$$\int_0^t \|\exp[A(t-s)]\|\,ds \le \int_0^\infty \|\exp[As]\|\,ds \le \int_0^\infty \exp[\alpha(A)t] \sum_{k=0}^\infty \frac{t^k v(A)^k}{(k!)^{3/2}}\,dt = j \qquad (t \ge 0)\,.$$

Hence, $\max_{t\ge 0}\|x(t)\| \le a\|x(0)\| + \max_{t\ge 0}\|x(t)\| j$ and we arrive at (4.3). ∎

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On molecules and fractional integrals on spaces of homogeneous type with finite measure

by

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Abstract. In this paper we prove the continuity of fractional integrals acting on non-homogeneous function spaces defined on spaces of homogeneous type with finite measure. A definition of the molecules which are used in the H^p theory is given. Results are proved for L^p, H^p, BMO, and Lipschitz spaces.

1. Definitions and statement results. We shall follow the definitions and notation of [GV], and we assume that the reader is familiar with that paper. In the present paper (X,δ,μ) is a normal space of homogeneous type of finite measure and of order γ, $0<\gamma\le 1$. In this case the diameter of the space is finite and will be denoted by D. We may and will assume that $\mu(X)=1$.

For the sake of completeness we will repeat the definitions of normality and order. (X,δ,μ) is a *normal space* if there are positive constants A_1 and A_2 such that for all x in X

$$A_1 r \le \mu(\mathcal{B}_r(x)) \quad \text{if } 0 < r \le R_x\,, \tag{1.1}$$

$$\mu(\mathcal{B}_r(x)) \le A_2 r \quad \text{if } r > r_x\,, \tag{1.2}$$

where $\mathcal{B}_r(x)$ denotes the ball of radius r and center x, and where $R_x = \inf\{r>0 : \mathcal{B}_r(x) = X\}$, and $r_x = \sup\{r>0 : \mathcal{B}_r(x) = \{x\}\}$ if $\mu(\{x\}) \ne 0$, and $r_x = 0$ if $\mu(\{x\}) = 0$. Note that $\sup\{R_x : x\in X\} = D < \infty$, that (1.1) holds for $0<r<2D$ with constant $A_1/2$ instead of A_1, and that (1.2) holds for $r=r_x$ if $r_x\ne 0$. The space (X,δ,μ) is said to be of *order* γ, $0<\gamma\le 1$, if there exists a positive constant M such that for every x, y, and z in X,

$$|\delta(x,z)-\delta(y,z)| \le M\delta(x,y)^\gamma(\max\{\delta(x,z),\delta(y,z)\})^{1-\gamma}\,.$$

We will consider on (X,δ,μ) the following function spaces and norms. If $0<p\le\infty$ then L^p and $\|f\|_p$ have their usual meaning. For a measurable

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function f the distribution function λ_f is defined for $t > 0$ by $\lambda_f(t) = \mu(\{x : |f(x)| > t\})$, and f is said to belong to weak L^p $(= wL^p)$ for $1 \le p < \infty$ if $[f]_p^p = \sup\{t^p \lambda_f(t) : t > 0\}$ is finite. For $\beta > 0$, the space $\mathrm{Lip}[\beta]$ is defined as the set of functions in $\mathrm{Lip}(\beta)$ as defined in [GV] with the norm

$$\|\varphi\|_{\mathrm{Lip}[\beta]} = \|\varphi\|_{\mathrm{Lip}(\beta)} + \|\varphi\|_\infty . \tag{1.3}$$

The fact that $\varphi \in \mathrm{Lip}(\beta)$ is bounded is a consequence of $\mu(X) < \infty$.

On the space BMO as defined in [GV] we introduce the following norm:

$$\|\psi\|^* = \|\psi\|_* + \|\psi\|_1 . \tag{1.4}$$

The fact that $\psi \in \mathrm{BMO}$ is in L^1 follows from $\mu(X) < \infty$. The norms defined in (1.3) and (1.4) are referred to as *nonhomogeneous norms*, and the corresponding spaces as *nonhomogeneous function spaces.*

The H^p spaces are defined in terms of atoms. The function identically one is a p-atom for any p, $(1+\gamma)^{-1} < p \le 1$. A measurable function $a \ne 1$ is called a *p-atom*, $(1+\gamma)^{-1} < p \le 1$, if a is bounded with support in a ball $\mathcal{B}$ and such that

$$\|a\|_\infty \le \mu(\mathcal{B})^{-1/p} , \qquad \int_X a \, d\mu = 0 . \tag{1.5}$$

The space H^1 is defined as follows. A measurable function f on X belongs to H^1 if it is in L^1 and if there exist a sequence $\{a_i\}$ of 1-atoms and a sequence of numbers $\{\lambda_i\}$ with $\sum_{i=1}^\infty |\lambda_i| < \infty$ such that

$$f = \sum_{i=1}^\infty \lambda_i a_i , \tag{1.6}$$

convergence being in L^1. The H^1 norm of f is defined by

$$\|f\|_{H^1} = \inf \sum_{i=1}^\infty |\lambda_i| ,$$

where the infimum is taken over all possible representations of the form (1.6).

