Since $\{A_n\}$ is increasing, (3.7) gives

$$a_n \ge A_n. \tag{3.8}$$

Then, since $b_n = n(A_n - a_n)$, we have $b_n \le 0$. From (3.5), we have $\eta(x)/x = a_n + b_n/x$, for x in $[n, n+1)$, $n = 0,1,\ldots$ (put $\eta(0)/0 = a_0$). Since $b_n \le 0$, this shows that $\eta(x)/x$ is increasing on $[n, n+1)$ for each n; since $\eta(x)/x$ is also continuous, we have $\eta(x)/x$ increasing on $[0,\infty)$, so $w(x) = e^{-\eta(x)}$ will be a star-shaped weight. Also, $\eta(x)$ continuous implies that $w(x)$ is continuous.

Now, suppose μ is a Radon measure on $[0,\infty)$ satisfying $\mu(\{0\}) = 0$, and first suppose also that $\mathrm{supp}(\mu) \subseteq [0,1]$. We are trying to find a weight so that (3.3) holds, so we may suppose that μ is a positive measure. Since μ has bounded support, μ is a finite measure. Since $\mu([0,\varepsilon)) \to 0$ as $\varepsilon \to 0+$, and since $e^{-Ay} \to 0$ as $A \to \infty$, uniformly on $[\varepsilon,\infty)$, for any $\varepsilon > 0$, we may, by taking each A_n sufficiently large, guarantee

$$\int \exp(-A_n y)\,d\mu(y) < \frac{1}{n+1} \quad \text{for } n = 0,1,\ldots \tag{3.9}$$

Also, because of (3.7), by successively choosing the numbers A_n sufficiently large, we can ensure that the sequence $\{a_n\}$ is increasing. Now suppose $x \in [n, n+1)$, and $y \in [n+1-x, 1]$, so that $x + y \in [n+1, n+2)$. Then we have

$$\begin{aligned}\eta(x+y) - \eta(x) &= \eta(x+y) - \eta(n+1) + \eta(n+1) - \eta(x)\\ &= a_{n+1}[x+y-(n+1)] + a_n(n+1-x)\\ &= (a_{n+1} - a_n)[x+y-(n+1)] + a_n y \ge a_n y,\end{aligned} \tag{3.10}$$

where we have used the continuity of η, (3.5), and the fact that $\{a_n\}$ is increasing. Also, for y in $[0, n+1-x)$, so that $x+y$ is in $[n, n+1)$, we have $\eta(x+y) - \eta(x) = a_n y$, by (3.5). Therefore, we have, for x in $[n, n+1)$,

$$\begin{aligned}\frac{x}{w(x)}\int w(x+y)\,d\mu(y) &\le (n+1)\int_{[0,1]} \exp(-a_n y)\,d\mu(y)\\ &\le (n+1)\int \exp(-A_n y)\,d\mu(y) < 1,\end{aligned}$$

using $w(t) = \exp(-\eta(t))$, (3.5), (3.10), (3.8) and (3.9). Since n was arbitrary, (3.3) holds, as required.

Next, suppose $\mathrm{supp}(\mu) \subseteq [1,\infty)$. As before, we can suppose that μ is positive. Again, by successively choosing $\{A_n\}$ sufficiently large, we ensure that $\{a_n\}$, defined by (3.7), is increasing; thus, again, (3.10) holds whenever $x \in [n, n+1)$ and $y \in [n+1-x, 1]$; in this case, we shall only use (3.10) in the case $y = 1$. As well, we shall, by choosing the numbers A_n sufficiently large, guarantee

$$a_n > \log(n+1) \quad \text{for } n = 0,1,\ldots, \tag{3.11}$$

and (recall $w(n) = \exp(-nA_n)$)

$$\sum_{j=1}^{\infty} w(j-1)\mu([j,j+1]) < \infty . \tag{3.12}$$

Now suppose $n \geq 0$ and $x \in [n, n+1)$. Then

$$\begin{aligned}\frac{x}{w(x)} \int w(x+y)\,d\mu(y) &= x\frac{w(x+1)}{w(x)} \int_{[1,\infty)} \frac{w(x+y)}{w(x+1)}\,d\mu(y) \\ &\leq (n+1)\exp[-(\eta(x+1)-\eta(x))] \int_{[1,\infty)} w(y-1)\,d\mu(y) \\ &\leq (n+1)\exp(-a_n)\sum_{j=1}^{\infty} \int_{[j,j+1)} w(y-1)\,d\mu(y) \\ &< \sum_{j=1}^{\infty} w(j-1)\mu([j,j+1)),\end{aligned} \tag{3.13}$$

where we have used (3.10) (with $y = 1$), (3.11), submultiplicativity of $w(t)$, and the fact that $w(x)$ is decreasing (since $\eta(x)$ is increasing). Since n was arbitrary, (3.12) and (3.13) now imply that (3.3) holds.

