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On the distribution function of the majorant of ergodic means

by

LASHA EPREMIDZE (Tbilisi)

Abstract. Let T be a measure-preserving ergodic transformation of a measure space $(X,\mathbb{S},\mu)$ and, for $f\in L(X)$, let

$$f^*=\sup_N\frac{1}{N}\sum_{m=0}^{N-1}f\circ T^m\,.$$

In this paper we mainly investigate the question of whether

$$\text{(i)}\qquad \int_a^\infty\Big|\mu(f^*>t)-\frac{1}{t}\int_{(f^*>t)}f\,d\mu\Big|\,dt<\infty$$

and whether

$$\text{(ii)}\qquad \int_a^\infty\Big|\mu(f^*>t)-\frac{1}{t}\int_{(f>t)}f\,d\mu\Big|\,dt<\infty$$

for some $a>0$. It is proved that (i) holds for every $f\ge 0$. (ii) holds if $f\ge 0$ and $f\log\log(f+3)\in L(X)$ or if $\mu(X)=1$ and the random variables $f\circ T^m$ are independent.

Related inequalities are proved. Some examples and counterexamples are constructed. Several known results are obtained as corollaries.

1. Introduction. Let T be a measure-preserving ergodic transformation of a measure space $(X,\mathbb{S},\mu)$. For a function $f:X\to\mathbb{R}$ the majorant of ergodic means will be denoted by f^*,

$$f^*(x)=\sup_N\frac{1}{N}\sum_{m=0}^{N-1}f\circ T^m(x)\,,\qquad x\in X\,.$$

In this paper we shall investigate the problem of isolating the principal part of the distribution function $t\to\mu(f^*>t)$, $t>0$ (we write $(f^*>t)=\{x:f^*(x)>t\}$), which is formulated as follows: Find a function $\gamma(t)$, $t>0$, of the simplest form possible such that the difference $\mu(f^*>t)-\gamma(t)$ be integrable in the neighbourhood of infinity.

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It is well known that the Maximal Ergodic Theorem asserts that for any integrable function f ($f \in L(X)$),

$$\mu(f^* > t) \leq \frac{1}{t} \int\limits_{(f^*>t)} f\, d\mu\,, \qquad t > 0\,. \tag{1}$$

(At present there exist several different proofs of this theorem; see [12], [6], [9], [7] or Theorem 1.2 of this paper.)

Naturally, a question arises if one can take

$$\gamma(t) = \frac{1}{t} \int\limits_{(f^*>t)} f\, d\mu\,. \tag{2}$$

It is easy to show that without any restrictions on T this is not true, i.e. the inequality

$$\int\limits_1^\infty \left|\mu(f^* > t) - \frac{1}{t} \int\limits_{(f^*>t)} f\, d\mu\right| dt < \infty \tag{3}$$

may be false. Indeed, if $T = \mathrm{id}_X$, $f \in L(X)$, $f \geq 1$ and

$$\int\limits_X f \log f\, d\mu = \int\limits_1^\infty \frac{dt}{t} \int\limits_{(f>t)} f\, d\mu = \infty\,,$$

then $f^* = f$ and (3) fails.

O. Tsereteli has raised the following question: Assuming that T is ergodic (i.e. $\mu(A \triangle T^{-1}(A)) = 0 \Rightarrow \mu(A) = 0$ or $\mu(X \setminus A) = 0$, $A \in \mathbb{S}$), can it be asserted that γ has the form (2), i.e. (3) holds at least for nonnegative $f \in L(X)$?

Theorem 1.8 of this paper gives an affirmative answer to this question (see Corollary 1.9). It is also shown that (3) is not true for an arbitrary integrable f (see Example 1.12).

In some cases it is possible to improve the form of the principal part so as to make it independent of the function f^* and we can claim that

$$\gamma(t) = \frac{1}{t} \int\limits_{(f>t)} f\, d\mu \tag{4}$$

(Example 1.17 shows that this is impossible for an arbitrary nonnegative integrable f). Such cases are: (a) $f \in L \log\log(L+3)$ (i.e. $|f| \log\log(|f|+3) \in L(X)$), $f \geq 0$ and T is an arbitrary ergodic transformation (see Theorem 1.15); (b) $(X, \mathbb{S}, \mu)$ is a probability space, $f \in L(X)$ and ergodic T are such that $f, f \circ T, f \circ T^2, \ldots$ form a sequence of independent random variables (see Theorem 2.1; we emphasize that f is not necessarily nonnegative in that theorem).

