Automorphisms and derivations of a Fréchet algebra of locally integrable functions

by

F. GHAHRAMANI † and J. P. McCLURE ‡ (Winnipeg, Man.)

Abstract. We find representations for the automorphisms, derivations and multipliers of the Fréchet algebra $L_{loc}^{1}$ of locally integrable functions on the half-line $\mathbb{R}^+$. We show, among other things, that every automorphism $\theta$ of $L_{loc}^{1}$ is of the form $\theta = \varphi_{a}e^{|X|}e^{D}$, where $D$ is a derivation, $X$ is the operator of multiplication by coordinate, $\lambda$ is a complex number, $a > 0$, and $\varphi_{a}$ is the dilation operator $(\varphi_{a}f)(x) = a f(ax)$ ($f \in L_{loc}^{1}$, $x \in \mathbb{R}^+$). It is also shown that the automorphism group is a topological group with the topology of uniform convergence on bounded sets and is the semidirect product of a connected subgroup and a discrete group which is isomorphic to the discrete group of real numbers.

1. Introduction. Weighted convolution algebras on the half-line $\mathbb{R}^+ = [0, \infty)$ were identified as examples of commutative Banach algebras early in the development of the subject [7]. More recently, in connection with the development of automatic continuity theory, radical weighted convolution algebras on $[0, \infty)$ have received a lot of attention [2]. These algebras are all subalgebras of the algebra of locally integrable functions on $[0, \infty)$, which is the main object of study in this paper. We denote this algebra by $L_{loc}^{1}(\mathbb{R}^+)$, or just $L_{loc}^{1}$. As a vector space, $L_{loc}^{1}$ consists of all (equivalence classes, with respect to equality almost everywhere, of) Lebesgue measurable, complex-valued functions $f$ on $[0, \infty)$, for which

$$P_{a}(f) = \int_{0}^{a} |f(x)| \, dx$$

is finite, for all positive real numbers $a$; addition and scalar multiplication are defined pointwise. $L_{loc}^{1}$ becomes a commutative, associative algebra, if

1991 Mathematics Subject Classification: Primary 46J45; Secondary 43A20, 43A22.

† Supported by NSERC grant OGP0038440.
‡ Supported by NSERC grant A8060.
we take the convolution product

\[(f * g)(x) = \int_0^x f(x - y)g(y) \, dy \quad (x \in [0, \infty)) .\]

In the next section, we shall see that the functionals \(P_a\) of (1.1) give \(L^1_{loc}\) a topology in which it becomes a Fréchet algebra. We shall show that the closed ideals of \(L^1_{loc}\) are just the standard ideals \(I_a\): for \(a > 0\),

\[I_a = \{f \in L^1_{loc} \mid f(x) = 0 \text{ a.e. on } [0, a]\}\]

and we also put \(I_0 = L^1_{loc}\) and \(I_\infty = \{0\}\). We deduce that \(L^1_{loc}\) is a radical Fréchet algebra. We shall also give some preliminary remarks about the continuity of linear maps on \(L^1_{loc}\), and about a topology on the algebra of continuous linear maps on \(L^1_{loc}\), which will be relevant to the later sections of the paper. In those sections, we explore, in turn, the multipliers, the derivations, and the automorphisms of \(L^1_{loc}\). It turns out that these maps are all continuous, and have representations analogous to the corresponding operators on Banach convolution algebras on \([0, \infty)\). In particular, we obtain a representation of automorphisms as products of dilatation automorphisms and exponentials of derivations. This leads to a description of the automorphism group of \(L^1_{loc}\) as the semidirect product of a discrete subgroup isomorphic with \(\mathbb{R}\), with a connected subgroup. We also consider the relationship between automorphisms of \(L^1_{loc}\) and automorphisms of its weighted (Banach) subalgebras. Automorphisms of Fréchet (and Banach) algebras of power series were studied in [13].

2. Preliminaries. It is easy to check, and well known, that each of the functionals \(P_a\), defined in (1.1), is a seminorm on \(L^1_{loc}\) and is submultiplicative with respect to the product (1.2). The null space of \(P_a\) is exactly the ideal \(I_a\) defined in (1.3). The quotient algebra \(L^1_{loc}/I_a\) is isomorphic with the algebra \(L^1[0, a]\) of (equivalence classes of) Lebesgue integrable functions on \([0, a]\). \(L^1[0, a]\) is a radical Banach algebra, normed by \(P_a\) and with product given by (1.2), restricted to \(x\) in \([0, a]\). As Banach algebras, the algebras \(L^1[0, a]\) are all isomorphic with the Volterra algebra \(V = L^1[0, 1]\), which has been studied by a number of authors [10], [11], and [15]. We shall identify \(L^1_{loc}/I_a\) with \(L^1[0, a]\); the quotient map is then represented by the restriction

\[R_a f = f|_{[0, a]} \quad (f \in L^1_{loc}) .\]

When \(0 < a < b < \infty\), it will also be convenient to have a notation for the restriction (or quotient) map from \(L^1[0, b]\) to \(L^1[0, a]\); we put

\[R_{a,b} f = f|_{[0, a]} \quad (f \in L^1[0, b]) .\]

Note that \(R_{a,b}R_a = R_a\). In fact, the system of algebras \(L^1[0, a]\), with the homomorphisms \(R_{a,b}\) (when \(0 < a < b < \infty\)) is a projective system of commutative (radical) Banach algebras and continuous homomorphisms, and \(L^1_{loc}\) is (isomorphic with) the projective limit of this system. The seminorms \(\{P_a : a \in [0, \infty)\}\) clearly form a separating family on \(L^1_{loc}\), and the countable subset \(\{P_n : n \in \mathbb{N}\}\) is cofinal in \(\{P_a\}\). Thus, the following result is an immediate consequence of the work of Michael [16].

**Proposition (2.3).** With the topology determined by the seminorms \(\{P_a : 0 < a < \infty\}\), \(L^1_{loc}\) is a commutative Fréchet algebra.

Sometimes, it will be convenient to think of functions in \(L^1[0, a]\) as defined on \([0, \infty)\). For \(a > 0\), and \(f \in L^1[0, a]\), we define \(S_a f\) in \(L^1_{loc}\) as follows:

\[(S_a f)(x) = \begin{cases} f(x) & \text{if } x < a, \\ 0 & \text{if } x \geq a . \end{cases}\]

It is easy to see that \(S_a\) is a linear map on \(L^1[0, a]\) into \(L^1_{loc}\). The image \(S_a(L^1[0, a])\) is a closed subspace of \(L^1_{loc}\), but not a subalgebra. Note that \(R_aS_a\) is the identity map on \(L^1[0, a]\), while \(S_aR_a\) is the projection of \(L^1_{loc}\) onto its subspace of functions vanishing on \([a, \infty)\). It may be worth emphasizing that we are using the same symbol \(P_a\) for a seminorm on \(L^1_{loc}\) and for the norm on \(L^1[0, a]\). Thus, for \(f \in L^1_{loc}\) we have \(P_a(f) = P_a(R_a f)\), while for \(f \in L^1[0, a]\), we have \(P_a(f) = (S_aR_a f)\). Recall that, for \(f \in L^1_{loc}\), \(\alpha(f)\) is the infimum of the support of \(f\). Conventionally, we take \(\alpha(0) = \infty\). With this notation, for \(a \geq 0\), we have \(I_a = \{f : \alpha(f) \geq a\}\), where \(I_a\) was defined in (1.3). As null spaces of the seminorms \(P_a\) defining the topology of \(L^1_{loc}\), the ideals \(I_a\) are closed in \(L^1_{loc}\). These are called standard ideals, and we now show that every closed ideal is standard.

**Proposition (2.5).** (a) Every closed ideal in \(L^1_{loc}\) is a standard ideal.

(b) \(L^1_{loc}\) is a radical algebra.