Finally, consider an arbitrary Radon measure μ with $\mu(\{0\}) = 0$. Again, we can suppose that μ is positive. Put $\mu_1 = \chi_{[1,\infty)}\mu$ and $\mu_0 = \mu - \mu_1$. Then $\mu_0(\{0\}) = 0$, $\operatorname{supp}(\{\mu_0\}) \subseteq [0,1]$, and $\operatorname{supp}(\mu_1) \subseteq [1,\infty)$. By the arguments above, there are continuous, star-shaped weights w_0 and w_1 such that

$$\sup_{x>0} \frac{x}{w_q(x)} \int w_q(x+y)\,d\mu_q(y) < \infty , \tag{3.14}$$

for $q = 0,1$. We can suppose $w_q(x) \leq w_q(0) = 1$. Now put $w(x) = w_0(x)w_1(x)$. Then it is straightforward to verify that $w(x)$ is continuous, star-shaped and satisfies (3.3).

4. Automorphisms. In this section, we study the automorphisms of L^1_{loc}. Our first result is simple, and answers the natural question whether automorphisms of L^1_{loc} are continuous.

PROPOSITION (4.1). *Each automorphism of L^1_{loc} is continuous.*

Proof. Let θ be an automorphism of L^1_{loc}. Since L^1_{loc} is the projective limit of the algebras $L^1[0,a)$, it is enough to show that $R_a\theta$ is continuous for each $a > 0$. But L^1_{loc} is a Fréchet algebra, $L^1[0,a)$ is (isomorphic with) the Volterra algebra, and $R_a\theta$ is an epimorphism; so the continuity of $R_a\theta$ follows from an extension of [14, Remark 3a] to Fréchet algebras [3].

Next, we review some particular automorphisms of L^1_{loc}.

Let D be a derivation on L^1_{loc}. It follows from Theorem (3.1) (see (3.2)) that D is in B_1. Thus, there are bounded linear maps D_a on $L^1[0,a)$ defined by $R_aD = D_aR_a$, and there is a linear map e^D in B_1 determined by $R_ae^D = \exp(D_a)R_a$ for all a in $(0,\infty)$ (see Definition (2.13)). We show that each D_a is a derivation on $L^1[0,a)$. By Theorem (3.1), there is a measure μ in M_{loc} such that $Df = Xf * \mu$ for all f in L^1_{loc}. Thus, for f in $L^1[0,a)$,

$$\begin{aligned}D_a(f) &= (D_aR_aS_a)(f) = (R_aD)(S_af) \\ &= R_a(XS_af * \mu) = R_a(XS_af) * R_a\mu = Xf * R_a\mu .\end{aligned}$$

Thus, D_a is the derivation determined on $L^1[0,a)$ by the measure $R_a\mu$. It follows that $\exp(D_a)$ is an automorphism of $L^1[0,a)$, for each a in $(0,\infty)$. Then, using $R_ae^D = \exp(D_a)R_a$, it is easy to see that e^D is a homomorphism on L^1_{loc}. Since e^{-D} is, similarly, a homomorphism, and is easily seen to invert e^D, the latter is an automorphism of L^1_{loc}. Note the special case where $\mu = \lambda\delta_0$, a multiple of the Dirac measure; i.e., $Df = \lambda Xf$. In this case, e^D is exactly multiplication by the function $t \to e^{\lambda t}$, and we shall write $e^{\lambda X}$ for this automorphism.

Recall that, in Section 2, we noted that for each $a > 0$, there is a dilation automorphism φ_a on L^1_{loc}, where $(\varphi_af)(t) = af(at)$. In fact, these dilations, together with the automorphisms e^D, determine all automorphisms of L^1_{loc}.

THEOREM (4.2). *Let θ be an automorphism of L^1_{loc}. Then there are a positive real number a, a complex number λ, and a derivation D, defined by $Df = Xf * \mu$ with the Radon measure μ satisfying $\mu(\{0\}) = 0$, such that*

$$\theta = \varphi_a e^{\lambda X} e^D . \tag{4.3}$$

Conversely, any map θ of the form (4.3), *with a, λ, and D as described, is an automorphism. Finally, for a given automorphism θ, the numbers a and λ, and the derivation D which satisfy* (4.3) *are unique.*

Proof. That a map θ of the form (4.3) is an automorphism was shown in the discussion preceding the statement of the theorem. To prove the non-trivial part, let θ be any automorphism. By Propositions (4.1) and (2.5), the image by θ of each standard ideal is a standard ideal.