Now we define the space H^p for $(1+\gamma)^{-1} < p < 1$. If $\varphi \in \mathrm{Lip}[1/p-1]$ and a is a p-atom, then

$$\langle a, \varphi\rangle = \int_X a\varphi \, d\mu$$

defines a bounded linear functional on $\mathrm{Lip}[1/p-1]$. If $\{a_i\}$ is a sequence of p-atoms and $\{\lambda_i\}$ a sequence of numbers such that $\sum_{i=1}^\infty |\lambda_i|^p < \infty$ then the series $\sum_{i=1}^\infty \lambda_i \langle a, \varphi\rangle$ converges absolutely for every φ in $\mathrm{Lip}[1/p-1]$ and

defines a bounded linear functional f on $\mathrm{Lip}[1/p-1]$, and

$$\langle f, \varphi\rangle = \sum_{i=1}^\infty \lambda_i \langle a_i, \varphi\rangle . \tag{1.7}$$

The space H^p, $(1+\gamma)^{-1} < p < 1$, is defined as the space of continuous linear functionals on $\mathrm{Lip}[1/p-1]$ which can be represented in the form (1.7) with $\sum_{i=1}^\infty |\lambda_i|^p < \infty$. One defines $\|f\|_{H^p}$ by

$$\|f\|_{H^p} = \inf \Big(\sum_{i=1}^\infty |\lambda_i|^p\Big)^{1/p} , \tag{1.8}$$

where the infimum is taken over all representations of the form (1.7). The expression (1.8) is not a norm but its pth power defines a metric relative to which H^p is complete.

We shall now define the molecules that are needed in this paper. Other definitions of molecules were given in [CW], [MS1], and [TW]. We first consider the case $s < 1$.

Let $\varepsilon > 0$ and $(1+\gamma)^{-1} < s < 1$. A measurable function M on X is an *s-molecule* if there exist a point x_0 in X and constants $L > 0$ and $r > 0$ such that

$$|M(x)| \le L\mu(\mathcal{B}_r(x_0))^{-1/s} \quad \text{for all } x , \tag{1.9}$$

$$|M(x)|\delta(x, x_0)^{1/s+\varepsilon} \le L\mu(\mathcal{B}_r(x_0))^{\varepsilon} \quad \text{for all } x , \tag{1.10}$$

$$\Big|\int_X M \, d\mu\Big| \le L . \tag{1.11}$$

Now we consider the case $p = 1$. Let $\varepsilon > 0$. A measurable function M on X is a *1-molecule* if there exist a point x_0 in X and constants $L > 0$ and $r > 0$ such that

$$|M(x)| \le L\mu(\mathcal{B}_r(x_0))^{-1} \quad \text{for all } x , \tag{1.12}$$

$$|M(x)|\delta(x, x_0)^{1+\varepsilon} \le L\mu(\mathcal{B}_r(x_0))^{\varepsilon} \quad \text{for all } x , \tag{1.13}$$

$$\Big|\int_X M \, d\mu\Big| \le \frac{L}{1 + |\log \mu(\mathcal{B}_r(x_0))|} . \tag{1.14}$$

The following estimates will occur many times in this paper, and we state them for easy reference:

$$\int_{0<\delta(x,y)<r} \delta(x,y)^{\alpha-1} \, d\mu(y) \le C_1 r^\alpha \quad \text{for all } r > 0 \text{ and } \alpha > 0 , \tag{1.15}$$

$$\int_{r\le\delta(x,y)} \delta(x,y)^{-\alpha-1} \, d\mu(y) \le C_2 r^{-\alpha} \quad \text{for all } r > 0 \text{ and } \alpha > 0 , \tag{1.16}$$

$$\int_{0<\delta(x,y)} \delta(x,y)^{\alpha-1}\,d\mu(y) \le C_1 D^{\alpha} \quad \text{for all } \alpha > 0\,. \tag{1.17}$$

The inequalities (1.15) and (1.16) are the inequalities of Lemma II.1 of [GV]. The inequality (1.17) follows from (1.15) by taking $\mathcal{B}_{D+\varepsilon}(x) = X$ with $\varepsilon > 0$, and letting ε tend to zero.

In order to define the kernel of the fractional integral without having to distinguish the case when the measure μ has atoms we shall adopt the following abuse of notation. If $0 < \alpha < 1$ we define

$$\frac{1}{\delta(x,y)^{1-\alpha}} = \begin{cases} 1/\delta(x,y)^{1-\alpha} & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases}$$

The *fractional integral of order* α, $0 < \alpha < 1$, on measurable functions f is defined by

$$I_\alpha f(x) = \int_X \frac{f(y)\,d\mu(y)}{\delta(x,y)^{1-\alpha}}\,.$$

The letters c and c_i will denote constants, not necessarily the same at each occurrence.

We now state the main results.

THEOREM 1. *Let $(1+\gamma)^{-1} < s \le 1$ and let M be an s-molecule. Then M belongs to H^s and*

$$\|M\|_{H^s} \le C\,, \tag{1.18}$$

where C is a constant which depends only on s, ε, and L for $s < 1$, and on ε and L for $s = 1$.

THEOREM 2. *Let $0 < \alpha < 1$. Then the following statements hold:*

(i) *For f in $wL^{1/\alpha}$, $I_\alpha f(x)$ converges absolutely for a.e. x in X, and*

$$\|I_\alpha f\|^* \le c[f]_{1/\alpha}$$

with a constant c independent of f.

(ii) *For f in wL^p with $0 < \alpha - 1/p < \gamma$, $I_\alpha f(x)$ converges absolutely for every x in X, and*

$$\|I_\alpha f\|_{\mathrm{Lip}[\alpha-1/p]} \le c[f]_p$$

with a constant c independent of f.

THEOREM 3. *Let $0 < \alpha + \beta < \gamma$ and $f \in \mathrm{Lip}[\beta]$. Then the following statements are equivalent:*

(i) $I_\alpha 1 \in \mathrm{Lip}[\alpha+\beta]$.

