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On an estimate for the norm of a function of a quasihermitian operator

by

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Abstract. Let A be a closed linear operator acting in a separable Hilbert space. Denote by $\mathrm{co}(A)$ the closed convex hull of the spectrum of A. An estimate for the norm of $f(A)$ is obtained under the following conditions: f is a holomorphic function in a neighbourhood of $\mathrm{co}(A)$, and for some integer p the operator $A^p - (A^*)^p$ is Hilbert–Schmidt. The estimate improves one by I. Gelfand and G. Shilov.

1. Introduction. Notations. Let H be a separable Hilbert space, and let A be a closed linear operator acting on H with domain $D(A)$. Then A is called *quasihermitian* if $D(A) \subseteq D(A^*)$ and the imaginary component $A_J = (A - A^*)/2i$ is completely continuous. Denote by $\mathrm{co}(A)$ the closed convex hull of the spectrum $\sigma(A)$ of A. In this paper we obtain an estimate for the norm of $f(A)$ if f is a holomorphic function in a neighbourhood of $\mathrm{co}(A)$, and A is a quasihermitian operator with

$$A_J \in C_2 \tag{1.1}$$

where C_2 is the Hilbert–Schmidt ideal [9]. Moreover, this estimate is generalized to the case

$$A^p - (A^*)^p \in C_2 \tag{1.2}$$

for some integer p.

Singular integral and integral-differential operators are examples of operators which satisfy (1.1) and (1.2).

We recall that I. M. Gelfand and G. E. Shilov [5, Ch. 2] obtained an estimate for the norm of a matrix-valued function with equality being attained for no matrix (finite-dimensional operator). In [6] we obtained a sharp estimate for matrix-valued functions. This estimate becomes equality in the case of a normal matrix. In [7] an estimate for the norm of a function of a Hilbert–Schmidt operator is obtained.

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Below we generalize and improve the results from [6, 7] and also supplement Carleman's estimate for the resolvent $R_\mu(A)$ of $A \in C_2$ [4, Ch. XI].

Let I denote the identity operator in H, and let

$$v(A) = \Big[|A_J|_2^2 - \sum_{k=1}^{\infty} |\operatorname{Im} \mu_k(A)|^2\Big]^{1/2} \sqrt{2}$$

where $|B|_2$ is the Hilbert–Schmidt norm of a Hilbert–Schmidt operator B, and $\mu_1(A), \mu_2(A), \ldots$ are all nonreal eigenvalues of A counted with their multiplicity. If A is a normal operator, then $v(A) = 0$ (see [8]).

We define $f(A)$ by

$$f(A) = -\frac{1}{2\pi i} \int_\Gamma f(\mu) R_\mu(A)\, d\mu + f(\infty) I \tag{1.3}$$

where Γ is a smooth contour encircling $\sigma(A)$.

2. Main result

THEOREM 1. *Let A be a quasihermitian operator satisfying* (1.1) *and let f be a holomorphic function in a neighbourhood of* $\operatorname{co}(A)$. *Then*

$$\|f(A)\| \le \sum_{k=0}^{\infty} \sup_{\mu \in \operatorname{co}(A)} |f^{(k)}(\mu)| \frac{v(A)^k}{(k!)^{3/2}}. \tag{2.1}$$

First we prove a few lemmata.

LEMMA 1. *Let the imaginary part A_J of a quasihermitian operator A belong to the Matsaev ideal C_ω* [10, Ch. 4.3], *i.e.*

$$\sum_{k=1}^{\infty} (2k-1)^{-1} \mu_k(A_J) < \infty \quad (\mu_k(A_J) \in \sigma(A_J)).$$

Then there are an orthogonal resolution of the identity $E(t)$ $(-\infty < t < \infty)$, a normal operator N and a Volterra (completely continuous quasinilpotent) operator V such that for all $t \in (-\infty, \infty)$

$$NE(t) = E(t)N, \tag{2.2}$$

$$E(t)VE(t) = VE(t), \tag{2.3}$$

$$A = N + V. \tag{2.4}$$

Proof. As is shown in [2],

$$A = \int_{-\infty}^{\infty} h(t)dP(t) + i \int_{-\infty}^{\infty} P(t) A_J\, dP(t). \tag{2.5}$$