**Proof.** (a) It is sufficient to show that every principal closed ideal is standard. We shall show that, for each \(f \in L^1_{loc}\), \((f \ast L^1_{loc})^- = I_a(f)\). A function \(g\) belongs to \((f \ast L^1_{loc})^-\) if and only if, for each \(a > 0\), and every \(x \geq 0\), there is \(h \in L^1_{loc}\) such that \(P_a(g - f * h) < \epsilon\). Since \(P_a(g - f * h) = P_a(R_a h - (R_a f) \ast (R_a h))\), it follows that \(g\) belongs to \((f \ast L^1_{loc})^-\) if and only if, for each \(a > 0\), \(R_a h\) belongs to \((R_a f \ast L^1[0, a])^-\). But, the closed ideals in the Volterra-type algebras \(L^1[0, a]\) are known to be standard [6]; thus, \(g\) belongs to \((f \ast L^1_{loc})^-\) if and only if, for each \(a > 0\), we have \(\alpha(R_a g) \geq \alpha(R_a f)\), and the latter condition is clearly equivalent to \(\alpha(g) \geq \alpha(f)\). Thus \((f \ast L^1_{loc})^- = I_a(f)\).

(b) From (a), there is no character on \(L^1_{loc}\) (i.e., no nonzero, continuous, multiplicative linear functional). So, Corollary 5.6 of [16] implies that \(L^1_{loc}\) is a radical algebra.
Next, we recall some basic information about continuous linear maps on $L^1_{loc}$ into itself. First, a linear map $T$ on $L^1_{loc}$ into itself is continuous if and only if for each $a > 0$, there are $b > 0$ and $K > 0$ such that

(2.6) \[ P_a(Tf) \leq K P_b(f) \]

for all $f$ in $L^1_{loc}$. It follows from (2.6) that

(2.7) \[ T(I_b) \subseteq I_a, \]

and that there is a unique, bounded linear map $T_{a,b}$ in $B_{a,b} \equiv B(L^1[0,b), L^1[0,a])$ satisfying

(2.8) \[ R_a T = T_{a,b} R_b. \]

We remark in passing that, for continuous $T$, (2.7), and (2.8) with bounded $T_{a,b}$, are equivalent. The operator norm $\|T_{a,b}\|$ of $T_{a,b}$ in $B_{a,b}$ is exactly the infimum of the values of $K$ for which (2.6) holds. Note that we also have, for any $f$ in $L^1_{loc}$,

(2.9) \[ (R_a T)(f) = (R_a (T b R_b) f) = (T (S_b R_b) f) = (T f) S_b R_b, \]

that is, $T f$ on $[0,a]$ is determined by $f$ on $[0,b)$.

Write $B(L^1_{loc})$ for the algebra of all continuous linear operators on $L^1_{loc}$.

The topology $\tau_0$ of uniform convergence on bounded sets is defined on $B(L^1_{loc})$ by the seminorms $P_{a,b}$, where, for $a > 0$ and $b$ a bounded subset of $L^1_{loc}$, $P_{a,b}(T) = \sup \{P_a(Tf) : f \in B\}$, for each $T$ in $B(L^1_{loc})$. This (or, an equivalent) definition is given in [17, p. 68] and in [19, p. 337]. More information about topologies on spaces of continuous operators may also be found in these sources; however, we shall need only the definition, and we shall use the topology only on certain subspaces of $B(L^1_{loc})$, related to the automorphism group.

For each $r > 0$, we write $B_r$ for the set of all linear operators on $L^1_{loc}$ such that, for each $a > 0$, (2.6) holds with $b = ar$, for some $K$. It is easy to see that $B_r$ is a linear subspace of $B(L^1_{loc})$, and it is not the trivial subspace: the dilation operator $\varphi_r$ defined by

(2.10) \[ (\varphi_r f)(x) = rf(\alpha x) \]

for $f$ in $L^1_{loc}$ and $x$ in $[0,\infty)$, is easily seen to belong to $B_r$. Another easy verification shows that each $\varphi_r$ is, in fact, an automorphism of $L^1_{loc}$; the inverse operator to $\varphi_r$ is $\varphi_{1/r}$. Now, each $T$ in $B_r$ determines, and is determined by, a set of bounded linear maps $\{T_{a,ar} : a \in (0,\infty)\}$, where $T_{a,ar}$ in $B_{a,ar}$, satisfies (2.8) with $b = ar$. Of course, $T_{a,ar}$ depends linearly on $T$, so for each $a$ in $(0,\infty)$, we can define a seminorm $q_{a,r}$ on $B_r$ by

(2.11) \[ q_{a,r}(T) = \|T_{a,ar}\| \]

for $T$ in $B_r$, where the norm symbol means the operator norm on $B_{a,ar}$. The following lemma will be used in our study of the automorphism group of $L^1_{loc}$.

**Lemma (2.12).** If $\{a(n) : n \in \mathbb{N}\}$ is any sequence of real numbers tending to infinity, then the topology $\tau_0$ on $B_r$ is determined by the seminorms $q_{a(n),r}$. In particular, $\tau_0$ is metrizable on $B_r$.

**Proof.** Take any sequence $a(n)$ such that $a(n) \to \infty$ as $n \to \infty$. Then, for any $a$ in $(0,\infty)$ and any bounded set $B$ in $L^1_{loc}$, we can pick and fix $n$ so that $a(n) \geq a$, and we can take a constant $K$ such that $P_{a(n)}(f) \leq K$ if $f$ is in $B$. Let $B(n) = \{f \in L^1_{loc} : P_{a(n)}(f) \leq K$ and $f$ vanishes on $[a(n),\infty]\}$. Then $B(n)$ is a bounded set in $L^1_{loc}$, and $(S_{a(n)} R_{a(n)}(B(n)) \subseteq B(n)$. Using (2.9), we have, for any $T$ in $B_r$,

(2.9) \[ P_{a(n),b}(T) \leq P_{a(n),b}(B(n)) \leq K \|T_{a(n),r}\| K q_{a(n),r}(T). \]

This calculation shows that, on $B_r$, each seminorm $P_{a,n}$ is dominated by a multiple of a seminorm $q_{a(n),r}$ and also that each $q_{a(n),r}$ is a $P_{a(n),b}$ (take $K = 1$ and $B = B(n)$). Thus, on $B_r$, $\tau_0$ is determined by the countably many seminorms $q_{a(n),r}$.

It is an easy consequence of the definition of $B_r$ that $B_r \subseteq B_s$ whenever $r \leq s$; thus, when $r \leq s$, the seminorms $q_{a,s}$ are defined on $B_r$, and can be used to determine $\tau_0$ on $B_r$.

The subspace $B_1$ will be of particular interest. Note that an operator $T$ in $B(L_{loc})$ belongs to $B_1$ if and only if $T(I_b) \subseteq I_a$ for all $a$ in $(0,\infty)$; it follows easily that $B_1$ is a subalgebra of $B(L_{loc})$. For $T$ in $B_1$ (and only for this case), we write $T_a$ for the operator $T_{a,a}$ in $B_{a,a}$ determined by (2.8), for each $a$ in $(0,\infty)$. Since each $B_{a,a}$ is a Banach algebra, we may define $T_a$ in $B_{a,a}$. If $a \leq b$, we have $T_b R_a = T_a R_b$ and $R_a T = R_b T R_a$. It follows that $T_a R_b = R_b T_b$, and from that we get $exp(T_a) R_b = R_b exp(T_a)$.

**Theorem (2.13).** For $T$ in $B_1$, we define $exp(T) \equiv e^T$ on $L^1_{loc}$ by requiring $R_a exp(T) = exp(T_a) R_a$ for all $a$ in $(0,\infty)$.

For $T$ in $B_1$, the operator $e^T$ is continuous (since each $exp(T_a)$ is), and belongs to $B_1$.

Finally, in this preliminary section, we mention the algebra $M_{loc}$ of all Radon measures on $(0,\infty)$. $M_{loc}$ is a Fréchet algebra, the topology being determined by the seminorms

$$ P_\alpha(\mu) = |\mu|([0,a]) $$

for $a > 0$, and the product being defined (indirectly) by

$$ \int f(t) d(\mu \ast \nu)(t) = \int \int f(x + y) d \mu(x) d \nu(y) $$

for $f$ in $C_c$ and $\mu, \nu$ in $M_{loc}$. 

From the point of view of projective limits, we note that, by the Riesz representation theorem, the dual of \( C_b[0, a] \) is the space \( M[0, a] \) of bounded Borel measures on \([0, a]\). It is well known that, with the convolution product (restricted to \([0, a]\)), \( M[0, a] \) is (isomorphic with) the multiplier algebra of \( L^1[0, a] \) [15, Remark 10]. If, for \( b > a \), we again write \( R_{a, b} \) for the restriction map from \( M[0, b] \) to \( M[0, a] \), then the maps \( R_{a, b} \) are epimorphisms of convolution algebras, and with these epimorphisms, the algebras \( M[0, a] \) form a projective system, whose projective limit is \( M_{loc} \). As with \( R_{a, b} \), we will use the same symbol \( R_{a, b} \) already used for \( L^1_{loc} \) to denote the restriction (or quotient) mapping from \( M_{0c} \) to \( M[0, a] \), for each \( a > 0 \).