First consider the special case when we have $\theta(I_1) = I_1$. Let n be a positive integer, and suppose $\theta(I_n) = I_b$. Take f in I_1 such that $\alpha(\theta f) = 1$. We have f^{*n} in I_n so by assumption, $\theta(f^{*n})$ is in I_b. Also, by the Titchmarsh convolution theorem, $\alpha(\theta(f^{*n})) = n\alpha(\theta(f)) = n$; therefore $n \geq b$. Next, since θ^{-1} is also an automorphism and $\theta^{-1}(I_b) = I_n$, an argument like the one above shows that if g is in $I_{b/n}$, then $\theta^{-1}(g)$ is in I_1; that is, $\theta^{-1}(I_{b/n}) \subseteq I_1$. So, $I_{b/n} = \theta(\theta^{-1}(I_{b/n})) \subseteq \theta(I_1) = I_1$; so $b/n \geq 1$. Therefore $b = n$, and we have shown $\theta(I_n) = I_n$ for every positive integer n. Hence, $\theta^{-1}(I_n) = I_n$ also, and it follows (cf. (2.8)) that for each n, there is an automorphism θ_n

of $L^1[0,n)$ determined by $\theta_n R_n = R_n\theta$. If $n > m$ and f is in $L^1[0,m)$,

$$(4.4)\qquad \begin{aligned}\theta_m(f) &= (\theta_m R_m S_m)(f) = (R_m\theta S_m)(f)\\ &= (R_{m,n}R_n\theta S_m)(f) = (R_{m,n}\theta_n R_n S_m)(f)\,.\end{aligned}$$

Now we use again the isomorphism of each $L^1[0,n)$ with the Volterra algebra: by the results of [11], for each n, there are a complex number λ_n and a quasinilpotent derivation D_n on $L^1[0,n)$ such that $\theta_n = \exp(\lambda_n X)\exp(D_n)$. The derivation D_n is determined by a measure μ_n on $[0,n)$, satisfying $D_n f = Xf * \mu_n$ for f in $L^1[0,n)$, and the quasinilpotence of D_n is equivalent to $\mu_n(\{0\}) = 0$ [15, Remark 1]. Moreover, λ_n and D_n, hence also μ_n, are uniquely determined by θ_n [15, Lemma 13]. For $n > m$, let $D_{m,n}$ be the derivation defined on $L^1[0,m)$ by $D_{m,n}(f) = Xf * R_{m,n}\mu_n$. Observe that for f in $L^1[0,n)$, we have $D_{m,n}R_{m,n}f = R_{m,n}D_n f$; it follows that $(D_{m,n})^k = R_{m,n}D_n^k R_n S_m$ for any non-negative integer k. This, together with (4.4), and the representation $\theta_n = \exp(\lambda_n X)\exp(D_n)$, shows that $\theta_m = \exp(\lambda_n X)\exp(D_{m,n})$. By the uniqueness of the constants λ_n, the derivations D_n, and their corresponding measures μ_m [15, Lemma 13], we now conclude that $\lambda_n = \lambda_m$ for all n and m, and $R_{m,n}\mu_n = \mu_n$ whenever $n > m$. Now let λ be the common value of the numbers λ_n, and let μ be the Radon measure on $[0,\infty)$ defined by $R_n\mu = \mu_n$ for all n. If D is the derivation on L^1_{loc} defined by $Df = Xf * \mu$, and if ψ is the automorphism $e^{\lambda X}e^D$ of L^1_{loc}, then we have $R_n\psi = R_n\theta$ for all n, and therefore $\psi = \theta$. Thus, in the case $\theta(I_1) = I_1$, we have shown that θ is representable as $e^{\lambda X}e^D$, the derivation D being of the form $Df = Xf * \mu$, with $\mu(\{0\}) = 0$.

Now suppose $\theta(I_1) = I_a$, where a is any positive number. It is easy to check that the dilation φ_a satisfies $\varphi_a(I_a) = I_1$; thus $(\varphi_a\theta)(I_1) = I_1$. By the first part of the proof, $\varphi_a\theta = e^{\lambda X}e^D$ for some complex number λ, and some derivation D determined by a measure μ with $\mu(\{0\}) = 0$. Therefore $\theta = \varphi_{1/a}e^{\lambda X}e^D$.

Finally, suppose that some automorphism θ has two representations in the form (4.3); i.e., for some numbers a, b, λ, ϱ, and some derivations D and Δ, each determined by a measure with no mass at the origin, we have $\varphi_a e^{\lambda X}e^D = \varphi_b e^{\varrho X}e^{\Delta}$. Then $e^{\lambda X}e^D = \varphi_{1/a}\varphi_b e^{\varrho X}e^{\Delta}$. Therefore $I_1 = e^{\lambda X}e^D(I_1) = \varphi_{1/a}\varphi_b e^{\varrho X}e^{\Delta}(I_1) = \varphi_{1/a}\varphi_b(I_1) = I_{a/b}$. Therefore $a/b = 1$, or $a = b$. So $e^{\lambda X}e^D = e^{\varrho X}e^{\Delta}$. It follows that, for each a in $(0,\infty)$, $e^{\lambda X}\exp(D_a) = e^{\varrho X}\exp(\Delta_a)$. By the uniqueness of the representation of an automorphism of the Volterra algebra in the form $e^{\lambda X}e^q$, with q a quasinilpotent derivation [15, Lemma 13], we have $\lambda = \varrho$, and $D_a = \Delta_a$ for all a in $(0,\infty)$. But then $D = \Delta$, as required.