Here $P(t)$ is an orthogonal resolution of the identity and h is a nondecreasing scalar-valued function. The second integral in (2.5) is the limit in the C_ω-norm of the sums

$$\frac{1}{2}\sum_{k=1}^{n} [P(t_k) + P(t_{k-1})] A_J \Delta P_k = S_n + U_n$$

$$(t_k = t_k^{(n)};\ \Delta P_k = P(t_k) - P(t_{k-1}),\ -\infty < t_0 < t_1 < \ldots < t_n < \infty)$$

where

$$U_n = \sum_{k=1}^{n} P(t_{k-1}) A_J \Delta P_k, \qquad S_n = \sum_{k=1}^{n} \Delta P_k A_J \Delta P_k. \tag{2.6}$$

The sequence $\{S_n\}$ is norm convergent by Lemma 1.5.1 of [10]. We denote its limit by S. By Theorem 2.5.2 of [9, p. 77], each S_n belongs to C_ω. According to Theorem 3.5.1 of [9, p. 113], so does S. It is clear that the $P(t)$ $(-\infty < t < \infty)$ are projectors of H onto invariant subspaces of the selfadjoint operator S. We arrive at (2.2) when $E(t) = P(t)$ and $N = \int_{-\infty}^{\infty} h(t) dP(t) + iS$. Further, U_n is a nilpotent operator: $(U_n)^n = 0$. The sequence $\{U_n\}$ converges in the C_ω-norm because so do the second integral in (2.5) and $\{S_n\}$. We denote the limit by U. Then U is a Volterra operator by Lemma 2.17 of [3]. From (2.5) we obtain (2.4). ∎

By Neumann's theorem [1, p. 314] there exists a bounded scalar-valued function ψ such that $S = \int_{-\infty}^{\infty} \psi(t)\, dE(t)$ since $E(t)S = SE(t)$. Hence

$$N = \int_{-\infty}^{\infty} \varphi(t)\, dE(t) \tag{2.7}$$

where $\varphi = h + i\psi$.

DEFINITION 1. Suppose there are an orthogonal resolution of the identity $E(t)$, a scalar-valued function φ and a Volterra operator V such that (2.3), (2.4) and (2.7) hold. Then we call $E(t)$, N, V and (2.4) a *spectral function*, a *diagonal part*, a *nilpotent part* and a *triangular representation* of A, respectively.

Our definition of spectral function is analogous to the corresponding definitions in [2, 3].

LEMMA 2. *Let a bounded operator A have a triangular representation and a spectral function $P(t)$ which consists of $n < \infty$ projectors $0 = P_0 < P_1 < \ldots < P_n = I$. Suppose its nilpotent part V is in C_2. Then*

$$\|f(A)\| \le \sum_{k=0}^{n-1} \sup_{\mu \in \operatorname{co}(A)} |f^{(k)}(\mu)| \frac{|V|_2^k}{(k!)^{3/2}} \tag{2.8}$$

for every function f holomorphic in a neighbourhood of $\operatorname{co}(A)$.

Proof. (2.7) has the form $N=\sum_{k=1}^n \varphi_k \Delta P_k$ in this case. Here φ_k ($k=1,\dots,n$) are eigenvalues of A. Let $\{e_j(m)\}$ ($m=1,2,\dots$) be an orthonormal basis in $\Delta P_j H$. We set

$$a_{ij}(m)=(Ae_i(m),e_j(m)),\qquad N_m=\sum_{j=1}^n \varphi_j(\cdot,e_j(m))e_j(m),$$

$$V_m=\sum_{1\le i<j\le n} a_{ij}(m)(\cdot,e_j(m))e_i(m),\qquad A_m=N_m+V_m\,.$$

Clearly, $a_{ij}(m)=0$ when $i>j$. The operators A, N, V and $f(A)$ are the orthogonal sums of A_m, N_m, V_m and $f_m(A)$ ($m=1,2,\dots$), respectively. Therefore

$$\sigma(A_m)\subseteq\sigma(A),\quad |V_m|_2\le|V|_2,\quad \max_m\|f(A_m)\|=\|f(A)\|\,. \tag{2.9}$$