(ii) *$I_\alpha f(x)$ converges absolutely for every x and there is a constant c independent of f such that $\|I_\alpha f\|_{\mathrm{Lip}[\alpha+\beta]} \le c\|f\|_{\mathrm{Lip}[\beta]}$.*

THEOREM 4. (i) *Let $0 < \alpha < \gamma$. If $f \in L^\infty$ then $I_\alpha f(x)$ converges absolutely for every x in X, and there is a constant c independent of x such that*

$$\|I_\alpha f\|_{\mathrm{Lip}[\alpha]} \le c\|f\|_\infty\,.$$

(ii) *Let $0 < \alpha < \gamma$ and $f \in \mathrm{BMO}$. If $I_\alpha 1$ satisfies*

$$\sup_{x,y} \frac{|I_\alpha 1(x) - I_\alpha 1(y)|(1+|\log\delta(x,y)|)}{\delta(x,y)^\alpha} \le C_{I_\alpha 1} < \infty\,, \tag{1.19}$$

then $I_\alpha f(x)$ converges absolutely for every x in X, and there exists a constant c independent of f such that

$$\|I_\alpha f\|_{\mathrm{Lip}[\alpha]} \le c\|f\|^*\,.$$

THEOREM 5. *Let $(1+\gamma)^{-1} < p < 1$, $0 < \alpha < \gamma$ and $1 < 1/q = 1/p - \alpha$. Then the following statements are equivalent:*

(i) $I_\alpha 1 \in \mathrm{Lip}[1/p - 1]$.

(ii) *For any p-atom a, $I_\alpha a$ belongs to H^q, there exists a constant c independent of a such that*

$$\|I_\alpha a\|_{H^q} \le c\,,$$

and I_α extends to a continuous linear map from H^p to H^q.

THEOREM 6. *Let $0 < \alpha < \gamma$. If $I_\alpha 1$ satisfies* (1.19) *then for any $(1+\alpha)^{-1}$-atom a, $I_\alpha a$ belongs to H^1, there is a constant c independent of a such that*

$$\|I_\alpha a\|_{H^1} \le c\,,$$

and I_α extends to a continuous linear map from $H^{1/(1+\alpha)}$ to H^1.

Remark 1. Since the assumptions of this paper do not change the L^p spaces, the Sobolev Theorem for $1 \le p < 1/\alpha$ is the one obtained in Theorem 1 of [GV]. Similarly when $1/(1+\gamma) < p \le 1$ the theorem which asserts that $I_\alpha : H^p \to L^q$ for $1/q = 1/p - \alpha < 1$ has the same proof as in [GV] except for the fact that $\|I_\alpha 1\|_q \le c$, which follows immediately from $\mu(X) < \infty$.

Remark 2. The main novelty in Theorems 3–6 is that the cancellation property introduced in [GV] is no longer necessary when $\mu(X) < \infty$. This cancellation property for $\mu(X) < \infty$ states that $I_\alpha 1$ is constant. This condition has been replaced by a condition on the smoothness of $I_\alpha 1$ in each of these theorems.

Remark 3. Observe that $I_\alpha 1 \in \mathrm{Lip}[\alpha+\eta]$ for some $\eta > 0$ implies that $I_\alpha 1$ satisfies (1.19).

2. Proofs of the theorems

Proof of Theorem 1. To prove that M belongs to H^s and that (1.18) holds we write M as

$$M(x)=M_0(x)+I\frac{1}{\mu(\mathcal{B})}\chi_{\mathcal{B}}(x)\,, \tag{2.1}$$

where $I=\int M\,d\mu$, $\mathcal{B}=\mathcal{B}_r(x_0)$ and $\chi_{\mathcal{B}}$ denotes the characteristic function of $\mathcal{B}$. Let $g=I\mu(\mathcal{B})^{-1}\chi_{\mathcal{B}}$. It is easy to see that g satisfies (1.9)–(1.11) with constant L' that depends only on L, ε, and s if $s<1$, and (1.12)–(1.14) with constant L'' that depends only on L and ε if $s=1$. Consequently, M_0 also satisfies the same conditions with a constant that depends only on L, ε, and s, and $\int M_0\,d\mu=0$. Therefore by a known result (see [CW]), M_0 belongs to H^s and $\|M_0\|_{H^s}\le C_1$, where C_1 is a constant that depends only on L, ε, and s.

Now we will show that $g\in H^s$ and $\|g\|_{H^s}\le C_2$, where C_2 depends only on L, ε, and s. If $\mathcal{B}=X$ then $g(x)=I$ for all x and $\|g\|_{H^s}\le |I|\le L$. Assume now that $\mathcal{B}\ne X$, and let $\mathcal{B}_k=\mathcal{B}_{2^k r}(x_0)$, $k=0,\dots,K$, where $\mathcal{B}_K=X$ but $\mathcal{B}_{K-1}\ne X$. It is easy to see that $K\le c_1(1+\log(1/\mu(\mathcal{B})))$, where c_1 depends only on the space. For $k=1,\dots,K$, let

$$b_k=\frac{1}{\mu(\mathcal{B}_{k-1})}\chi_{\mathcal{B}_{k-1}}-\frac{1}{\mu(\mathcal{B}_k)}\chi_{\mathcal{B}_k}\,.$$

Then $b_k=\lambda_k a_k$ with

$$\lambda_k=\frac{\mu(\mathcal{B}_k)^{1/s}}{\mu(\mathcal{B}_{k-1})}\,,\qquad a_k=\frac{1}{\mu(\mathcal{B}_k)^{1/s}}\left[\chi_{\mathcal{B}_{k-1}}-\frac{\mu(\mathcal{B}_{k-1})}{\mu(\mathcal{B}_k)}\chi_{\mathcal{B}_k}\right].$$

Then a_k is an s-atom, and

$$g=\sum_{k=1}^{K}I\lambda_k a_k+I\chi_X\,.$$

Therefore, for $0<s<1$, using normality we obtain

$$\begin{aligned}\|g\|_{H^s}^s&\le\sum_{k=1}^{K}|I\lambda_k|^s+|I|^s\le|I|^s\sum_{k=1}^{K}\frac{A_2 2^k r}{A_1^s(2^{k-1}r)^s}+|I|^s\\&\le|I|^s\frac{A_2}{A_1^s}r^{1-s}2\left[\frac{2^{(1-s)K}-1}{2^{1-s}-1}\right]+|I|^s\\&\le|I|^s c_2(2^K r)^{1-s}+|I|^s\le c_3|I|^s\left[\frac{\mu(X)}{A_1}\right]^{1-s}+|I|^s\\&\le c_4|I|^s\le c_4L^s=C_2^s\,.\end{aligned}$$