The algebra \( L^1_{loc} \) can be identified with a closed ideal in \( M_{loc} \), by identifying a locally integrable function \( f \) with the absolutely continuous measure \( f(x) \, dx \). In fact, with this identification, \( L^1_{loc} \) becomes a closed ideal in \( M_{loc} \), and we see that not every closed ideal in \( M_{loc} \) is (the analogue of) a standard ideal. We write \( \delta_x \) for the point mass at \( x \) (a point of \([0, \infty)\)). These point masses form a (multiplicative) semigroup in \( M_{loc} \), and \( \delta_0 \) is the multiplicative identity in \( M_{loc} \).

Next we give a description of the multipliers of \( L^1_{loc} \). Recall that a linear map \( T \) on \( L^1_{loc} \) is a multiplier if \( Tf = \mu \ast f \) for every \( f \) in \( L^1_{loc} \). In particular, every multiplier on \( L^1_{loc} \) is continuous.

**Theorem (2.14).** (a) For each \( \mu \) in \( M_{loc} \), the mapping \( T_\mu f = \mu \ast f \) is a continuous multiplier on \( L^1_{loc} \).

(b) If \( T \) is any multiplier on \( L^1_{loc} \), then there is a measure \( \mu \) in \( M_{loc} \) such that \( Tf = \mu \ast f \) for all \( f \) in \( L^1_{loc} \). In particular, every multiplier on \( L^1_{loc} \) is continuous.

**Proof.** (a) That \( T_\mu \) is a multiplier is implicit in the fact, mentioned in the previous section, that \( L^1_{loc} \) is an ideal in \( M_{loc} \). The continuity of \( T_\mu \) is shown by the calculation

\[
P_\mu (f \ast g) = P_\mu (R_{a, b} \ast R_a f) \leq P_\mu (R_{a, b}) P_\mu (R_a f) = P_\mu (R_{a, b}) P_\mu (f).
\]

(b) Let \( g \) be any element of \( L^1_{loc} \) with \( \alpha(g) = 0 \). Then, for any \( f \) in \( L^1_{loc} \), \( T g = f \ast T g \), so the Titchmarsh convolution theorem implies \( \alpha(T f) = \alpha(T f) + \alpha(g) = \alpha(f) \ast \alpha(g) \geq \alpha(f) \). It follows that \( T(\delta_x) \leq \delta_x \) for all \( x > a \), and thus there are unique maps \( T_\mu \) on \( L^1[0, a] \) satisfying \( T_a R_a = R_a T \) for all \( a > 0 \). It is routine to check that \( T_\mu \) is a multiplier on \( L^1[0, a] ; \) thus, there is a measure \( \mu_a \) in \( M[0, a] \) such that \( T_\mu f = \mu_a \ast f \) for all \( f \) in \( L^1[0, a] \) [15, Remark 10].

3. **Derivations.** In this section, we describe the derivations on \( L^1_{loc} \). Let \( D \) be such a derivation. Since \( M_{loc} \) is the multiplier algebra of \( L^1_{0c} \), we can extend \( D \) to a derivation on \( M_{0c} \), as follows. Given \( \mu \) in \( M_{loc} \), we define a mapping \( D_\mu \) on \( L^1_{loc} \) by \( D_\mu (f) = \mu \ast D(f) \). It is easy to check that \( D_\mu \) is a derivation on \( L^1_{loc} \). Thus, by Theorem (2.14), there is a measure \( \mu_\mu \) in \( M_{loc} \), which we denote by \( \Delta(\mu) \), which satisfies \( D(\mu) = \Delta(\mu) \ast f \) for all \( f \) in \( L^1_{loc} \). Further routine calculations show that the map \( \Delta \) thus defined on \( M_{loc} \) is a derivation, and that \( \Delta L^1_{loc} = D \).

Next let \( X \) denote the mapping on \( M_{loc} \) (or \( L^1_{loc} \)) of multiplication by the coordinate function: for \( x \) in \( M_{loc} \), \( d(x) = t \delta_0 \) (or, for \( f \) in \( L^1_{loc} \), \( x f(t) = t f(t) \)). Then it is a result of H. G. Diamond [4, 5] that \( \Delta \) is a derivation on \( M_{loc} \). If \( \mu \) is a measure \( \mu \) in \( M_{loc} \), then \( \Delta(\mu) = (\mu \ast 4) \ast \mu \). If \( \mu \) is a derivation, then \( \Delta D = D \) is a derivation on \( M_{loc} \). In particular, every derivation \( \Delta \) is continuous.

**Theorem (3.1).** \( D \) is a derivation on \( L^1_{loc} \) if and only if there is a measure \( \mu \) in \( M_{loc} \) such that \( D f = X f \ast \mu \) for all \( f \) in \( L^1_{loc} \). In particular, every derivation \( D \) is continuous.

**Proof.** If \( \mu \) is given, it is easy to check that \( D f = X f \ast \mu \) defines a derivation. On the other hand, if \( D \) is a given derivation, the existence of \( \mu \) satisfying \( D f = X f \ast \mu \) was established in the discussion preceding the
statement of the theorem. The continuity follows from the inequality

\[ P_a(xf + \mu) \leq aP_a(\mu)P_a(f) \]

so the theorem is proved.

In a later section, it will be convenient to know, for a given derivation \( D \) on \( L^1(w) \), whether there is a weight \( w(x) \) such that \( D \) restricts to a derivation on \( L^1(w) \). It is shown in [8] that, for a given weight \( w(x) \) and measure \( \mu \), \( DF = Xf \ast \mu \) defines a derivation on \( L^1(w) \) if and only if

\[ \sup \left\{ \frac{x}{w(x)} \int w(x + y) d\mu[y] : x > 0 \right\} < \infty. \]

We first note that if \( \mu(\{0\}) \neq 0 \), then (3.3) cannot hold for any weight \( w(x) \), since, for all \( x \) in \((0, \infty)\),

\[ \frac{x}{w(x)} \int w(x + y) d\mu[y] \geq x|\mu(\{0\})|. \]

It turns out that this is the only barrier to the existence of a suitable weight, as shown in the following theorem. Recall that a weight \( w(x) \) is called \textit{star-shaped} if the function \( \eta(x) = -\log w(x) \) has the property that \( \eta(x)/x \) is a non-decreasing function of \( x \) in \([0, \infty)\).

\textbf{Theorem (3.4).} Let \( \mu \) be a Radon measure such that \( \mu(\{0\}) = 0 \). Then there is a continuous, star-shaped weight \( w(x) \) such that \( DF = Xf \ast \mu \) defines a derivation on \( L^1(w) \).

\textbf{Proof.} We are going to obtain weights \( w(x) \) by first constructing a function \( \eta(x) \), and then putting \( w(x) = e^{-\eta(x)} \). To construct \( \eta(x) \), we shall first choose a sequence \( \{A_n : n = 0, 1, \ldots\} \) such that \( \{A_n\} \) is positive and increasing, and then we shall take sequences \( \{a_n\} \) and \( \{b_n\} \) so that, by putting

\[ \eta(x) = a_n x + b_n \quad \text{for} \quad x \in [n, n+1), \quad n = 0, 1, \ldots, \]

we obtain \( \eta(x) \) continuous on \([0, \infty)\), and satisfying

\[ \eta(n) = nA_n \quad \text{for} \quad n = 0, 1, \ldots. \]

In fact, these conditions determine \( \{a_n\} \) and \( \{b_n\} \), once \( \{A_n\} \) has been chosen. For, we must have \( nA_n = na_n + b_n \), and \( (n+1)a_n + b_n = (n+1)A_{n+1} \), whence

\[ a_n = (n+1)A_{n+1} - nA_n. \]

Since \( \{A_n\} \) is increasing, (3.7) gives

\[ a_n \geq A_n. \]

Then, since \( b_n = n(A_n - a_n) \), we have \( b_n \leq 0 \). From (3.5), we have \( \eta(x)/x = a_n + b_n/x \), for \( x \) in \([n, n+1)\), \( n = 0, 1, \ldots \) (put \( \eta(0)/0 = \infty \)). Since \( b_n \leq 0 \), this shows that \( \eta(x)/x \) is increasing on \([n, n+1)\) for each \( n \); since \( \eta(x)/x \) is also continuous, we have \( \eta(x)/x \) increasing on \([0, \infty)\), so \( w(x) = e^{-\eta(x)} \) will be a star-shaped weight. Also, \( \eta(x) \) continuous implies that \( w(x) \) is continuous.