Corollary (4.5). *An automorphism θ of L^1_{loc} satisfies $\theta(I_1) = I_1$ if and only if $\theta = e^{\lambda X}e^D$, where λ is a complex number and D is a derivation determined by a measure μ with $\mu(\{0\}) = 0$. In this case, we have $\theta(I_a) = I_a$ for all a in $[0,\infty)$.*

Suppose θ is an automorphism of L^1_{loc} of the special form $e^{\lambda X}e^D$, where λ is a complex number and D is a derivation determined by a measure μ with $\mu(\{0\}) = 0$. The operator X is itself a derivation (determined by the Dirac measure, δ_0), and so is $\lambda X + D$ (determined by $\lambda\delta_0 + \mu$). Thus, it is natural to ask whether the automorphism $e^{\lambda X}e^D$ is representable in the form e^{Δ}, where Δ is some derivation on L^1_{loc}. In general, no such derivation Δ exists. This follows from an example in [15, Theorem 16], where it is shown that there are a derivation q on $L^1[0,1)$, determined by a measure μ on $[0,1)$ with $\mu(\{0\}) = 0$, and a complex number λ such that the automorphism $e^{\lambda X}e^q$ of $L^1[0,1)$ is not representable in the form e^Q, for any derivation Q on $L^1[0,1)$. If we let D be the derivation on L^1_{loc} defined by $D(f) = Xf * S_1(\mu)$, then $D_1 = q$, and it follows that $e^{\lambda X}e^D$ is not representable as e^{Δ}, for any derivation Δ on L^1_{loc}.

Now we consider the group structure, both algebraic and topological, of the group $\text{Aut}(L^1_{\text{loc}})$ of the automorphisms of L^1_{loc}. First, we note that every automorphism θ of L^1_{loc} extends to an automorphism $\overline{\theta}$ of M_{loc}. To see this, one first observes that, for given μ in M_{loc}, the map $f \to \theta(\mu * \theta^{-1}(f))$ of L^1_{loc} into itself is a multiplier, so by Theorem (2.14)(b), there is a measure $\overline{\theta}(\mu)$ in M_{loc} such that $\overline{\theta}(\mu) * f = \theta(\mu * \theta^{-1}(f))$ for all f in L^1_{loc}. That *defines* $\overline{\theta}(\mu)$; it is straightforward to check that the map $\overline{\theta}$ is an endomorphism of M_{loc}, and that $(\theta^{-1})^-$ inverts $\overline{\theta}$. Further routine calculations show that $(\theta\psi)^- = \overline{\theta}\,\overline{\psi}$ for any θ and ψ in $\text{Aut}(L^1_{\text{loc}})$, so that the map $\theta \to \overline{\theta}$ is a monomorphism from $\text{Aut}(L^1_{\text{loc}})$ into $\text{Aut}(M_{\text{loc}})$. Also, one may show that for any derivation D on L^1_{loc}, extended as at the beginning of Section 3 to a derivation $\overline{D}$ on M_{loc}, we have $(e^D)^- = e^{\overline{D}}$.

Since M_{loc} is identified with the multiplier algebra of L^1_{loc}, there is a natural notion of strong operator convergence in M_{loc}, and a corresponding notion of strong continuity for operators on M_{loc}. We make the following definitions.

Definition (4.6). A net $\{\mu_i\}$ of measures in M_{loc} *converges strongly* to a measure μ if $\mu_i * f \to \mu * f$ in L^1_{loc}, for every f in L^1_{loc}; write $\mu_i \to \mu$ (S) to signify strong convergence. Then, we say an operator T on M_{loc} is *strongly continuous* if $T(\mu_i) \to T(\mu)$ (S) whenever $\mu_i \to \mu$ (S).

We have the following lemmas, whose proofs are routine.

Lemma (4.7). *Let T be either a derivation or an automorphism of L^1_{loc}, and let $\overline{T}$ be its extension to M_{loc}. Then $\overline{T}$ is strongly continuous.*

For the next lemma, write $e_n = n\chi_{[0,1/n)}$ for $n = 1, 2, \ldots$, and note that, for each a in $(0,\infty)$, $R_a e_n$ is a bounded approximate identity for $L^1[0,a)$.

LEMMA (4.8). *Let T be a strongly continuous operator on M_{loc}. Then*

$$\sup\{P_a(T(\delta_x)) : x \in [0,\infty)\} \leq \sup\{P_a(T(\delta_x * e_n)) : n = 1, 2, \dots, \ x \in [0,\infty)\}$$

for any $a \in (0,\infty)$.