For $s=1$, using again normality and (1.14) we have

$$\begin{aligned}\|g\|_{H^1}&\le|I|\sum_{k=1}^{K}|\lambda_k|+|I|\le|I|\frac{A_2}{A_1}2K+|I|\\&\le c_1c_5|I|(1+|\log\mu(\mathcal{B})|)\le C_2'<\infty\,.\end{aligned}$$

Proof of Theorem 2. Let f belong to $wL^{1/\alpha}$. Then

$$\begin{aligned}\|f\|_1&=\int_X|f|\,d\mu=\int_0^\infty\lambda_f(t)\,dt\\&\le\int_0^\varepsilon\mu(X)\,dt+\int_\varepsilon^\infty[f]_{1/\alpha}^{1/\alpha}t^{-1/\alpha}\,dt\\&=\varepsilon+\frac{\alpha}{1-\alpha}[f]_{1/\alpha}^{1/\alpha}\varepsilon^{1-1/\alpha}\,.\end{aligned}$$

Setting $\varepsilon=[f]_{1/\alpha}$ in the above inequality we obtain $\|f\|_1\le c[f]_{1/\alpha}$.

Now it follows easily from Fubini's Theorem, (1.17) and the last inequality that $I_\alpha f$ belongs to L^1, and

$$\|I_\alpha f\|_1\le c_1\|f\|_1\le c_2[f]_{1/\alpha}\,, \tag{2.2}$$

where c_1 and c_2 are constants which only depend on (X,δ,μ).

On the other hand, by Theorem 1.2 of [GV] we know that $\|I_\alpha f\|_*\le c[f]_{1/\alpha}$ with c independent of f. This inequality together with (2.2) concludes the proof of part (i) of the theorem.

In order to prove (ii) let f belong to wL^p with $0<\alpha-1/p<\gamma$. If $1/\alpha<s<p$ then an argument similar to the one used in (i) shows that $\|f\|_s\le c[f]_p$, where c is independent of f. Now it follows from Hölder's inequality with s and s', (1.17), and the last inequality that

$$|I_\alpha f(x)|\le\left[\int_X\frac{d\mu(y)}{\delta(x,y)^{(1-\alpha)s'}}\right]^{1/s'}\|f\|_s\le c[f]_p\,, \tag{2.3}$$

where c is independent of f.

On the other hand, by Theorem I.3 of [GV] we know that $\|I_\alpha f\|_{\mathrm{Lip}(\alpha-1/p)}\le c[f]_p$ with c independent of f. This inequality together with (2.3) proves (ii).

Proof of Theorem 3. The fact that (ii) implies (i) is obvious because $1\in\mathrm{Lip}[\beta]$. In order to prove that (i) implies (ii) let f belong to $\mathrm{Lip}[\beta]$; since $\delta(x,\cdot)^{\alpha-1}$ is in $L^1(X)$ and f is in L^∞, $I_\alpha f(x)$ converges absolutely for every x, and using (1.17) we have

$$\|I_\alpha f\|_\infty\le c\|f\|_\infty\,. \tag{2.4}$$

Let now $x_1 \neq x_2$ be points of X, $\delta(x_1,x_2)=r$, and $\mathcal{B}=\mathcal{B}_{2\varkappa r}(x_1)$, where $\varkappa$ is the constant in the inequality $\delta(x,z)\le\varkappa(\delta(x,y)+\delta(y,z))$ valid for all x,y,z in X. Write

$$I_\alpha f(x_1)-I_\alpha f(x_2)=\int_X\left[\frac{1}{\delta(x_1,y)^{1-\alpha}}-\frac{1}{\delta(x_2,y)^{1-\alpha}}\right]f(y)\,d\mu(y)$$
$$=\int_X\left[\frac{1}{\delta(x_1,y)^{1-\alpha}}-\frac{1}{\delta(x_2,y)^{1-\alpha}}\right](f(y)-f(x_1))\,d\mu(y)$$
$$+f(x_1)(I_\alpha 1(x_1)-I_\alpha 1(x_2)).$$

Since $I_\alpha 1\in \mathrm{Lip}[\alpha+\beta]$, for the second term we have

$$|f(x_1)(I_\alpha 1(x_1)-I_\alpha 1(x_2))|\le\|f\|_\infty c_1\delta(x_1,x_2)^{\alpha+\beta}. \tag{2.5}$$

Now we rewrite the first term as follows:

$$\int_{\mathcal{B}}\frac{f(y)-f(x_1)}{\delta(x_1,y)^{1-\alpha}}\,d\mu(y)-\int_{\mathcal{B}}\frac{f(y)-f(x_1)}{\delta(x_2,y)^{1-\alpha}}\,d\mu(y)$$
$$+\int_{\mathcal{B}^c}(f(y)-f(x_1))\left[\frac{1}{\delta(x_1,y)^{1-\alpha}}-\frac{1}{\delta(x_2,y)^{1-\alpha}}\right]d\mu(y)=I_1-I_2+I_3.$$

By (1.15)

$$|I_1|\le\|f\|_{\mathrm{Lip}[\beta]}(2\varkappa r)^\beta\int_{\mathcal{B}}\delta(x_1,y)^{\alpha-1}\,d\mu(y)\le C_1\|f\|_{\mathrm{Lip}[\beta]}(2\varkappa r)^{\alpha+\beta}$$
$$\le c_2\|f\|_{\mathrm{Lip}[\beta]}\delta(x_1,x_2)^{\alpha+\beta}.$$