Now, suppose \( \mu \) is a Radon measure on \([0, \infty)\) satisfying \( \mu(\{0\}) = 0 \), and first suppose also that \( \text{supp}(\mu) \subseteq [0, 1) \). We are trying to find a weight so that (3.3) holds, so we may suppose that \( \mu \) is a positive measure. Since \( \mu \) has bounded support, \( \mu \) is a finite measure. Since \( \mu([0, \epsilon)) \to 0 \) as \( \epsilon \to 0^+ \), and 

\[ e^{-\epsilon w} \to 0 \quad \text{as} \quad A \to \infty, \quad \text{uniformly on} \quad [\epsilon, \infty), \]

for any \( \epsilon > 0 \), we may, by taking each \( A_n \) sufficiently large, guarantee

\[ \int \exp(-A_n y) d\mu(y) < \frac{1}{n+1} \quad \text{for} \quad n = 0, 1, \ldots. \]

Also, because of (3.7), by successively choosing the numbers \( A_n \) sufficiently large, we can ensure that the sequence \( \{a_n\} \) is increasing. Now suppose \( x \in [n, n+1) \), and \( y \in [n+1 - x, 1) \), so that \( x + y \in [n + 1, n+2) \). Then we have

\[ \eta(x + y) - \eta(x) = \eta(x + y) - \eta(n+1) + \eta(n+1) - \eta(x) \]

\[ = a_{n+1}(x + y - (n+1)) + a_n(n+1 - x) \]

\[ = (a_{n+1} - a_n)(x + y - (n+1)) + a_n y \geq a_n y, \]

where we have used the continuity of \( \eta \), (3.5), and the fact that \( \{a_n\} \) is increasing. Also, for \( y \in [0, n+1 - x] \), so that \( x + y \in [n, n+1) \), we have \( \eta(x + y) - \eta(x) = a_n y \), by (3.5). Therefore, we have, for \( x \in [n, n+1) \),

\[ \frac{x}{w(x)} \int w(x + y) d\mu(y) \leq (n+1) \int \exp(-a_n y) d\mu(y) \]

\[ \leq (n+1) \int \exp(-A_n y) d\mu(y) < 1, \]

using \( w(t) = \exp(-\eta(t)), \) (3.5), (3.10), (3.8) and (3.9). Since \( n \) was arbitrary, (3.8) holds, as required.

Next, suppose \( \text{supp}(\mu) \subseteq [1, \infty) \). As before, we can suppose that \( \mu \) is positive. Again, by successively choosing \( \{A_n\} \) sufficiently large, we ensure that \( \{a_n\} \), defined by (3.7), is increasing; thus, again, (3.10) holds whenever \( x \in [n, n+1) \) and \( y \in [n+1 - x, 1] \); in this case, we shall only use (3.10) in the case \( y = 1 \). As well, we shall, by choosing the numbers \( A_n \) sufficiently large, guarantee

\[ a_n > \log(n+1) \quad \text{for} \quad n = 0, 1, \ldots. \]
and (recall $w(n) = \exp(-nA_n)$)

\begin{equation}
\sum_{j=1}^{\infty} w(j-1)\mu([j, j+1]) < \infty.
\end{equation}

Now suppose $n \geq 0$ and $x \in [n, n+1)$. Then

\begin{equation}
\frac{x}{w(x)} \int w(x+y) d\mu(y) = \frac{x w(x+1)}{w(x)} \int_{[1, \infty)} \frac{w(y)}{w+1} d\mu(y)
\end{equation}

\begin{equation}
\leq (n+1)\exp[-(\eta(x+1) - \eta(x))]
\int w(y-1) d\mu(y)
\end{equation}

\begin{equation}
\leq (n+1)\exp[-a_n \sum_{j=1}^{\infty} w(y-1) d\mu(y)]
\end{equation}

\begin{equation}
< \sum_{j=1}^{\infty} w(j-1)\mu([j, j+1]),
\end{equation}

where we have used (3.10) (with $y = 1$), (3.11), submultiplicativity of $w(t)$, and the fact that $w(x)$ is decreasing (since $\eta(x)$ is increasing). Since $n$ was arbitrary, (3.12) and (3.13) now imply that (3.3) holds.

Finally, consider an arbitrary Radon measure $\mu$ with $\mu(\{0\}) = 0$. Again, we can suppose that $\mu$ is positive. Put $\mu_1(\{0\}) = \mu_0(\{0\}) \leq 0, \mu_0(\{0\}) \leq 1$, and $\mu(\{1\}) \leq 1$. By the arguments above, there are continuous, star-shaped weights $w_0$ and $w_1$ such that

\begin{equation}
sup_{q \geq 0} \frac{x}{w_q(x)} \int w_q(x+y) d\mu_q(y) < \infty,
\end{equation}

for $q = 0, 1$. We can suppose $w_q(x) \leq w_q(0) = 1$. Now put $w(x) = w_0(x)w_1(x)$. Then it is straightforward to verify that $w(x)$ is continuous, star-shaped and satisfies (3.3).

4. Automorphisms. In this section, we study the automorphisms of $L^r_{\text{loc}}$. Our first result is simple, and answers the natural question whether automorphisms of $L^r_{\text{loc}}$ are continuous.

**Proposition (4.1).** Each automorphism of $L^r_{\text{loc}}$ is continuous.

**Proof.** Let $\theta$ be an automorphism of $L^r_{\text{loc}}$. Since $L^r_{\text{loc}}$ is the projective limit of the algebras $L^r[0, a)$, it is enough to show that $R_a \theta$ is continuous for each $a > 0$. But $L^r_{\text{loc}}$ is a Fréchet algebra, $L^r[0, a)$ is (isomorphic with) the Volterra algebra, and $R_a \theta$ is an epimorphism; so the continuity of $R_a \theta$ follows from an extension of [14, Remark 3a] to Fréchet algebras [3].

Next, we review some particular automorphisms of $L^r_{\text{loc}}$.

Let $D$ be a derivation on $L^r_{\text{loc}}$. It follows from Theorem (3.1) (see (3.2)) that $D$ is in $B_1$. Thus, there are bounded linear maps $D_a$ on $L^r[0, a]$ defined by $R_aD = D_aR_a$, and there is a linear map $e^D$ in $B_1$ determined by $\rho_0e^D = \exp(D_a)R_a$ for all $a$ in $(0, \infty)$. By (2.13), we show that each $D_a$ is a derivation on $L^r[0, a]$.

Thus, $D_a$ is the derivation determined on $L^r[0, a]$ by the measure $R_a\mu$. It follows that $\exp(D_a)$ is an automorphism of $L^r[0, a]$ for each $a$ in $(0, \infty)$. Thus, using $e^D = \exp(D_a)R_a$, it is easy to see that $e^D$ is a homomorphism on $L^r_{\text{loc}}$. Since $e^{-D}$ is, similarly, a homomorphism, and is easily seen to invert $e^D$, the latter is an automorphism of $L^r_{\text{loc}}$. Note the special case where $\mu = \lambda \delta_0$, a multiple of the Dirac measure; i.e., $DF = \lambda FX$. In this case, $e^D$ is exactly multiplication by the function $t \rightarrow e^{\lambda t}$, and we shall write $e^{\lambda X}$ for this automorphism.

Recall that, in Section 2, we noted that for each $a > 0$, there is a dilation automorphism $\varphi_a$ on $L^r_{\text{loc}}$, where $\varphi_a(f)(t) = af(at)$. In fact, these dilations, together with the automorphisms $e^D$, determine all automorphisms of $L^r_{\text{loc}}$.