COROLLARY (4.9). *Let θ_i be a net of automorphisms of L^1_{loc}, converging in the topology τ_b to an automorphism θ. Then for each a in $(0,\infty)$, $\lim_i \sup\{P_a((\overline{\theta}_i - \overline{\theta})(\delta_x)) : x \in [0,\infty)\} = 0$.*

Proof. The topology τ_b was introduced in Section 2. Since $\{\delta_x * e_n : n = 1, 2, \dots, \ x \in [0,\infty)\}$ is a bounded set in L^1_{loc}, and since $\theta_i \to \theta$ in τ_b, we have $\lim_i \sup\{P_a((\theta_i - \theta)(\delta_x * e_n)) : n = 1, 2, \dots, \ x \in [0,\infty)\} = 0$. The corollary now follows from Lemmas (4.7) and (4.8).

For the next result, we use the representation of automorphisms given in Theorem (4.2). We note that, in view of remarks about extensions $\overline{\theta}$ made before Definition (4.6), if $\theta = \varphi_a e^{\lambda X} e^D$ as in (4.3), then $\overline{\theta} = \overline{\varphi}_a e^{\lambda X} e^{\overline{D}}$. We shall also use the facts that, for any $a > 0$ and any $x \geq 0$, $\overline{\varphi}_a(\delta_x) = \delta_{x/a}$, and for any measure μ, $\alpha(\overline{\varphi}_a(\mu)) = \alpha(\mu)/a$.

LEMMA (4.10). *Suppose $\theta_i = \varphi_{a(i)} e^{\lambda(i)X} e^{D(i)}$ is a net of automorphisms, converging in the topology τ_b to an automorphism $\theta = \varphi_a e^{\lambda X} e^D$. Then there is an index j such that $a(i) = a$ for all i beyond j in the directed system of the net θ_i.*

Proof. Using the representation (4.3) for θ and θ_i, and the remarks preceding the lemma, we can compute $(\overline{\theta}_i - \overline{\theta})(\delta_x) = (e^{\lambda(i)x}\delta_{x/a(i)} + \nu(i)) - (e^{\lambda x}\delta_{x/a} + \nu)$, where $\nu(i)$ (respectively ν) is a measure in M_{loc} with $\alpha(\nu(i)) \geq x/a(i)$ and $\nu(i)(\{x/a(i)\}) = 0$ (respectively, $\alpha(\nu) \geq x/a$ and $\nu(\{x/a\}) = 0$). If $a > a(i)$, then the above expression implies $P_b((\overline{\theta}_i - \overline{\theta})(\delta_x)) \geq |e^{\lambda x}|$ for any $b > x/a$, and therefore $\sup\{P_b((\overline{\theta}_i - \overline{\theta})(\delta_x)) : x \in [0,\infty)\} \geq 1$, for any $b > 0$. Similarly, if $a < a(i)$, we get $P_b((\overline{\theta}_i - \overline{\theta})(\delta_x)) \geq |e^{\lambda(i)x}|$ if $b > x/a > x/a(i)$, so again $\sup\{P_b((\overline{\theta}_i - \overline{\theta})(\delta_x)) : x \in [0,\infty)\} \geq 1$, if $b > 0$. However, by Corollary (4.9), $\theta_i \to \theta$ in the topology τ_b implies $\lim_i \sup\{P_b((\overline{\theta}_i - \overline{\theta})(\delta_x)) : x \in [0,\infty)\} = 0$ for all $b > 0$. Therefore, $a(i) = a$ eventually.

THEOREM (4.11). *Let $H = \{\varphi_a : a \in (0,\infty)\}$ and let $N = \{\theta : \theta \in \text{Aut}(L^1_{\text{loc}})$ and $\theta(I_1) = I_1\}$.*

(a) *H is a subgroup of $\text{Aut}(L^1_{\text{loc}})$, isomorphic with the multiplicative group of positive reals, and discrete with the topology τ_b.*

(b) *N is a normal subgroup of $\text{Aut}(L^1_{\text{loc}})$, and is a connected topological group with τ_b.*

(c) *$\text{Aut}(L^1_{\text{loc}})$ is a topological group with τ_b, and is the semidirect product of H and N.*

Proof. (a) It is easy to show that H is algebraically isomorphic with the multiplicative group of positive reals. Suppose $\varphi_{a(i)}$ is a net of dilations, converging in τ_b to an automorphism $\varphi_a e^{\lambda X} e^D$. By Lemma (4.10), $a(i) = a$ eventually, which shows that H is discrete in τ_b.

(b) It is immediate from the definition that N is a subgroup of $\text{Aut}(L^1_{\text{loc}})$. Let ψ be any element of N. By Corollary (4.5), $\psi(I_c) = I_c$ for all $c > 0$. Let $\theta = \varphi_a e^{\lambda X} e^D$ be any automorphism of L^1_{loc}, represented as in (4.3). Using Corollary (4.5) again, and the fact that $\varphi_a(I_b) = I_{b/a}$ for any positive a and b, we have $(\theta^{-1}\psi\theta)(I_1) = (\theta^{-1}\psi)(I_{1/a}) = \theta^{-1}(I_{1/a}) = I_1$; that is, $\theta^{-1}\psi\theta \in N$.