To estimate I_2 set $\widetilde{\mathcal{B}}=\mathcal{B}_{\varkappa(2\varkappa+1)r}(x_2)$, and note that $\mathcal{B}\subset\widetilde{\mathcal{B}}$. We have

$$|I_2|\le\int_{\widetilde{\mathcal{B}}}\frac{|f(y)-f(x_2)|}{\delta(x_2,y)^{1-\alpha}}\,d\mu(y)+|f(x_2)-f(x_1)|\int_{\widetilde{\mathcal{B}}}\frac{d\mu(y)}{\delta(x_2,y)^{1-\alpha}}.$$

Both terms are estimated as I_1, and hence

$$|I_2|\le c_3\|f\|_{\mathrm{Lip}[\beta]}\delta(x_1,x_2)^{\alpha+\beta}.$$

Now we estimate I_3. Using Lemma II.3 of [GV], $\alpha+\beta<\gamma$, the fact that $f\in\mathrm{Lip}[\beta]$ and (1.16) we have

$$\|I_\alpha f\|_{\mathrm{Lip}[\beta]}\delta(x_1,x_2)^\gamma\int_{\mathcal{B}^c}\delta(x_1,y)^{\alpha-1-\gamma+\beta}\,d\mu(y)\le c_4\|f\|_{\mathrm{Lip}[\beta]}\delta(x_1,x_2)^{\alpha+\beta}.$$

Combining (2.4) and the estimates for I_1, I_2, and I_3 we finally obtain

$$\|I_\alpha f\|_{\mathrm{Lip}[\alpha+\beta]}\le c\|f\|_{\mathrm{Lip}[\alpha]}.$$

Proof of Theorem 4. To prove part (i) let f belong to L^∞, let $x_1\neq x_2$ be points of X and let $\mathcal{B}=\mathcal{B}_{2\varkappa r}(x_1)$, where $r=\delta(x_1,x_2)$. Then

we have

$$I_\alpha f(x_1)-I_\alpha f(x_2)=\int_{\mathcal{B}}\frac{f(y)\,d\mu(y)}{\delta(x_1,y)^{1-\alpha}}-\int_{\mathcal{B}}\frac{f(y)\,d\mu(y)}{\delta(x_2,y)^{1-\alpha}}$$
$$+\int_{\mathcal{B}^c}\left[\frac{1}{\delta(x_1,y)^{1-\alpha}}-\frac{1}{\delta(x_2,y)^{1-\alpha}}\right]f(y)\,d\mu(y)=I_1-I_2+I_3.$$

Now $|I_1|\le C_1(2\varkappa r)^\alpha\|f\|_\infty$ by (1.15). Let $\widetilde{\mathcal{B}}=\mathcal{B}_{\varkappa(2\varkappa+1)r}(x_2)$ and note that $\mathcal{B}\subset\widetilde{\mathcal{B}}$. Using (1.15) again we have

$$|I_2|\le\int_{\widetilde{\mathcal{B}}}\frac{|f(y)|\,d\mu(y)}{\delta(x_2,y)^{1-\alpha}}\le C_1(\varkappa(2\varkappa+1)r)^\alpha\|f\|_\infty.$$

By Lemma II.2 of [GV], $\alpha<\gamma$ and (1.16) we have

$$|I_3|\le c_1\delta(x_1,x_2)^\gamma\int_{\mathcal{B}^c}\delta(x_1,y)^{\alpha-1-\gamma}|f(y)|\,d\mu(y)\le c_1C_2(2\varkappa r)^\alpha\|f\|_\infty.$$

On the other hand, by (1.17), $\|I_\alpha f\|_\infty\le C_1D^\alpha\|f\|_\infty$. This concludes the proof of (i).

To prove (ii) let f belong to BMO and let x be a point of X. Then

$$|I_\alpha f(x)|\le\int_X\frac{|f(y)-m_X(f)|}{\delta(x,y)^{1-\alpha}}\,d\mu(y)+\int_X\frac{|m_X(f)|\,d\mu(y)}{\delta(x,y)^{1-\alpha}},$$

where $m_X(f)$ is the average of f over X, and therefore $|m_X(f)|\le\|f\|_1$. Setting $r=D+\varepsilon$ with $\varepsilon>0$ in Lemma II.5(i) of [GV], and then letting ε tend to zero we find that the first integral is majorized by $c_2D^\alpha\|f\|_*$. By (1.15) the second integral is majorized by $C_1D^\alpha\|f\|_1$. These two estimates imply

$$\|I_\alpha f\|_\infty\le c\|f\|^*. \tag{2.6}$$

Now let $x_1\neq x_2$ be points of X, $r=\delta(x_1,x_2)$ and $\mathcal{B}=\mathcal{B}_{2\varkappa r}(x_1)$. Note that $x_2\in\mathcal{B}$; thus we have

$$I_\alpha f(x_1)-I_\alpha f(x_2)=\int_X\left[\frac{1}{\delta(x_1,y)^{1-\alpha}}-\frac{1}{\delta(x_2,y)^{1-\alpha}}\right]f(y)\,d\mu(y)$$
$$=\int_X\left[\frac{1}{\delta(x_1,y)^{1-\alpha}}-\frac{1}{\delta(x_2,y)^{1-\alpha}}\right](f(y)-m_{\mathcal{B}}(f))\,d\mu(y)$$
$$+m_{\mathcal{B}}(f)[(I_\alpha 1)(x_1)-(I_\alpha 1)(x_2)].$$

We first estimate the second term. To do this we need the following inequality:

$$|m_{\widetilde{\mathcal{B}}}(f)|\le c_1\|f\|^*(1+|\log\mu(\widetilde{\mathcal{B}})|),$$

where $\overline{\mathcal{B}}$ is any ball in X, f is a function in BMO, and c_1 is independent of $\overline{\mathcal{B}}$ and f. To prove this inequality write

$$m_{\overline{\mathcal{B}}}(f) = \int_X f(y) \frac{1}{\mu(\overline{\mathcal{B}})} \chi_{\overline{\mathcal{B}}}(y)\, d\mu(y)$$

and use that f belongs to BMO, duality, and the fact proved in Theorem 1 that

$$\left\| \frac{1}{\mu(\overline{\mathcal{B}})} \chi_{\overline{\mathcal{B}}} \right\|_{H^1} \le c_2(1 + |\log \mu(\overline{\mathcal{B}})|) .$$

This inequality and normality imply that

$$(2.7) \qquad |m_{\mathcal{B}}(f)((I_\alpha 1)(x_1) - (I_\alpha 1)(x_2))| \le c_3 \|f\|^* \delta(x_1, x_2)^\alpha$$

with c_3 independent of f and $\mathcal{B}$.