**Theorem (4.2).** Let $\theta$ be an automorphism of $L^r_{\text{loc}}$. Then there are a positive real number $a$, a complex number $\lambda$, and a derivation $D$, defined by $DF = X \theta \mu$ with the Radon measure $\mu$ satisfying $\mu(\{0\}) = 0$, such that

\begin{equation}
\theta = \varphi_a e^{\lambda X} e^D.
\end{equation}

Conversely, any map $\theta$ of the form (4.3), with $a$, $\lambda$, and $D$ as described, is an automorphism. Finally, for a given automorphism $\theta$, the numbers $a$ and $\lambda$, and the derivation $D$ which satisfy (4.3) are unique.

**Proof.** That a map $\theta$ of the form (4.3) is an automorphism was shown in the discussion preceding the statement of the theorem. To prove the non-trivial part, let $\theta$ be an automorphism. By Propositions (4.1) and (2.5), the image by $\theta$ of each standard ideal is a standard ideal.

First consider the special case when we have $\theta(I_1) = I_1$. Let $n$ be a positive integer, and suppose $\theta(I_n) = I_n$. Take $f$ in $I_1$ such that $\alpha(\theta(f)) = 1$. We have $f^n$ in $I_n$ by assumption, so $\theta(f^n)$ is in $I_n$. Also, by the Itô-McKean convolution theorem, $\alpha(\theta(f^n)) = n\alpha(\theta(f)) = n$; therefore $n \geq 1$. Next, since $\theta^{-1}$ is also an automorphism and $\theta^{-1}(I_1) = I_n$, an argument like the one above shows that if $g$ is in $I_{1/n}$, then $\theta^{-1}(g)$ is in $I_1$; that is, $\theta^{-1}(I_{1/n}) \subseteq I_1$. So, $I_{1/n} = \theta^{-1}(I_{1/n}) \subseteq \theta(I_1) = I_1$; therefore $b/n \geq 1$. Therefore $b = n$, and we have shown $\theta(I_n) = I_n$ for every positive integer $n$. Hence, $\theta^{-1}(I_n) = I_n$ also, and it follows (cf. (2.8)) that for each $n$, there is an automorphism $\theta_n$. 

of $L^1[0,n]$ determined by $\theta_n \mu = \mu_n \lambda$. If $n > m$ and $f$ is in $L^1[0,m]$,
\begin{equation}
\theta_n(f) = (\theta_m \mu_n \lambda)(f) = (\mu_n \lambda \theta_m)(f) = (\mu_n \lambda \theta_m)(f) = (\mu_n \lambda \theta_m)(f).
\end{equation}

Now we use again the isomorphism of each $L^1[0,n]$ with the Volterra algebra: by the results of [11], for each $n$, there is a complex number $\lambda_m$ and a quasinilpotent derivation $D_n$ on $L^1[0,n]$ such that $\mu_n = \exp(\lambda_n X) \exp(D_n)$. The derivation $D_n$ is determined by a measure $\mu_n$ on $[0,n]$, satisfying $D_n f = X f + \mu_n$, for $f$ in $L^1[0,n]$, and the quasinilpotence of $D_n$ is equivalent to $\mu_n(\{0\}) = 0$. Therefore, $\lambda_n$ and $D_n$ are uniquely determined by $\mu_n$ [15, Lemma 13]. For $n > m$, let $D_m,n$ be the derivation defined on $L^1[0,m]$ by $D_m,n = D_m, \mu_n(n) = \mu_n$. Observe that for $f$ in $L^1[0,n]$, we have $D_m,n \mu_n \lambda_m \theta_n = \mu_n \lambda_m \theta_n$; it follows that $(D_m,n)^k = R_m, R_m D_m,n \theta_n$ for any non-negative integer $k$. This, together with (4.4), and the representation $\mu_n = \exp(\lambda_n X) \exp(D_n)$, shows that $\theta_n = \exp(\lambda_n X) \exp(D_n)$, by the uniqueness of the constants $\lambda_n$, the derivations $D_n$, and their corresponding measures $\mu_n$ [15, Lemma 13], we now conclude that $\lambda_n = \lambda_m$ for all $n$ and $m$, and $R_m, \mu_n \lambda_m = \mu_n$ whenever $n > m$. Now let $\lambda$ be the common value of the numbers $\lambda_n$, and let $\mu$ be the Radon measure on $[0,\infty]$ defined by $R_m, \mu = \mu_n$ for all $n$. If $D$ is the derivation on $L^1_{\text{loc}}$ defined by $D f = X f + \mu$, and if $\psi$ is the automorphism $e^{X \psi} e^{D}$ of $L^1_{\text{loc}}$, then we have $R_m, \psi = \lambda_m$ for all $n$, and therefore $\psi = \psi$. Thus, in the case $\theta(1) = 1$, we have shown that $\theta$ is representable as $e^{X \psi} e^{D}$, the derivation $D$ being of the form $D f = X f + \psi$, with $\psi(\{0\}) = 0$.

Now suppose $\theta(1) = 1$, where $\theta$ is a measurable function. It is easy to check that the dilation $\phi : \theta \mapsto \phi \theta$ satisfies $I_\phi(1) = 1_G$, thus $\phi(\psi) \theta(1) = 1_G$. By the first part of the proof, $\theta(0) = e^{X \theta} e^{D}$ for some complex number $\lambda$, and some derivation $D$ determined by a measure $\mu$ with $\mu(\{0\}) = 0$. Therefore $\theta = \phi \theta \mu = e^{X \theta} e^{D}$.

Finally, suppose that some automorphism $\theta$ has two representations in the form (4.3); i.e., for some numbers $a$, $b$, $\lambda$, $\psi$, and some derivations $D$ and $\Delta$, each determined by a measure with no mass at the origin, we have $\phi \theta e^{X \theta} e^{D} = \phi \theta e^{X \psi} e^{D}$. Then $e^{X \psi} e^{D} = \phi \theta e^{X \psi} e^{D}$. Therefore $I_\phi = e^{X \psi} e^{D} = \phi \theta e^{X \psi} e^{D}$. Therefore $\lambda = 0$, and $\theta = \phi$. So $e^{X \theta} e^{D} = e^{X \theta} e^{D}$. It follows that, for each $a$ in $(0,\infty)$, $e^{X \theta} e^{D} = e^{X \theta} e^{D}$. By the uniqueness of the representation of an automorphism of the Volterra algebra in the form $e^{X \theta} e^{D}$, with $\theta$ a quasinilpotent derivation [15, Lemma 13], we have $\lambda = \phi$, and $D = \Delta$, for all $a$ in $(0,\infty)$. But then $D = \Delta$, as required.

Corollary (4.5). An automorphism $\theta$ of $L^1_{\text{loc}}$ satisfies $\theta(1) = 1_G$ if and only if $\theta = e^{X \theta} e^{D}$, where $\lambda$ is a complex number and $D$ is a derivation determined by a measure $\mu$ with $\mu(\{0\}) = 0$. In this case, we have $\theta(I_\mu = I_\mu$ for all $a$ in $(0,\infty)$.

Suppose $\theta$ is an automorphism of $L^1_{\text{loc}}$ of the special form $e^{X \theta} e^{D}$, where $\lambda$ is a complex number and $D$ is a derivation determined by a measure $\mu$ with $\mu(\{0\}) = 0$. The operator $X$ is itself a derivation (determined by the Dirac measure, $\delta_0$), and so is $\lambda X + D$ (determined by $\lambda \delta_0 + \mu$). Thus, it is natural to ask whether the automorphism $e^{X \lambda + D}$ is representable in the form $e^{\theta}$, where $\theta$ is some derivation on $L^1_{\text{loc}}$. In general, no such derivation $\theta$ exists. This follows from an example in [15, Theorem 16], where it is shown that there is a derivation $q$ on $L^1[0,1]$, determined by a measure $\mu$ on $[0,1]$ with $\mu(\{1\}) = 0$, and a complex number $\lambda$ such that the automorphism $e^{X \lambda + q}$ of $L^1[0,1]$ is not representable in the form $e^{\theta}$, for any derivation $q$ on $L^1[0,1]$.

If we let $D$ be the derivation on $L^1_{\text{loc}}$ defined by $D(f) = X f + \theta(1)\mu$, then $D = q$, and it follows that $e^{X \lambda + q}$ is not representable as $e^{\theta}$, for any derivation $\theta$ on $L^1_{\text{loc}}$.