Let B be an $(n\times n)$-matrix. We apply the estimate [7]

$$\|f(B)\|\le\sum_{k=0}^{n-1}\sup_{\mu\in\mathrm{co}(B)}|f^{(k)}(\mu)|\frac{|V_B|_2^k}{(k!)^{3/2}}$$

where V_B is the nilpotent part of B. Hence

$$\|f(A_m)\|\le\sum_{k=0}^{n-1}\sup_{\mu\in\mathrm{co}(A_m)}|f^{(k)}(\mu)|\frac{|V_m|_2^k}{(k!)^{3/2}}$$

since A_m is a finite-dimensional operator and V_m is its nilpotent part. From this and from (2.9) we obtain (2.8). ■

COROLLARY 1. *Under the conditions of Lemma* 2,

$$\|f(A)\|\le\sum_{k=0}^{n-1}\sup_{\mu\in\mathrm{co}(A)}|f^{(k)}(\mu)|\frac{v(A)^k}{(k!)^{3/2}}\,.$$

This follows from Lemma 2 and the equality

$$v(A)=|V|_2\,, \tag{2.10}$$

which is proved in [8, p. 164].

LEMMA 3. *Let A admit a triangular representation, and let N be its diagonal part. Then $\sigma(A)=\sigma(N)$.*

Proof. By (2.4),

$$R_\mu(A)=R_\mu(N)(I+VR_\mu(N))^{-1}.$$

Now, $VR_\mu(A)$ ($\mu\notin\sigma(A)$) is a Volterra operator by Corollary 2 of Theorem

17.1 of [3, p. 121]. Hence

$$(I+VR_\mu(N))^{-1}=\sum_{k=0}^{\infty}(VR_\mu(N))^k(-1)^k\,,$$

i.e.

$$R_\mu(A)=R_\mu(N)\sum_{k=0}^{\infty}(VR_\mu(N))^k(-1)^k\,,$$

which clearly implies our assertion. ■

Proof of Theorem 1. (2.3), (2.4) and (2.7) hold by Lemma 1. Define

$$V_n=\sum_{k=1}^n P(t_{k-1})V\Delta P_k\,,\quad N_n=\sum_{k=1}^n\varphi(t_k)\Delta P_k,\quad B_n=N_n+V_n\,.$$

First, suppose that A is bounded. Then $\{B_n\}$ strongly converges to A. By (1.3), $\{f(B_n)\}$ strongly converges to $f(A)$. The inequality

$$\|f(A)\|\le\sup_n\|f(B_n)\| \tag{2.11}$$

follows from the Banach–Steinhaus theorem. Since the spectral function of B_n consists of $n<\infty$ projectors, Lemma 2 yields

$$\|f(B_n)\|\le\sum_{k=0}^{n-1}\sup_{\mu\in\mathrm{co}(B_n)}|f^{(k)}(\mu)|\frac{|V_n|_2^k}{(k!)^{3/2}}\,.$$

By Lemma 3, $\sigma(B_n)=\sigma(N_n)$. Clearly, $\sigma(N_n)\subset\sigma(N)$. Hence, $\sigma(B_n)\subset\sigma(A)$. By Theorem 3.6.3 of [9, p. 119], $|V_n|_2\to|V|_2$ as $n\to\infty$. (2.1) holds by (2.10) and (2.11).

Now, let A be an unbounded operator. Let $Q_n=P(n)-P(-n)$. Then AQ_n is bounded for each $n<\infty$. We have $(A-\mu I)^{-1}Q_n(AQ_n-\mu Q_n)=Q_n$. By (1.3), $\|f(AQ_n)\|=\|f(A)Q_n\|$. Moreover, AQ_n is a restriction of A onto its invariant subspace. Hence $\sigma(AQ_n)\subset\sigma(A)$. Now, we obtain by (2.1) the estimate

$$\|f(AQ_n)\|\le\sum_{k=0}^{\infty}\sup_{\mu\in\mathrm{co}(A)}|f^{(k)}(\mu)|\frac{v(AQ_n)^k}{(k!)^{3/2}}\,. \tag{2.12}$$

Since $|VQ_n|_2\to|V|_2$ and $v(AQ_n)=|VQ_n|_2$ by (2.10), we have $v(AQ_n)\to v(A)$ as $n\to\infty$. From this and from (2.12) we get (2.1). ■

Theorem 1 is sharp: (2.1) turns into the equality $\|f(A)\|=\sup_{\sigma(A)}|f(\mu)|$ if A is a normal operator and $\sup_{\mathrm{co}(A)}|f(\mu)|=\sup_{\sigma(A)}|f(\mu)|$.