The proof of the main result of Section 1 (Theorem 1.4) uses the filling scheme method (see [11], 3.7). Recently we have been made aware that for finite measure spaces Theorem 1.4 may be deduced from the proof of Lemma 1 in [13]. Nevertheless we have not made any change in our proof. This makes our paper self-contained.

The following notation will be used: $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$; $T^{-m}(A) = (T^m)^{-1}(A)$, $A \in \mathbb{S}$. $L_+(X)$ is the class of $\mathbb{S}$-measurable functions for which $\int_X f^+\, d\mu < \infty$. For $f \in L_+(X)$ let

$$E(f) = \frac{1}{\mu(X)} \int\limits_X f\, d\mu\,,$$

where it is assumed that $E(f) = 0$ if $\mu(X) = \infty$ and $E(f) = -\infty$ if $\mu(X) < \infty$ and $\int_X f\, d\mu = -\infty$.

1. The proof of the main results is based on

LEMMA 1.1. *Let $T : X \to X$ be a transformation of a space X, and let $f : X \to \mathbb{R}$. Define*

$$f_0 = f, \qquad f_{n+1} = -f_n^- + f_n^+ \circ T\,, \qquad n = 0, 1, \ldots\,,$$

$$E_n = \Big\{x : \max_{0 \leq k \leq n} \sum_{m=0}^{k} f \circ T^m(x) > 0\Big\}\,. \tag{5}$$

Then

$$(f_n > 0) \subset E_n\,, \tag{6}$$

$$E_n \subset (f_n \geq 0)\,, \qquad n = 0, 1, \ldots \tag{7}$$

Proof. Since

$$\sum_{m=0}^{N} f_{n+1} \circ T^m = \sum_{m=0}^{N} -f_n^- \circ T^m + \sum_{m=0}^{N} f_n^+ \circ T^{m+1}$$
$$= -f_n^- + \sum_{m=1}^{N} f_n \circ T^m + f_n^+ \circ T^{N+1}\,,$$

we have

$$f_n \circ T^{N+1}(x) > 0 \;\Rightarrow\; \sum_{m=0}^{N} f_{n+1} \circ T^m(x) \leq \sum_{m=0}^{N+1} f_n \circ T^m(x)\,; \tag{8}$$

$$f_n(x) < 0 \;\Rightarrow\; \sum_{m=0}^{N} f_{n+1} \circ T^m(x) \geq \sum_{m=0}^{N+1} f_n \circ T^m(x)\,, \tag{9}$$
$$n = 0, 1, \ldots,\; N = 0, 1, \ldots$$

From (5) it follows that $f_{n+1}(x)>0 \Rightarrow f_n\circ T(x)>0$ and $f_{n+1}(x)<0 \Rightarrow f_n(x)<0$, $n=0,1,\ldots$ Hence if $f_n(x)>0$, then $f_{n-1}\circ T(x)$, $f_{n-2}\circ T^2(x)$, $\ldots, f\circ T^n(x)$ are also positive and by (8) we have

$$0<f_n(x)\le f_{n-1}(x)+f_{n-1}\circ T(x)\le\ldots\le\sum_{m=0}^{n} f\circ T^m(x).$$

Thus $x\in E_n$ and (6) holds.

If $f_n(x)<0$, then $f_{n-1}(x), f_{n-2}(x),\ldots,f(x)$ are also negative and by (9) we have

$$\sum_{m=0}^{N} f\circ T^m(x)\le\sum_{m=0}^{N-1} f_1\circ T^m(x)\le\ldots\le f_N(x)<0$$

for each $N\le n$. Thus $x\notin E_n$ and (7) holds.

Lemma 1.1 is proved.

To make the paper complete and also to illustrate the application of Lemma 1.1 we give the proof of the Maximal Ergodic Theorem.