Now we consider the topological structure of N. By Corollary (4.5), $N \subseteq B_1$, so each $\theta \in N$ determines automorphisms θ_a of $L^1([0,a))$ by $R_a\theta = \theta_a R_a$, and by Lemma (2.12), the topology τ_b on B_1 is determined by seminorms $q_{a,1}$, where $q_{a,1}(\theta)$ is the operator norm in $B_{a,a}$ of θ_a. Now, the maps $\theta \to \theta_a$ are homomorphisms of automorphism groups. Since, for each $a > 0$, $\text{Aut}(L^1([0,a)))$ is a topological group with respect to the operator norm in $B_{a,a}$ it follows that multiplication and inversion are continuous in N for the topology τ_b.

Finally, to see that N is connected, simply note that for any $\theta = e^{\lambda X} e^D$ in N, the map $\psi(t) = e^{\lambda t X} e^{tD}$ maps $[0,1]$ continuously into N, with $\psi(0)$ the identity and $\psi(1) = \theta$. The continuity of ψ follows from the fact that, for each $a > 0$, $\psi_a(t) = e^{\lambda t X} \exp(tD_a)$ is a continuous map into $B_{a,a}$ with respect to the operator norm.

(c) First we show that $\text{Aut}(L^1_{\text{loc}})$ is (algebraically) the semidirect product of H and N. By Theorem (4.2) and Corollary (4.5), every automorphism θ of L^1_{loc} is (uniquely) a product $\varphi_a\psi$ for some $a > 0$ and some $\psi = e^{\lambda X} e^D$ in N. If $\theta \in H \cap N$, then $\theta = \varphi_a$ for some $a > 0$, so that $\theta(I_1) = I_{1/a}$. But also, $\theta \in N$, so $\theta(I_1) = I_1$, so $a = 1$, and $\theta = \varphi_1$ is the identity.

Next, we show that inversion is continuous in $\text{Aut}(L^1_{\text{loc}})$. Let $\theta_i = \varphi_{a(i)}\psi_i$ ($a(i) > 0$, $\psi_i \in N$) be a net of automorphisms, converging in τ_b to $\theta = \varphi_a\psi$. By Lemma (4.10), $a(i) = a$ eventually, so we can assume $a(i) = a$ for all i. Now, for any $b > 0$, any $x > 0$, and any F in L^1_{loc}, we have $P_b(\varphi_x F) = P_{xb}(F)$, and thus, for any bounded set B in L^1_{loc}, $P_{b,B}(\psi_i - \psi) = P_{b,B}[\varphi_{1/a}(\theta_i - \theta)] = P_{b/a,B}(\theta_i - \theta) \to 0$. Therefore, $\psi_i \to \psi$ in N for τ_b, and then part (b) implies $\psi_i^{-1} \to \psi^{-1}$ for τ_b. From that, we conclude that for any $b > 0$ and any bounded set B in L^1_{loc}, $P_{b,B}(\theta_i^{-1} - \theta^{-1}) = P_{b,B}[(\psi_i^{-1} - \psi^{-1})\varphi_{1/a}] = P_{b,C}(\psi_i^{-1} - \psi^{-1}) \to 0$, where we have written $C = \varphi_{1/a}(B)$, a bounded set, since $\varphi_{1/a}$ is a continuous map.

Finally, we show the continuity of multiplication in $\mathrm{Aut}(L^1_{\mathrm{loc}})$. First we note that arguments used in proving the continuity of inversion show that if θ_i is any net of automorphisms converging in τ_b to an automorphism θ, and if $x > 0$, then $\varphi_x\theta_i \to \varphi_x\theta$ and $\theta_i\varphi_x \to \theta\varphi_x$. Now let $\varphi_{x(i)}\theta_i$ and $\varphi_{y(i)}\psi_i$ be nets converging, respectively, to $\varphi_x\theta$ and $\varphi_y\psi$; we are assuming θ_i, θ, ψ_i and ψ all belong to N. Using Lemma (4.10), we can assume $x(i) = x$ and $y(i) = y$ for all i. We have $\theta_i = \varphi_{1/x}\varphi_x\theta_i \to \theta$ and, similarly, $\psi_i \to \psi$ in N, by the remarks above. Therefore, $\varphi_x\theta_i\varphi_{1/x} \to \varphi_x\theta\varphi_{1/x}$, and $\varphi_{xy}\psi_i\varphi_{1/xy} \to \varphi_{xy}\psi\varphi_{1/xy}$, again using the remarks above. Now, all the automorphisms $\varphi_x\theta_i\varphi_{1/x}$, etc., belong to N, and N is a topological group with τ_b, by part (b). Therefore,

$$(\varphi_x\theta_i)(\varphi_y\psi_i) = (\varphi_x\theta_i\varphi_{1/x})(\varphi_{xy}\psi_i\varphi_{1/xy})\varphi_{xy}$$
$$\to (\varphi_x\theta\varphi_{1/x})(\varphi_{xy}\psi\varphi_{1/xy})\varphi_{xy} = (\varphi_x\theta)(\varphi_y\psi),$$

where we have used the continuity of multiplication in N, and the fact that multiplication by the fixed dilation φ_{xy} is a continuous operation.