Now we consider the group structure, both algebraic and topological, of the group $\text{Aut}(L^1_{\text{loc}})$ of the automorphisms of $L^1_{\text{loc}}$. First, we note that every automorphism $\theta$ of $L^1_{\text{loc}}$ extends to an automorphism $\theta$ of $M^1_{\text{loc}}$. To see this, first observe that, for given $\mu$ in $M^1_{\text{loc}}$, the map $f \mapsto \theta(\mu * \theta^{-1}(f))$ of $L^1_{\text{loc}}$ into itself is a multiplier, so by Theorem (2.14)(b), there is a measure $\theta(\mu)$ in $M^1_{\text{loc}}$ such that $\theta(\mu) * f = \theta(\mu * \theta^{-1}(f))$ for all $f$ in $L^1_{\text{loc}}$. That defines $\theta(\mu)$; it is straightforward to check that the map $\theta$ is an endomorphism of $M^1_{\text{loc}}$, and that $(\theta^{-1})^{-1}$ inverts $\theta$. Further routine calculations show that $(\theta(\mu))^{-1} = \theta^{-1}(\mu)$ for any $\theta$ and $\mu$ in $M^1_{\text{loc}}$, so that the map $\theta \mapsto \theta$ is a monomorphism from $\text{Aut}(L^1_{\text{loc}})$ into $\text{Aut}(M^1_{\text{loc}})$. Also, one may show that for any derivation $D$ on $L^1_{\text{loc}}$, extended as at the beginning of Section 3 to a derivation $\overline{D}$ on $M^1_{\text{loc}}$, we have $(\overline{D})^{-1} = e^{\overline{D}}$.

Since $M^1_{\text{loc}}$ is identified with the multiplier algebra of $L^1_{\text{loc}}$, there is a natural notion of strong operator convergence in $M^1_{\text{loc}}$, and a corresponding notion of strong continuity for operators on $M^1_{\text{loc}}$. We make the following definitions.

Definition (4.6). A net $\{\mu_n\}$ of measures in $\text{Aut}(L^1_{\text{loc}})$ converges strongly to a measure $\mu$ if $\mu_n * f \to \mu * f$ in $L^1_{\text{loc}}$, for every $f$ in $L^1_{\text{loc}}$, write $\mu_n \to \mu$ (S) to signify strong convergence. Then, we say an operator $T$ on $M^1_{\text{loc}}$ is strongly continuous if $T(\mu_n) \to T(\mu)$ (S) whenever $\mu_n \to \mu$ (S).

We have the following lemmas, whose proofs are routine.

Lemma (4.7). Let $T$ be either a derivation or an automorphism of $L^1_{\text{loc}}$, and let $\overline{T}$ be its extension to $M^1_{\text{loc}}$. Then $\overline{T}$ is strongly continuous.