THEOREM 1.2. *Let T be a measure-preserving transformation of $(X,\mathbb{S},\mu)$, and let $f\in L_+(X)$ and $t>0$. Then*

$$\mu(f^*>t)\le\frac{1}{t}\int_{(f^*>t)} f\,d\mu. \tag{10}$$

LEMMA 1.3 (Maximal Ergodic Theorem). *Let T be a measure-preserving transformation of $(X,\mathbb{S},\mu)$ and $f\in L_+(X)$. Then*

$$\int_{(f^*>0)} f\,d\mu\ge 0. \tag{11}$$

Proof. Let f_n and E_n, $n=0,1,\ldots$, be the same as in Lemma 1.1. Since $E_0\subset E_1\subset\ldots$ and $\bigcup_{n=0}^\infty E_n=(f^*>0)$, to prove (11) it is sufficient to show that

$$\int_{E_n} f\,d\mu\ge 0 \tag{12}$$

for each $n\ge 0$.

(6) implies that $(f_k>0)\subset E_k\subset E_n$ for $k\le n$. Hence

$$\int_{E_n} f_k\,d\mu=\int_{E_n}-f_k^-\,d\mu+\int_X f_k^+\,d\mu=\int_{E_n}-f_k^-\,d\mu+\int_X f_k^+\circ T\,d\mu$$
$$\ge\int_{E_n} f_{k+1}\,d\mu,\qquad k=0,1,\ldots,n-1.$$

Thus $\int_{E_n} f\,d\mu\ge\int_{E_n} f_n\,d\mu$, and (7) yields (12).

Proof of Theorem 1.2. Since $f-t\in L_+(X)$,

$$\int_{((f-t)^*>0)}(f-t)\,d\mu\ge 0,$$

by Lemma 1.3. But $((f-t)^*>0)=(f^*>t)$, and we obtain (10).

THEOREM 1.4. *Let T be a measure-preserving ergodic transformation of a measure space $(X,\mathbb{S},\mu)$, let $f\in L_+(X)$, and suppose $\mu(f^*>0)<\infty$ and*

$$\int_X f\,d\mu\le 0. \tag{13}$$

Then

$$\int_{(f>0)\cup(f^*\circ T>0)} f\,d\mu\le 0. \tag{14}$$

To prove this theorem we need

LEMMA 1.5. *Let T be a measure-preserving ergodic transformation of a measure space $(X,\mathbb{S},\mu)$. Let A be a measurable set of finite measure which is not of full measure (i.e. $A\in\mathbb{S}$, $\mu(X\setminus A)>0$, $\mu(A)<\infty$) and let $(B_n)_{n=0}^\infty$ be a sequence of measurable sets such that $B_0\subset A$ and $B_{n+1}\subset(T^{-1}(B_n)\cap A)$, $n=0,1,\ldots$ Then*

$$\lim_{n\to\infty}\mu(B_n)=0.$$

Proof. Let $A_n=\bigcap_{m=0}^n T^{-m}(A)$, $n=0,1,\ldots$, $A_\infty=\bigcap_{m=0}^\infty T^{-m}(A)$. Then $\mu(A_0)<\infty$ and $A_n\downarrow A_\infty$. Hence $\mu(A_n)\to\mu(A_\infty)$. Since $\mu(A_\infty)<\infty$ and $T^{-1}(A_\infty)=\bigcap_{m=1}^\infty T^{-m}(A)\supset A_\infty$, we have $\mu(T^{-1}(A_\infty)\triangle A_\infty)=\mu(T^{-1}(A_\infty))-\mu(A_\infty)=0$. Since T is ergodic, we have either $\mu(A_\infty)=0$ or $\mu(X\setminus A_\infty)=0$. But the latter is impossible because $X\setminus A_\infty\supset X\setminus A$ and $\mu(X\setminus A)>0$ by assumption. Thus $\mu(A_\infty)=0$ and $\mu(A_n)\to 0$.