COROLLARY 2. *Let A satisfy* (1.2) *and let $g(\lambda) = f(\lambda^{1/p})$ be an analytic function on* $\operatorname{co}(A^p)$. *Then*

$$\|f(A)\| \le \sum_{k=0}^{\infty} \sup_{\mu\in\operatorname{co}(A^p)} |g^{(k)}(\mu)| \frac{v(A^p)^k}{(k!)^{3/2}}.$$

COROLLARY 3. *Let A satisfy* (1.1). *Then*

$$\|\exp(At)\| \le \exp[\alpha(A)t] \sum_{k=0}^{\infty} \frac{v(A)^k}{(k!)^{3/2}} t^k \quad (t \ge 0)$$

where $\alpha(A) = \sup \operatorname{Re} \sigma(A)$.

COROLLARY 4. *Let A satisfy* (1.1) *and let $\sigma(A) = [a,b]$* $(-\infty \le a < b \le \infty)$. *Then*

$$\|R_\mu(A)\| \le \sum_{k=0}^{\infty} \frac{v(A)^k}{d(\mu,A)^{k+1}\sqrt{k!}}$$

where $d(\mu,A)$ is the distance between $\sigma(A)$ and μ on the complex plane.

This supplements Carleman's estimate [4, Ch. XI] and also generalizes the author's estimate [8] in the case $\operatorname{Im}\sigma(A) = 0$.

3. Perturbation of the spectrum

LEMMA 4. *Let A, B be linear operators acting in a Banach space and suppose*

(3.1) $$q = \|A - B\| < \infty,$$

(3.2) $$\|R_\mu(A)\| \le b(d(\mu,A)^{-1})$$

where $b(y)$ is an increasing function of $y > 0$. Then $\sup\{\operatorname{dist}(\lambda,\sigma(A)) : \lambda\in\sigma(B)\} \le 1/\psi(q^{-1})$ *where ψ is the inverse function to b; $\psi(b(y)) = y$.*

Proof. We have $R_\mu(A) - R_\mu(B) = R_\mu(B)(B-A)R_\mu(A)$. Let $q\|R_\mu(A)\| < 1$. Then $\|R_\mu(B)\| \le \|R_\mu(A)\|(1 - q\|R_\mu(A)\|)^{-1}$, hence $\mu \notin \sigma(B)$. Therefore $1 \le q\|R_\mu(A)\| \le qb(d(\mu,A)^{-1})$ if $\mu\in\sigma(B)$. This implies $d(\mu,A) \le 1/\psi(q^{-1})$ for each $\mu\in\sigma(B)$. ■

Lemma 4 and Corollary 4 give:

COROLLARY 5. *Let A satisfy* (1.1) *and* (3.1), *and suppose $\sigma(A) = [a,b]$, $-\infty \le a < b \le \infty$. Then*

(3.3) $$\sup\{\operatorname{dist}(\lambda,\sigma(A)) : \lambda\in\sigma(B)\} \le 1/\psi_A(q^{-1})$$

where ψ_A is the inverse function to

$$b_A(y) \equiv \sum_{k=0}^{\infty} \frac{v(A)^k}{\sqrt{k!}} y^{k+1}.$$

Let $A = A^*$. Then $v(A) = 0$, $b_A(y) = y$. In this case under the condition (3.1), $\operatorname{dist}\{\sigma(B),\sigma(A)\} \le q$, i.e. (3.3) generalizes the well-known result for selfadjoint operators with $\sigma(A) = [a,b]$ [12, Ch. V].

Remark. Schwarz's inequality gives

$$b_A(y)^2 \le \sum_{j=0}^{\infty} (1/2)^j y^2 \sum_{k=0}^{\infty} \frac{(yv(A))^{2k}}{k!} 2^k = 2y^2 \exp[2v(A)^2y^2].$$

By Corollary 4 under (1.1) and $\sigma(A) = [a,b]$ we have

$$\|R_\mu(A)\| \le \sqrt{2}\, d(\mu,A)^{-1} \exp[v(A)^2/d(\mu,A)^2].$$

4. Nonlinear perturbation of a linear semigroup. Consider the equation

(4.1) $$du/dt = Au + F(u,t) \quad (0 \le t \le \infty)$$

where A is a linear operator in H and F maps $H\times[0,\infty)$ into H.