Now we rewrite the first term as follows:

$$\int_{\mathcal{B}} \frac{f(y) - m_{\mathcal{B}}(f)}{\delta(x_1, y)^{1-\alpha}}\, d\mu(y) - \int_{\mathcal{B}} \frac{f(y) - m_{\mathcal{B}}(f)}{\delta(x_2, y)^{1-\alpha}}\, d\mu(y)$$
$$+ \int_{\mathcal{B}^c} (f(y) - m_{\mathcal{B}}(f)) \left\{ \frac{1}{\delta(x_1, y)^{1-\alpha}} - \frac{1}{\delta(x_2, y)^{1-\alpha}} \right\} d\mu(y) = I_1 - I_2 + I_3 .$$

By Lemma II.5(i) of [GV], $|I_1| \le c_4(2\varkappa r)^\alpha \|f\|_*$. To estimate I_2 note that $\mathcal{B} \subset \widetilde{\mathcal{B}} = \mathcal{B}_{\varkappa(2\varkappa+1)r}(x_2)$, and hence

$$|I_2| \le \int_{\widetilde{\mathcal{B}}} \frac{|f(y) - m_{\widetilde{\mathcal{B}}}(f)|}{\delta(x_2, y)^{1-\alpha}}\, d\mu(y) + |m_{\mathcal{B}}(f) - m_{\widetilde{\mathcal{B}}}(f)| \int_{\widetilde{\mathcal{B}}} \frac{d\mu(y)}{\delta(x_2, y)^{1-\alpha}} .$$

Again by Lemma II.5(i) of [GV] the first integral is majorized by $c_5 \times (\varkappa(2\varkappa + 1)r)^\alpha \|f\|_*$. Now note that $|m_{\mathcal{B}}(f) - m_{\widetilde{\mathcal{B}}}(f)| \le c_6 \|f\|_*$, where c_6 is independent of x_1, x_2, r, and f. Therefore by (1.15) the second term is majorized by $c_6 C_1 r^\alpha \|f\|_*$. Combining these estimates we have

$$|I_2| \le c_7 \delta(x_1, x_2)^\alpha \|f\|_* .$$

Now we estimate I_3. Using Lemmas II.3, II.5(i) of [GV] and the fact that $\alpha < \gamma$ we have

$$|I_3| \le c_8 \delta(x_1, x_2)^\gamma \int_{\mathcal{B}^c} \frac{|f(y) - m_{\mathcal{B}}(f)|}{\delta(x_1, y)^{1-\alpha+\gamma}}\, d\mu(y) \le c_9 \|f\|_* \delta(x_1, x_2)^\alpha .$$

Combining the estimates for I_1, I_2 and I_3 with (2.7) we obtain $\|I_\alpha f\|_{\mathrm{Lip}(\alpha)} \le c_{10} \|f\|^*$. Finally, using (2.6) we have $\|I_\alpha f\|_{\mathrm{Lip}[\alpha]} \le c\|f\|^*$.

Proof of Theorem 5. We shall prove first that (i) implies (ii). Let a be a p-atom with support in $\mathcal{B} = \mathcal{B}_r(x_0)$. Since the diameter D of X is finite, $\mathcal{B}_r(x) = X$ when $r > D$, therefore we can assume without loss of generality that a p-atom is supported in a ball of radius r less than or equal to $2D$.

If a is a p-atom and $r \le r_{x_0}$, then $\int a\, d\mu = 0$ forces a to be identically zero and consequently $I_\alpha a \equiv 0$. If a is not identically 1 and $r_{x_0} < r \le 2D$ we shall prove (1.9)–(1.11) with $s = q$ and $\varepsilon = 1 + \gamma - 1/p$. To prove (1.9) we consider two cases: $\delta(x, x_0) < 2\varkappa r$ and $\delta(x, x_0) \ge 2\varkappa r$. If $\delta(x, x_0) < 2\varkappa r$ we have

$$|I_\alpha a(x)| \le \int_{\mathcal{B}} \frac{1}{\delta(x, y)^{1-\alpha}} |a(y)|\, d\mu(y) \le \int_{\delta(x,y) \le \varkappa(2\varkappa+1)r} |a(y)|\, d\mu(y)$$
$$\le C_1 \mu(\mathcal{B})^{-1/p} [\varkappa(2\varkappa + 1)r]^\alpha \le (2/A_1)^\alpha C_1 [\varkappa(2\varkappa + 1)]^\alpha \mu(\mathcal{B})^{-1/q}$$
$$= L_1 \mu(\mathcal{B})^{-1/q} .$$

If $\delta(x, x_0) \ge 2\varkappa r$, then

$$|I_\alpha a(x)| \le \int_{\mathcal{B}} \frac{1}{\delta(x, y)^{1-\alpha}} |a(y)|\, d\mu(y) \le \frac{1}{(2\varkappa r)^{1-\alpha}} \mu(\mathcal{B})^{-1/p} \mu(\mathcal{B})$$
$$\le (A_2/(2\varkappa))^{1-\alpha} \mu(\mathcal{B})^{\alpha-1} \mu(\mathcal{B})^{1-1/p} \le L_2 \mu(\mathcal{B})^{-1/q} .$$