Now it remains to show that $B_n\subset A_n$, $n=0,1,\ldots$ This can be easily done by induction: $B_0\subset A_0$ and if $B_n\subset A_n$, then $B_{n+1}\subset(T^{-1}(B_n)\cap A)\subset(T^{-1}(A_n)\cap A)=A_{n+1}$, $n=0,1,\ldots$

Proof of Theorem 1.4. First note that

$$(f^*>0)\subset((f>0)\cup(f^*\circ T>0))\equiv G. \tag{15}$$

Indeed, if $x\in(f^*>0)$, then there exists N such that $\sum_{m=0}^N f\circ T^m(x)>0$. Thus either $f(x)>0$ or $\sum_{m=1}^N f\circ T^m(x)=\sum_{m=0}^{N-1} f\circ T^m(T(x))>0$. Hence $x\in G$.

If $(f^*>0)$ is a full measure set, then (14) follows from (15) and (13). Assume now that $\mu(X\setminus(f^*>0))>0$. Let f_n, $n=0,1,\ldots$, be the functions defined by (5). Let us show that

$$\int_G f_n\,d\mu = \int_G f\,d\mu\,, \qquad n=0,1,\ldots \tag{16}$$

Since

$$(f_n>0)\subset(f^*>0) \tag{17}$$

(see (6)), we have $(f_n\circ T>0)\subset(f^*\circ T>0)$. Thus

$$((f_n>0)\cup(f_n\circ T>0))\subset G\,. \tag{18}$$

This implies

$$\int_G f_n\,d\mu = \int_G -f_n^-\,d\mu + \int_X f_n^+\,d\mu = \int_G -f_n^-\,d\mu + \int_X f_n^+\circ T\,d\mu$$
$$= \int_G -f_n^-\,d\mu + \int_G f_n^+\circ T\,d\mu = \int_G f_{n+1}\,d\mu\,, \qquad n=0,1,\ldots,$$

and since $f_0=f$, (16) holds.

We shall now show that

$$\lim_{n\to\infty}\int_X f_n^+\,d\mu = 0\,. \tag{19}$$

This will complete the proof on account of (16).

Since

$$f_{n+1}^+\le f_n^+\circ T \tag{20}$$

(see (5)) and (17) holds, we have

$$(f_{n+1}>0)\subset T^{-1}(f_n>0)\cap(f^*>0)\,, \qquad n=0,1,\ldots$$

Hence the sets $A=(f^*>0)$ and $B_n=(f_n>0)$, $n=0,1,\ldots$, satisfy the conditions of Lemma 1.5 and thus

$$\lim_{n\to\infty}\mu(f_n>0)=0\,. \tag{21}$$

(20) implies that $f_n^+\le f^+\circ T^n$. Therefore for any $t>0$ we have

$$\int_{(f_n>0)} f_n^+\,d\mu \le \int_{(f_n>0)} f^+\circ T^n\,d\mu$$
$$= \Big(\int_{(f_n>0)\cap(f^+\circ T^n>t)} + \int_{(f_n>0)\cap(f^+\circ T^n\le t)}\Big) f^+\circ T^n\,d\mu$$
$$\le \int_{(f>t)} f\,d\mu + t\mu(f_n>0)\,, \qquad n=0,1,\ldots$$

Thus, taking into account (21), we conclude that (19) holds.

Theorem 1.4 is proved.

Remark 1.6. The condition $\mu(f^*>0)<\infty$ is necessary in Theorem 1.4. Indeed, let X be the set of all integers, and let μ be the counting measure on it. Let $T(m)=m+1$, $m\in\mathbb{Z}$, and define

$$f(m)=\begin{cases}1 & \text{if } m=-1,\\ -1 & \text{if } m=0,\\ 0 & \text{if } m\ne -1,0.\end{cases}$$

Then $f^*(m)=-1/m>0$ for $m<0$ and $f^*(m)=0$ for $m\ge 0$, $G=(f>0)\cup(f^*\circ T>0)=\{m\in\mathbb{Z}: m<0\}$ and $\int_G f\,d\mu=1$.