We remark here that the automorphism group of each $L^1(0,a)$ is connected [11]. The theorem above shows that this property does not transfer to the inductive limit of the algebra $L^1(0,a)$.

Finally, we consider the relationship between automorphisms of L^1_{loc}, and those of the weighted convolution subalgebras $L^1(w)$. In order to state the following lemma, we recall that, for a radical weight w on $[0,\infty)$, D is a derivation on $L^1(w)$ if and only if there is a measure μ in M_{loc} satisfying (3.3) such that $Df = Xf * \mu$ for all f in $L^1(w)$. The derivation D is then bounded on $L^1(w)$, and $E(D) = \sum_{n=0}^{\infty} D^n/n!$ converges in $B(L^1(w))$ and defines an automorphism of $L^1(w)$.

LEMMA (4.12). *Suppose that a derivation $Df = Xf * \mu$ of L^1_{loc} restricts to a derivation of $L^1(w)$, for some radical weight w. Then the exponential $E(D)$ of D, defined by the exponential series in $B(L^1(w))$, agrees with the restriction to $L^1(w)$ of the automorphism e^D of L^1_{loc} defined by* (2.13). *Thus, we may use the notation e^D without ambiguity.*

Proof. We have $(R_aD)(f) = R_a(Xf * \mu) = XR_af * R_a\mu$, for any f in L^1_{loc}, and in particular, for any f in $L^1(w)$. Since R_a defines an epimorphism from $L^1(w)$ onto $L^1[0,a)$, the derivation D_a on $L^1[0,a)$ determined by $D_aR_a = R_aD$ is the same whether we regard D as a derivation on $L^1(w)$ or on L^1_{loc}. It is easy to see that $T \to R_aT$ defines a continuous linear map from $B(L^1(w))$ to $B(L^1(w), L^1[0,a))$. Thus, $R_aE(D) = R_a\sum_{n=0}^{\infty} D^n/n! = \sum_{n=0}^{\infty} R_aD^n/n! = \sum_{n=0}^{\infty}(D_a)^nR_a/n! = \exp(D_a)R_a = R_ae^D$, where we have used (2.13) to obtain the last equality. Since a in $(0,\infty)$ is arbitrary, this shows that $E(D)$ agrees with e^D on $L^1(w)$.

THEOREM (4.13). (a) *For every automorphism θ of L^1_{loc}, there are radical weights w_1 and w_2 such that θ restricts to an isomorphism of $L^1(w_1)$ onto $L^1(w_2)$.*

(b) *If ψ is an automorphism of $L^1(w)$ for some radical weight w, then ψ extends to an automorphism θ of L^1_{loc}.*

(c) *An automorphism θ of L^1_{loc} extends an automorphism of some $L^1(w)$ if and only if $\theta = e^{icX}e^D$ for some real number c and some derivation D determined by a measure μ with $\mu(\{0\}) = 0$.*

Proof. (a) By Theorem (4.2), $\theta = \varphi_ae^{\lambda X}e^D$, for some a in $(0,\infty)$, some complex number λ, and some derivation D determined by a measure μ with $\mu(\{0\}) = 0$. By Theorem (3.4), there is a radical weight w_1 such that D restricts to a derivation on $L^1(w_1)$. By Lemma (4.12), e^D restricts to an automorphism of $L^1(w_1)$. Since, for any weight w, $e^{\lambda X}$ gives an isometric isomorphism from $L^1(w)$ onto $L^1(e^{-\mathrm{Re}(\lambda)X}w)$, and φ_a gives an isometric isomorphism from $L^1(w)$ onto $L^1(d_aw)$, where $(d_aw)(x) = w(ax)$, the given automorphism θ restricts to an isomorphism of $L^1(w_1)$ onto $L^1(d_a(e^{-\mathrm{Re}(\lambda)X}w_1))$.