For the next lemma, write $e_n = \nu(0,1/n)$ for $n = 1, 2, \ldots$, and note that, for each $a$ in $(0,\infty)$, $R_a e_n$ is a bounded approximate identity for $L^1[0,a]$. 

```
Lemma (4.8). Let $T$ be a strongly continuous operator on $M_{loc}$. Then
\[ \sup \{ P_a(T(\delta_x)) : x \in [0, \infty) \} \leq \sup \{ P_a(T(\delta_x * \eta_n)) : n = 1, 2, \ldots, x \in [0, \infty) \} \]
for any $a \in (0, \infty)$.

Corollary (4.9). Let $\theta_1$ be a net of automorphisms of $L^1_{loc}$, converging in the topology $\tau_{\theta}$ to an automorphism $\theta$. Then for each $a \in (0, \infty)$,
\[ \lim_{n \to \infty} \sup \{ P_a((\theta_1 - \theta)(\delta_x * \eta_n)) : x \in [0, \infty) \} = 0. \]

Proof. The topology $\tau_{\theta}$ was introduced in Section 2. Since $\delta_x * \eta_n : n = 1, 2, \ldots, x \in [0, \infty)$ is a bounded set in $L^1_{loc}$, and since $\theta_1 \to \theta$ in $\tau_{\theta}$, we have
\[ \lim_{n \to \infty} \sup \{ P_a((\theta_1 - \theta)(\delta_x * \eta_n)) : x \in [0, \infty) \} = 0. \]

The corollary now follows from Lemmas (4.7) and (4.8).

For the next result, we use the representation of automorphisms given in Theorem (4.2). We note that, in view of remarks about extensions $\theta$ made before Definition (4.6), if \( \theta = \varphi_{a} e^{\lambda X} \varphi_{D} \) as in (4.3), then $\theta = \varphi_{a} e^{\lambda X} \varphi_{D}$. We shall also use the facts that, for any $a > 0$ and any $x \geq 0$, $\varphi_{a}(\delta_x) = \delta_{a/x}$, and for any measure $\mu$, $\alpha(\varphi_{a}(\mu)) = \alpha(\mu)/a$.

Lemma (4.10). Suppose $\theta_1 = \varphi_{a} e^{\lambda X} \varphi_{D}$ is a net of automorphisms, converging in the topology $\tau_{\theta}$ to an automorphism $\theta = \varphi_{a} e^{\lambda X} \varphi_{D}$. Then there is an index $i$ such that $a(i) = a$ for all $i$ beyond $j$ in the directed system of the net $\theta_1$.

Proof. Using the representation (4.3) for $\theta$ and $\theta_1$, and the remarks preceding the lemma, we can compute
\[ (\theta_1 - \theta)(\delta_x) = (e^{\lambda X} \delta_x * \eta_n + \xi) \cdot (\eta_n) \in L^1_{loc} \]
with $\alpha(\xi) \geq \alpha(x/a(i)) = \alpha(x/a(i)) = 0$, so that $\alpha(\xi) \geq \alpha(x/a(i)) = 0$. Given $x/a(i)$, take $b \geq x/a(i)$, and then $\sup \{ P_b((\theta_1 - \theta)(\delta_x)) : \delta_x \leq x \} = 0$. If $a > a(i)$, then the above expression implies $P_b((\theta_1 - \theta)(\delta_x)) \geq \varphi_{a} e^{\lambda X} \varphi_{D}$ for any $b > x/a(i)$, and therefore $\sup \{ P_b((\theta_1 - \theta)(\delta_x)) : \delta_x \leq x \} = 0$. If $a < a(i)$, we get $P_b((\theta_1 - \theta)(\delta_x)) \geq \varphi_{a} e^{\lambda X} \varphi_{D}$ for any $b > x/a(i)$, so again $\sup \{ P_b((\theta_1 - \theta)(\delta_x)) : \delta_x \leq x \} = 0$. Therefore, $\alpha(\xi) = \alpha(x/a(i)) = 0$ for all $b > 0$. However, by Corollary (4.9), $\theta_1 \to \theta$ in the topology $\tau_{\theta}$ implies
\[ \lim_{n \to \infty} \sup \{ P_b((\theta_1 - \theta)(\delta_x)) : x \in [0, \infty) \} = 0 \]
for all $b > 0$. Therefore, $\alpha(\xi) = a$.

Theorem (4.11). Let $H = \{ \varphi_{a} : a \in (0, \infty) \}$ and let $N = \{ \theta : \theta \in \text{Aut}(L^1_{loc}) \}$.

(a) $H$ is a subgroup of $\text{Aut}(L^1_{loc})$, isomorphic with the multiplicative group of positive reals, and discrete with the topology $\tau_{\theta}$.

(b) $N$ is a normal subgroup of $\text{Aut}(L^1_{loc})$, and is a connected topological group with $\tau_{\theta}$.

(c) $\text{Aut}(L^1_{loc})$ is a topological group with $\tau_{\theta}$, and is the semidirect product of $H$ and $N$.

Proof. (a) It is easy to show that $H$ is algebraically isomorphic with the multiplicative group of positive reals. Suppose $\varphi_{a(i)}(\xi) = a(i)$, which is a net of dilations, converging in $\tau_{\theta}$ to an automorphism $\varphi_{a(i)} e^{\lambda X} \varphi_{D}$. By Lemma (4.10), $a(i) = a$ eventually, which shows that $H$ is discrete in $\tau_{\theta}$.

(b) It is immediate from the definition that $N$ is a subgroup of $\text{Aut}(L^1_{loc})$. Let $\psi$ be any element of $N$. By Corollary (4.5), $\psi[I_1] = I_1$ for all $c > 0$. Let $\theta = \varphi_{a} e^{\lambda X} \varphi_{D}$ be any automorphism of $L^1_{loc}$, represented as in (4.3). Using Corollary (4.5) again, and the fact that $\varphi_{a} e^{\lambda X} \varphi_{D}$ is a group homomorphism of $\text{Aut}(L^1_{loc})$, for any positive $a$ and $b$, we have
\[ \int_{I_1} \varphi_{a} e^{\lambda X} \varphi_{D}(x) \varphi_{b} e^{\lambda X} \varphi_{D}(x) = \int_{I_1} \varphi_{a} e^{\lambda X} \varphi_{D}(x) \varphi_{b} e^{\lambda X} \varphi_{D}(x) \]
for any $a$ and $b$.

Now we consider the topological structure of $N$. For Corollary (4.5), $N \subseteq B_1$, so each $\theta \\in N$ determines automorphisms $\varphi_{a} e^{\lambda X} \varphi_{D}$ of $B_1$. By Lemma (2.12), the topology $\tau_{\theta}$ on $B_1$ is determined by semiinvariants $g_{a,1}$, where $g_{a,1}$ is the operator norm in $B_{a,0}$ of $\varphi_{a}$. Now, for any automorphism $\theta = \varphi_{a} e^{\lambda X} \varphi_{D}$, $\varphi_{a}$ is a group homomorphism of $\text{Aut}(L^1_{loc})$. Since for each $a > 0$, $\text{Aut}(L^1_{loc})$ is a topological group with respect to the operator norm in $B_{a,0}$, it follows that multiplication and inversion are continuous in $N$ for the topology $\tau_{\theta}$.

Finally, to see that $N$ is connected, simply note that for any $\theta = e^{\lambda X} \varphi_{D}$ in $N$, the map $\psi(t) = e^{\lambda X} \varphi_{D} \exp(tD_{0})$ is a continuous map into $N$, with $\psi(0)$ the identity and $\psi(1) = \theta$. The continuity of $\psi$ follows from the fact that, for each $a > 0$, $\varphi_{a}(t) = e^{\lambda X} \exp(tD_{0})$ is a continuous map into $B_{a,0}$ with respect to the operator norm.

(c) First we show that $\text{Aut}(L^1_{loc})$ is (algebraically) the semidirect product of $H$ and $N$. By Theorem (4.2) and Corollary (4.5), every automorphism $\theta$ of $L^1_{loc}$ is (uniquely) a product $\varphi_{a} e^{\lambda X} \varphi_{D}$ for some $a > 0$ and some $\varphi_{a}$ of $L^1_{loc}$, so that $\theta = \varphi_{a} e^{\lambda X} \varphi_{D}$ in $N$. If $\theta \in H \cap N$, then $\theta = \varphi_{a}$ for some $a > 0$, so that $\theta(I_1) = I_1$. But also, $\theta \in N$, so $\theta(I_1) = I_1$, so $a = 1$, and $\theta = \varphi_{1}$ is the identity.

Next, we show that inversion is continuous in $\text{Aut}(L^1_{loc})$. Let $\theta_1 = \varphi_{a} \psi$, $(a(i) > 0, \psi \in N)$ be a net of automorphisms, converging in $\tau_{\theta}$ to $\theta = \varphi_{a} \psi$. By Lemma (4.10), $a(i) = a$ eventually, so we can assume $a(i) = a$ for all $i$. Now, for any $b > 0$, any $x > 0$, and any $F$ in $L^1_{loc}$, we have
\[ P_b([\varphi_{a} \psi] F) = P_b(\varphi_{a} F) \]
and, for any bounded set $B$ in $L^1_{loc}$, $P_b[\psi(\varphi_{a} \psi(\theta_1 - \theta)) = P_b(\varphi_{a} \psi(\theta_1 - \theta)) = P_b(\theta_1(\theta_1 - \theta)) = P_b(\theta_1(\theta_1 - \theta)) = 0$. Therefore, $\psi \to \psi$ in $N$ for $\tau_{\theta}$ and then part (b) implies $\psi^{-1} \to \psi^{-1}$ for $\tau_{\theta}$. From this, we conclude that for any $b > 0$ and any bounded set $B$ in $L^1_{loc}$, $P_b(\theta_1(\theta_1 - \theta)^{-1} = P_b(\theta_1(\theta_1 - \theta)^{-1} = 0$, where we have written $C = \varphi_{a} \psi(B)$, a bounded set, since $\varphi_{a}$ is a continuous map.
Finally, we show the continuity of multiplication in $\text{Aut}(L^1_{\text{loc}})$. First we note that arguments used in proving the continuity of inversion show that if $\theta_i$ is any net of automorphisms converging in $\tau_i$ to an automorphism $\theta$, and if $x > 0$, then $\varphi_{x^i} \theta_i \rightarrow \varphi_x \theta$ and $\theta_i \varphi_x \rightarrow \varphi_x \theta_i$. Now let $\varphi_{x^i} \theta_i$ and $\varphi_{x^i} \psi_i$ be nets converging, respectively, to $\varphi_x \theta$ and $\varphi_x \psi$; we are assuming $\theta_i, \theta, \psi_i$ and $\psi$ all belong to $N$. Using Lemma (4.10), we can assume $x(t) = x$ and $y(t) = y$ for all $t$. We have $\theta_i = \varphi_{1/x^i} \varphi_x \theta_i \rightarrow \theta$ and, similarly, $\psi_i \rightarrow \psi$ in $\tau_i$, by the remarks above. Therefore, $\varphi_x \theta \varphi_{1/x^i} \varphi_x \psi_i \rightarrow \varphi_x \theta \varphi_{1/x} \varphi_x \psi$, and $\varphi_x \psi \varphi_{1/x^i} \varphi_x \psi_i \rightarrow \varphi_x \psi \varphi_{1/x} \varphi_x \psi_i$, again using the remarks above. Now, all the automorphisms $\varphi_x \theta_i \varphi_{1/x^i}$, etc., belong to $N$, and $N$ is a topological group with $\tau_i$, by part (b). Therefore,