A *solution of the Cauchy problem* for (4.1) is a continuously differentiable function $u : [0,\infty) \to D(A)$ which satisfies (4.1) and an initial condition $u(0) = u_0 \in D(A)$. Assume

(4.2) $$\|F(x,t)\| \le q\|x\| \quad \text{for each } x\in D(A) \text{ and } t \ge 0.$$

THEOREM 2. *Let $x(t)$ be a solution of the Cauchy problem for* (4.1) *under the conditions* (1.1), (4.2), $\alpha(A) < 0$ *and*

$$j \equiv \sum_{k=0}^{\infty} \frac{v(A)^k}{|\alpha(A)|^{k+1}\sqrt{k!}} < 1/q.$$

Then

(4.3) $$\|x(t)\| \le a\|x(0)\|(1 - qj)^{-1} \quad (t \ge 0,\ a = \text{const}).$$

Proof. We have by (4.1)

$$x(t) = \exp[At]x(0) + \int_0^t \exp[A(t-s)]F(x(s),s)\,ds$$

(see [11, p. 53]). This implies

$$\|x(t)\| \le \|\exp[At]x(0)\| + \int_0^t \|\exp[A(t-s)]\|q\|x(s)\|\,ds.$$

By Corollary 3,

$$\|\exp[At]\| \le a \quad (t \ge 0),$$

$$\int_0^t \|\exp[A(t-s)]\|\,ds \le \int_0^\infty \|\exp[As]\|\,ds$$
$$\le \int_0^\infty \exp[\alpha(A)t]\sum_{k=0}^{\infty}\frac{t^k v(A)^k}{(k!)^{3/2}}\,dt = j \qquad (t\ge 0).$$

Hence, $\max_{t\ge 0}\|x(t)\| \le a\|x(0)\| + \max_{t\ge 0}\|x(t)\| j$ and we arrive at (4.3). ■

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On molecules and fractional integrals on spaces of homogeneous type with finite measure

by

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Abstract. In this paper we prove the continuity of fractional integrals acting on non-homogeneous function spaces defined on spaces of homogeneous type with finite measure. A definition of the molecules which are used in the H^p theory is given. Results are proved for L^p, H^p, BMO, and Lipschitz spaces.

1. Definitions and statement results. We shall follow the definitions and notation of [GV], and we assume that the reader is familiar with that paper. In the present paper (X,δ,μ) is a normal space of homogeneous type of finite measure and of order γ, $0<\gamma\le 1$. In this case the diameter of the space is finite and will be denoted by D. We may and will assume that $\mu(X)=1$.

For the sake of completeness we will repeat the definitions of normality and order. (X,δ,μ) is a *normal space* if there are positive constants A_1 and A_2 such that for all x in X

$$A_1 r \le \mu(\mathcal{B}_r(x)) \quad \text{if } 0<r\le R_x, \tag{1.1}$$

$$\mu(\mathcal{B}_r(x)) \le A_2 r \quad \text{if } r> r_x, \tag{1.2}$$

where $\mathcal{B}_r(x)$ denotes the ball of radius r and center x, and where $R_x=\inf\{r>0:\mathcal{B}_r(x)=X\}$, and $r_x=\sup\{r>0:\mathcal{B}_r(x)=\{x\}\}$ if $\mu(\{x\})\ne 0$, and $r_x=0$ if $\mu(\{x\})=0$. Note that $\sup\{R_x : x\in X\}=D<\infty$, that (1.1) holds for $0<r<2D$ with constant $A_1/2$ instead of A_1, and that (1.2) holds for $r=r_x$ if $r_x\ne 0$. The space (X,δ,μ) is said to be of *order* γ, $0<\gamma\le 1$, if there exists a positive constant M such that for every x, y, and z in X,

$$|\delta(x,z)-\delta(y,z)| \le M\delta(x,y)^\gamma(\max\{\delta(x,z),\delta(y,z)\})^{1-\gamma}.$$

We will consider on (X,δ,μ) the following function spaces and norms. If $0<p\le\infty$ then L^p and $\|f\|_p$ have their usual meaning. For a measurable

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