To prove (1.10), observe first that if $\delta(x, x_0) < 2\varkappa r$ then (1.10) follows from (1.9) using normality:

$$|I_\alpha a(x)| \delta(x, x_0)^{1/q+\varepsilon} \le L_1 \mu(\mathcal{B})^{-1/q} (2\varkappa r)^{1/q+\varepsilon} \le L_3 \mu(\mathcal{B})^\varepsilon .$$

If $\delta(x, x_0) \ge 2\varkappa r$, then since $\int a\, d\mu = 0$ we can write

$$I_\alpha a(x) = \int_{\mathcal{B}} \left[\frac{1}{\delta(x, y)^{1-\alpha}} - \frac{1}{\delta(x, x_0)^{1-\alpha}} \right] a(y)\, d\mu(y) .$$

Now using Lemma II.3 of [GV] and normality we have

$$|I_\alpha a(x)| \le c_1 \delta(x, x_0)^{\alpha-\gamma-1} \mu(\mathcal{B})^{-1/p} \int_{\mathcal{B}} \delta(y, x_0)^\gamma\, d\mu(y)$$
$$\le c_1 \delta(x, x_0)^{\alpha-\gamma-1} \mu(\mathcal{B})^{-1/p} r^\gamma \mu(\mathcal{B})$$
$$\le c_1 (2/A_1)^\gamma \delta(x, x_0)^{\alpha-\gamma-1} \mu(\mathcal{B})^\varepsilon = L_4 \delta(x, x_0)^{\alpha-\gamma-1} \mu(\mathcal{B})^\varepsilon .$$

In order to show (1.11), consider

$$\int I_\alpha a(x)\, d\mu(x) = \int_X \left(\int_X \frac{a(y)}{\delta(x, y)^{1-\alpha}}\, d\mu(y) \right) d\mu(x) .$$

Changing the order of integration, and using $\int a\, d\mu = 0$, the above integral is equal to

$$\int a(y) I_\alpha 1(y)\, d\mu(y) = \int a(y) [I_\alpha 1(y) - I_\alpha 1(x_0)]\, d\mu(y) .$$

Since $I_\alpha 1$ belongs to $\mathrm{Lip}[1/p-1]$ and a is a p-atom supported in $\mathcal{B}_r(x_0)$ we have

$$\left|\int I_\alpha a\,d\mu\right| \le \|I_\alpha 1\|_{\mathrm{Lip}[1/p-1]} r^{1/p-1} \int |a(y)|\,d\mu(y) \le \|I_\alpha 1\|_{\mathrm{Lip}[1/p-1]} D^{1/p-1} = L_5\,. \tag{2.8}$$

This proves that $I_\alpha a$ is a q-molecule with constant $L = \max\{L_i : 1 \le i \le 5\}$ which depends only on the space, α, and p. When a is identically 1, (1.17) implies that $|I_\alpha 1(x)| \le C_1 D^\alpha$, and therefore $I_\alpha a$ is a q-molecule for any $x_0 \in X$, $r = 2D$ and any $\varepsilon > 0$, and its constant L depends only on the space, α, and p. Applying now Theorem 1 we see that $I_\alpha a$ is in H^q and

$$\|I_\alpha a\|_{H^q} \le C\,. \tag{2.9}$$

Let now f be an element of H^p, and consider an atomic decomposition of f, i.e. $f = \sum_{i=1}^\infty \lambda_i a_i$ with $\sum_{i=1}^\infty |\lambda_i|^p \le 2^p \|f\|_{H^p}^p$. Then by (2.9) we have

$$\sum_{i=1}^\infty \|\lambda_i I_\alpha a_i\|_{H^q}^q \le C^q \sum_{i=1}^\infty |\lambda_i|^q \le C^q \Big(\sum_{i=1}^\infty |\lambda_i|^p\Big)^{q/p} < \infty\,. \tag{2.10}$$

Since H^q is complete, (2.10) implies that $\sum_{i=1}^\infty \lambda_i I_\alpha a_i$ converges in H^q to an element g of H^q. Furthermore, we have

$$\|g\|_{H^q} \le 2C\|f\|_{H^p}\,. \tag{2.11}$$

In order to extend I_α to all of H^p we must show that g does not depend on the particular atomic decomposition of f. Let a be a p-atom and let φ belong to $\mathrm{Lip}[1/q-1]$. Observe that by Theorem 3, $I_\alpha \varphi$ belongs to $\mathrm{Lip}[1/p-1]$ and

$$\langle I_\alpha a, \varphi\rangle = \langle a, I_\alpha \varphi\rangle\,; \tag{2.12}$$

in fact,

$$\begin{aligned}\langle I_\alpha a, \varphi\rangle &= \int \varphi(x)\Big[\int a(y)\frac{1}{\delta(x,y)^{1-\alpha}}\,d\mu(y)\Big]\,d\mu(x)\\ &= \int \Big[\int \varphi(x)\frac{1}{\delta(x,y)^{1-\alpha}}\,d\mu(x)\Big]\,a(y)\,d\mu(y)\\ &= \int I_\alpha\varphi(y)a(y)\,d\mu(y) = \langle a, I_\alpha\varphi\rangle\,,\end{aligned}$$

because $\varphi(x)a(y)\delta(x,y)^{\alpha-1}$ is in $L^1(X\times X)$. Now

$$\langle g, \varphi\rangle = \sum_{i=1}^\infty \lambda_i \langle I_\alpha a_i, \varphi\rangle = \sum_{i=1}^\infty \lambda_i\langle a_i, I_\alpha\varphi\rangle = \langle f, I_\alpha\varphi\rangle\,.$$

This shows that g is independent of the particular atomic decomposition of f, and defining $I_\alpha f = g$ by (2.11) we have

$$\|I_\alpha f\|_{H^q} \le 2C\|f\|_{H^p}\,,$$

for all f in H^p. This concludes the proof that (i) implies (ii). To prove that (ii) implies (i) we use the fact that $\mathrm{Lip}[1/p-1]$ is the dual space of H^p. Since the transpose $I_\alpha^t = I_\alpha$, (ii) implies that I_α is continuous from $\mathrm{Lip}[1/q-1]$ to $\mathrm{Lip}[1/p-1]$, consequently $I_\alpha 1 \in \mathrm{Lip}[1/p-1]$.