$$(\varphi_x \theta_i)(\varphi_{1/x} \psi_i) = (\varphi_x \theta_i \varphi_{1/x^i} \varphi_x \psi_i)(\varphi_{1/x} \psi_i) \rightarrow (\varphi_x \theta \varphi_{1/x} \psi)(\varphi_{1/x} \psi) = (\varphi_x \theta \varphi_{1/x} \psi) = (\varphi_x \psi)(\varphi_x \psi),$$

where we have used the continuity of multiplication in $N$, and the fact that multiplication by the fixed dilation $\varphi_x$ is a continuous operation.

We remark here that the automorphism group of each $L^1(0, a)$ is connected [11]. The theorem above shows that this property does not transfer to the inductive limit of the algebra $L^1(0, a)$.

Finally, we consider the relationship between automorphisms of $L^1_{\text{loc}}$ and those of the weighted convolution subalgebras $L^1(w)$. In order to state the following lemma, we recall that, for a radical weight $w$ on $[0, \infty)$, $D$ is a derivation on $L^1(w)$ if and only if there is a measure $\mu$ in $M_{\text{loc}}$ satisfying (3.3) such that $DF = FXf \ast \mu$ for all $f$ in $L^1(w)$. The derivation $D$ is then bounded on $L^1(w)$, and $E(D) = \sum_{n=0}^{\infty} D^n/n!$ converges in $B(L^1(w))$ and defines an automorphism of $L^1(w)$.

**Lemma (4.12).** Suppose that a derivation $DF = Xf \ast \mu$ of $L^1_{\text{loc}}$ restricts to a derivation of $L^1(w)$, for some radical weight $w$. Then, the exponential $E(D)$ of $D$, defined by the exponential (2.13), agrees with the restriction to $L^1_{\text{loc}}$ of the automorphism $e^D$ of $L^1_{\text{loc}}$ defined by (2.13). Thus, we may use the notation $e^D$ without ambiguity.

**Proof.** We have $(R_\alpha D)(f) = R_\alpha (Xf \ast \mu) = XR_\alpha f \ast R_\alpha \mu$, for any $f$ in $L^1_{\text{loc}}$, and in particular, for any $f$ in $L^1(w)$. Since $R_\alpha$ defines an epimorphism from $L^1(w)$ onto $L^1(0, a)$, the derivation $D_\alpha$ on $L^1(0, a)$ determined by $D_\alpha R_\alpha = R_\alpha D$ is the same whether we regard $D$ as a derivation on $L^1(w)$ or on $L^1_{\text{loc}}$. It is easy to see that $T \circ R_\alpha T$ defines a continuous linear map from $B(L^1(w))$ to $B(L^1(0, a))$. Thus, $R_\alpha E(D) = R_\alpha \sum_{n=0}^{\infty} D^n/n! = \sum_{n=0}^{\infty} R_\alpha D^n/n! = \sum_{n=0}^{\infty} (D_\alpha)^n R_\alpha n! = \exp(D_\alpha R_\alpha) R_\alpha = R_\alpha e^D$, where we have used (2.13) to obtain the last equality. Since $\alpha$ in $(0, \infty)$ is arbitrary, this shows that $E(D)$ agrees with $e^D$ on $L^1(w)$.

**Theorem (4.13).** (a) For every automorphism $\psi$ of $L^1_{\text{loc}}$, there are radical weights $w_1$ and $w_2$ such that $\psi$ restricts to an isomorphism of $L^1(w_1)$ onto $L^1(w_2)$.

(b) If $\psi$ is an automorphism of $L^1(w)$ for some radical weight $w$, then $\psi$ extends to an automorphism $\psi$ of $L^1_{\text{loc}}$.

(c) An automorphism $\theta$ of $L^1_{\text{loc}}$ extends an automorphism of some $L^1(w)$ if and only if $\theta = e^{cX}e^D$ for some real number $c$ and some derivation $D$ determined by a measure $\mu$ with $\mu(\{0\}) = 0$.

**Proof.** (a) By Theorem (4.2), $\theta = e^{cX}e^D$, for some $c$ in $(0, \infty)$, some complex number $\lambda$, and some derivation $D$ determined by a measure $\mu$ with $\mu(\{0\}) = 0$. By Theorem (3.4), there is a radical weight $w$ such that $D$ restricts to a derivation on $L^1(w)$. By Lemma (4.12), $e^D$ restricts to an automorphism of $L^1(w)$. Since, for any weight $w$, $e^{cX}$ gives an isometric isomorphism from $L^1(w)$ onto $L^1(e^{-\lambda X} e^D w)$, and $e^{\lambda X}$ gives an isometric isomorphism from $L^1(w)$ onto $L^1(e^{-\lambda X} e^D w)$, where $(e^{-\lambda X} e^D)(\varphi)(x) = w(x)$, the given automorphism $\theta$ restricts to an isomorphism of $L^1(w)$ onto $L^1(e^{-\lambda X} e^D w)$.

(b) Suppose $\psi$ is an automorphism of $L^1(w)$, for some radical weight $w$. Write $I_0(w)$ for the standard ideals in $L^1(w)$. By [9, Corollary 3], $\psi(I_0(w)) \subseteq I_0(w)$, for every $a$ in $(0, \infty)$. Since $R_\alpha L^1(0, a) = L^1(0, a)$ for all $\alpha$ in $(0, \infty)$, there are homomorphisms $\psi_a$ on $L^1(0, a)$ determined by $R_\alpha \psi = \psi_a R_\alpha$. As in the case of automorphisms of $L^1_{\text{loc}}$, it is easily checked that $\psi_a$ necessarily inverts $I_0$, so the maps $\psi_a$ are automorphisms of $L^1(0, a)$. It is also easily checked that $R_\alpha \psi_a = \psi_a R_\alpha$ whenever $\alpha < b$, and since $L^1_{\text{loc}}$ is the projective limit of the algebras $L^1(0, a)$, it follows that there is an automorphism $\theta$ of $L^1_{\text{loc}}$ determined by $R_\alpha \theta = \psi_a R_\alpha$ for all $\alpha$ in $(0, \infty)$. If $f$ belongs to $L^1(0, a)$, then, for any $\alpha$ in $(0, \infty)$, $(R_\alpha \theta)(f) = (R_\alpha \psi_a)(f) = (R_\alpha \psi)(f)$; so $\theta(f) = \psi(f)$; so $\theta$ extends $\psi$, as required.

(c) Let $\theta$ be an automorphism of $L^1_{\text{loc}}$ and suppose $\theta$ extends an automorphism $\psi$ of $L^1(w)$, for some $w$. By Theorem (4.2), $\theta = e^{cX}e^D$, for some $c$ in $(0, \infty)$, some complex $\lambda$, and some derivation $D$ determined by a measure $\mu$ with $\mu(\{0\}) = 0$. We have $\theta(I_0(w)) = \psi(I_0(w)) \subseteq I_0(w)$ for all $a$ in $(0, \infty)$, by [9, Corollary 3]. Since $I_0(w)$ is dense in $I_0$, and $\psi$ is continuous (Proposition (4.11)), $\theta(I_0(w)) \subseteq I_0$ for all $a$ in $(0, \infty)$. By Corollary (4.5), $a = 1$, so $\theta = e^{X}e^D$.

The automorphism of $L^1(w)$ extends to an automorphism $\psi$ of $M(w)$, where $\psi$ can be defined using the equation $\psi(\mu)(f) = \psi(\mu \ast f)(f)$ for $\mu$ in $M(w)$ and $f$ in $L^1(w)$; see [9, Proposition 1]. Similarly, since the multiplier algebra of $L^1_{\text{loc}}$ by $M_{\text{loc}}$ (Theorem (2.14)), $\theta$ extends to an automorphism $\theta$ of $M_{\text{loc}}$, where $\theta(\mu)(f) = \theta(\mu \ast f)(f)$ for $f$ in $L^1_{\text{loc}}$ and $\mu$ in $M_{\text{loc}}$. Since $\theta$ extends $\psi$ from $L^1(w)$ to $L^1_{\text{loc}}$, it is easy to check that $\theta$ extends $\psi$ from $M(w)$ to $M_{\text{loc}}$; i.e., $\theta(\mu) = \psi(\mu)$ if $\mu$ is in $M(w)$. Certainly, every point
mass $\delta_x$ belongs to $M(w)$, and by [10, Proof of Theorem 2.1] there is a real number $c$, and for each $x$ in $[0, \infty)$, a measure $\mu_x$ in $M(w)$ with $\alpha(\mu_x) \geq x$ and $\mu_x(\{x\}) = 0$, such that $\psi_x(\delta_x) = e^{i\alpha_x} \delta_x + \mu_x$, and therefore

$$\overline{\delta}(\delta_x) = \overline{\psi_x}(\delta_x) = e^{i\alpha_x} \delta_x + \mu_x$$

for every $x$ in $[0, \infty)$.

We also have $\theta = e^{i\lambda x} e^D$, where $\lambda$ is a complex number, and $D$ is a derivation on $L^1_0$, determined by a measure $\mu$ with $\mu(\{0\}) = 0$. By Theorem (3.4) and Lemma (4.12), there is a weight $w'$ such that $D$ restricts to a derivation, and therefore $e^D$ restricts to an automorphism, of $L^1(w')$. Arguing as in the preceding paragraph, we find that for each $x$ in $[0, \infty)$, there is a measure $\mu_x$ with $\alpha(\mu_x) \geq x$ and $\mu_x(\{x\}) = 0$, such that $(e^D)^{-1}(\delta_x) = \delta_x + \mu_x$, and therefore

$$\overline{\delta}(\delta_x) = (e^D)^{-1}(\delta_x) = e^{i\lambda x} \delta_x + e^{i\lambda x} \mu_x$$

for each $x$ in $[0, \infty)$. Comparing (4.14) and (4.15), we conclude that $\lambda = ic$, and therefore $\theta = e^{i\alpha} e^D$.

Conversely, suppose we are given $\theta$ of the form $e^{i\alpha} e^D$, with $\alpha$ real and the derivation $D$ determined by a measure $\mu$ with $\mu(\{0\}) = 0$. As above, by Theorem (3.4) and Lemma (4.12), there is a weight $w'$ such that $D$ restricts to a derivation, and $e^D$ restricts to an automorphism, of $L^1(w')$. Since $e^{i\alpha}$ restricts to an automorphism of $L^1(w)$ for any weight $w$, we conclude that $\theta = e^{i\alpha} e^D$ restricts to an automorphism of $L^1(w')$. Thus, (c) is proved.

References


DEPARTMENT OF MATHEMATICS AND ASTRONOMY
UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
CANADA R3T 2N2

Received August 16, 1991