(b) Suppose ψ is an automorphism of $L^1(w)$, for some radical weight w. Write $I_a(w)$ for the standard ideals in $L^1(w)$. By [9, Corollary 3], $\psi(I_a(w)) \subseteq I_a(w)$, for every a in $(0,\infty)$. Since $R_a(L^1(w)) = L^1[0,a)$ for all a in $(0,\infty)$, there are homomorphisms ψ_a on $L^1[0,a)$ determined by $R_a\psi = \psi_aR_a$. As in the case of automorphisms of L^1_{loc}, it is easily checked that $(\psi^{-1})_a$ inverts ψ_a, so the maps ψ_a are automorphisms of $L^1[0,a)$. It is also easily checked that $R_{a,b}\psi_b = \psi_aR_{a,b}$ whenever $a < b$, and since L^1_{loc} is the projective limit of the algebras $L^1[0,a)$, it follows that there is an automorphism θ of L^1_{loc} determined by $R_a\theta = \psi_aR_a$ for all a in $(0,\infty)$. If f belongs to $L^1(w)$, then, for any a in $(0,\infty)$, $(R_a\theta)(f) = (\psi_aR_a)(f) = (R_a\psi)(f)$; so $\theta(f) = \psi(f)$; so θ extends ψ, as required.

(c) Let θ be an automorphism of L^1_{loc}, and suppose θ extends an automorphism ψ of $L^1(w)$, for some w. By Theorem (4.2), $\theta = \varphi_ae^{\lambda X}e^D$, for some a in $(0,\infty)$, some complex λ, and some derivation D determined by a measure μ with $\mu(\{0\}) = 0$. We have $\theta(I_b(w)) = \psi(I_b(w)) \subseteq I_b(w)$ for all b in $(0,\infty)$, by [9, Corollary 3]. Since $I_b(w)$ is dense in I_b, and θ is continuous (Proposition (4.1)), $\theta(I_b) \subseteq I_b$ for all b in $(0,\infty)$. By Corollary (4.5), $a = 1$, so $\theta = e^{\lambda X}e^D$.

The automorphism ψ of $L^1(w)$ extends to an automorphism $\overline{\psi}$ of $M(w)$, where $\overline{\psi}$ can be defined using the equation $\overline{\psi}(\mu) * f = \psi(\mu * \psi^{-1}(f))$ for μ in $M(w)$ and f in $L^1(w)$; see [9, Proposition 1]. Similarly, since the multiplier algebra of L^1_{loc} is M_{loc} (Theorem (2.14)), θ extends to an automorphism $\overline{\theta}$ of M_{loc}, where $\overline{\theta}(\mu) * f = \theta(\mu * \theta^{-1}f)$ for f in L^1_{loc} and μ in M_{loc}. Since θ extends ψ from $L^1(w)$ to L^1_{loc}, it is easy to check that $\overline{\theta}$ extends $\overline{\psi}$ from $M(w)$ to M_{loc}; i.e., $\overline{\theta}(\mu) = \overline{\psi}(\mu)$ if μ is in $M(w)$. Certainly, every point mass δ_x belongs to $M(w)$, and by [10, Proof of Theorem 2.1] there is a real number c, and for each x in $[0,\infty)$, a measure μ_x in $M(w)$ with $\alpha(\mu_x) \geq x$ and $\mu_x(\{x\}) = 0$, such that $\overline{\psi}(\delta_x) = e^{icx}\delta_x + \mu_x$, and therefore

$$\overline{\theta}(\delta_x) = \overline{\psi}(\delta_x) = e^{icx}\delta_x + \mu_x \quad \text{for every } x \text{ in } [0,\infty)\,. \tag{4.14}$$

We also have $\theta = e^{\lambda X}e^D$, where λ is a complex number, and D is a derivation on L^1_{loc}, determined by a measure μ with $\mu(\{0\}) = 0$. By Theorem (3.4) and Lemma (4.12), there is a weight w' such that D restricts to a derivation, and therefore e^D restricts to an automorphism, of $L^1(w')$. Arguing as in the preceding paragraph, we find that for each x in $[0,\infty)$, there is a measure v_x with $\alpha(v_x) \geq x$ and $v_x(\{x\}) = 0$, such that $(e^D)^-(\delta_x) = \delta_x + v_x$, and therefore

$$\overline{\theta}(\delta_x) = (e^{\lambda X}e^D)^-(\delta_x) = e^{\lambda X}\delta_x + e^{\lambda X}v_x \tag{4.15}$$

for each x in $[0,\infty)$. Comparing (4.14) and (4.15), we conclude that $\lambda = ic$, and therefore $\theta = e^{icX}e^D$.

Conversely, suppose we are given θ of the form $e^{icX}e^D$, with c real and the derivation D determined by a measure μ with $\mu(\{0\}) = 0$. As above, by Theorem (3.4) and Lemma (4.12), there is a weight w' such that D restricts to a derivation, and e^D restricts to an automorphism, of $L^1(w')$. Since e^{icX} restricts to an automorphism of $L^1(w)$ for any weight w, we conclude that $\theta = e^{icX}e^D$ restricts to an automorphism of $L^1(w')$. Thus, (c) is proved.

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DEPARTMENT OF MATHEMATICS AND ASTRONOMY
UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
CANADA R3T 2N2

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