Proof of Theorem 6. Let a be a $1/(\alpha+1)$-atom supported in $\mathcal{B} = \mathcal{B}_r(x_0)$. The proofs of (1.12) and (1.13) are the same as the proofs of (1.9) and (1.10) in Theorem 5 if p and q are replaced respectively by $1/(1+\alpha)$ and 1. In order to prove (1.14) write

$$\int I_\alpha a(x)\,d\mu(x) = \int_X a(y)(I_\alpha 1(y) - I_\alpha 1(x_0))\,d\mu(y)$$

as in the proof of (1.11) in Theorem 5. Using condition (1.19) we majorize the absolute value of the last integral by

$$C_{I_\alpha 1}\int_X |a(y)|\frac{\delta(x_0,y)^\alpha}{1+|\log\delta(x_0,y)|}\,d\mu(y)\,.$$

Since $y \in \mathcal{B}_r(x_0)$ and $r \le 2D$ there is a constant c_1 such that the integral is less than or equal to

$$c_1\frac{r^\alpha}{1+|\log r|}\int_X |a(y)|\,d\mu(y)\,.$$

Using the fact that a is an $1/(1+\alpha)$-atom and normality in the last expression we finally obtain

$$\left|\int I_\alpha a\,d\mu\right| \le L_6\frac{1}{1+|\log\mu(\mathcal{B})|}\,.$$

Finally, if $a \equiv 1$ then (1.15) implies that $|I_\alpha 1(x)| \le C_1 D^\alpha$, and therefore $I_\alpha 1$ is a 1-molecule for any $x_0 \in X$, $r = 2D$, and any $\varepsilon > 0$, whose constant L depends only on the space, α, and ε.

Applying Theorem 1 we conclude that $I_\alpha a$ is in H^1 and

$$\|I_\alpha a\|_{H^1} \le C\,. \tag{2.13}$$

Let now f be an element of $H^{1/(1+\alpha)}$ and let $f = \sum_{i=1}^\infty \lambda_i a_i$ with $\sum_{i=1}^\infty |\lambda_i|^{1/(1+\alpha)} \le 2^{1/(1+\alpha)}\|f\|_{H^{1/(1+\alpha)}}^{1/(1+\alpha)}$ be an atomic decomposition of f. Then by (2.13)

$$\sum_{i=1}^\infty \|\lambda_i I_\alpha a\|_{H^1} \le C\sum_{i=1}^\infty |\lambda_i| \le C\Big(\sum_{i=1}^\infty |\lambda_i|^{1/(1+\alpha)}\Big)^{1+\alpha}. \tag{2.14}$$

Since H^1 is complete, (2.14) implies that $\sum_{i=1}^\infty \lambda_i I_\alpha a_i$ converges in H^1 to an element g of H^1, and $\|g\|_{H^1} \le 2C\|f\|_{H^{1/(1+\alpha)}}$. The extension of I_α to all of $H^{1/(1+\alpha)}$ is done as in the proof of Theorem 5, except that instead of using Theorem 3 we use part (ii) of Theorem 4.

Acknowledgements. Our original proof of Theorem 1 relied on the β-maximal function of Macías and Segovia [MS] for estimating $\|g\|_{H^s}$. Also, the condition (1.14) was originally stated as $|\int_X M\,d\mu| \le L\mu(\mathcal{B}_r(x_0))^\eta$ for some $\eta > 0$. We thank Professor M. Taibleson and the referee for suggesting the present proof of Theorem 1, which is shorter and direct, and condition (1.14).

3. A correction to [GV]. In [GV] the definition of the cancellation property (page 174, third paragraph) is incorrect if the measure μ has atoms. In order to state the cancellation property in a way that includes this case we have to define the kernel of the fractional integral as in this paper, i.e. for $0 < \alpha < 1$

$$\frac{1}{\delta(x,y)^{1-\alpha}} = \begin{cases} 1/\delta(x,y)^{1-\alpha} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then the fractional integrals are defined by

$$I_\alpha f(x) = \int_X \frac{f(y)}{\delta(x,y)^{1-\alpha}}\,d\mu(y),$$

$$\widetilde{I}_\alpha f(x) = \int_X \left[\frac{1}{\delta(x,y)^{1-\alpha}} - \frac{\psi_z(y)}{\delta(z,y)^{1-\alpha}}\right] f(y)\,d\mu(y),$$

where z and ψ_z are as in [GV], page 177 (cf. (I, 21) and (I, 22) of [GV]). Now the cancellation property can be stated as follows:

$$\widetilde{I}_\alpha 1 \equiv \text{const.}, \tag{3.1}$$

or equivalently

$$\int_X \left[\frac{1}{\delta(x_1,y)^{1-\alpha}} - \frac{1}{\delta(x_2,y)^{1-\alpha}}\right] d\mu(y) = 0 \tag{3.2}$$

for all x_1 and x_2 in X (cf. (I, 10) of [GV]). To show that (3.1) implies (3.2) observe that the integral in (3.2) is $\widetilde{I}_\alpha 1(x_1) - \widetilde{I}_\alpha 1(x_2)$. To see that (3.2) implies (3.1) add and subtract $\psi_z(y)/\delta(z,y)^{1-\alpha}$ in the integrand of (3.2). With these definitions all the theorems and their proofs in [GV] are valid when μ has atoms.

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