Theorem 1.7 (cf. [13]). *Let T be a measure-preserving ergodic transformation of a measure space $(X,\mathbb{S},\mu)$ and let $f\in L_+(X)$. Then for $t>\max(E(f),0)$*

$$\mu((f>t)\cup(f^*\circ T>t))\ge\frac{1}{t}\int_{(f>t)\cup(f^*\circ T>t)} f\,d\mu \tag{22}$$

and consequently

$$\mu(f^*>t)\ge\frac{1}{t}\int_{(f>t)\cup(f^*\circ T>t)} f\,d\mu-\mu(f>t)\,. \tag{23}$$

Proof. If $t>\max(E(f),0)$, then the function $f-t$ satisfies the conditions of Theorem 1.4 ($\mu((f-t)^*>0)=\mu(f^*>t)<\infty$ by Theorem 1.2). Applying inequality (14) to this function, taking into account that $(f^*\circ T>t)=T^{-1}(f^*>t)=T^{-1}((f-t)^*>0)=((f-t)^*\circ T>0)$ and performing elementary transformations, we obtain (22).

(23) immediately follows from (22) since

$$\mu((f>t)\cup(f^*\circ T>t))\le\mu(f>t)+\mu(f^*\circ T>t)=\mu(f>t)+\mu(f^*>t)\,.$$

Theorem 1.7 is proved.

Since

$$(f^*>t)\subset((f>t)\cup(f^*\circ T>t)) \tag{24}$$

(see (15)), (23) implies the validity of

Theorem 1.8. *Let T be a measure-preserving ergodic transformation of a measure space $(X,\mathbb{S},\mu)$, and let $f\in L(X)$ and $f\ge 0$. Then*

$$\mu(f^*>t)\ge\frac{1}{t}\int_{(f^*>t)} f\,d\mu-\mu(f>t)\,, \qquad t>E(f)\,. \tag{25}$$

Corollary 1.9. *Under the assumptions of Theorem 1.8 the function (2) is the principal part of the distribution function $t\to\mu(f^*>t)$.*

Proof. It follows from (10) and (25) that

$$\int_{E(f)}^{\infty}\left|\frac{1}{t}\int_{(f^*>t)} f\,d\mu-\mu(f^*>t)\right|dt\le\int_{E(f)}^{\infty}\mu(f>t)\le\int_X f\,d\mu<\infty. \tag{26}$$

Remark 1.10. It follows from (26) that if $\mu(X)=\infty$ (in this case $E(f)=0$), then the function (2) is the principal part of $t\to\mu(f^*>t)$ also in the neighbourhood of zero.

From (26) immediately follows

COROLLARY 1.11. *Under the assumptions of Theorem* 1.8,

$$\int_X f^*\,d\mu+\int_X f\,d\mu\ge\int_{E(f)}^{\infty}\frac{dt}{t}\int_{(f>t)} f\,d\mu$$
$$=\begin{cases}\displaystyle\int_X f\log^+\frac{f}{E(f)}\,d\mu & \text{if } \mu(X)<\infty,\\ \infty & \text{if } \mu(X)=\infty,\ \mu(f>0)>0.\end{cases}$$

This corollary contains the known result about the integrability of f^*:

THEOREM (Ornstein [10]). *Let T be a measure-preserving ergodic transformation of $(X,\mathbb{S},\mu)$, let $f\ge 0$ and suppose $f^*\in L(X)$. Then*:

(i) *if $\mu(X)<\infty$ then $f\in L\log^+L$;*
(ii) *if $\mu(X)=\infty$ then $f=0$ almost everywhere.*

EXAMPLE 1.12. The following example shows that if f is not nonnegative, then the functions $t\to t^{-1}\int_{(f^*>t)} f\,d\mu$ and $t\to t^{-1}\int_{(f>t)\cup(f^*\circ T>t)} f\,d\mu$ (see (23)) may not be the principal parts of $t\to\mu(f^*>t)$.

Let X be the interval $[0,1)$, ε any irrational number in $(0,1/2)$, $T(x)=x+\varepsilon \pmod 1$, $x\in[0,1)$, and f an integrable function with the properties: $f(x)<0$ and $f(x+\varepsilon)=-\frac{1}{2}f(x)$ for $x\in[0,\varepsilon)$, $f(x)=0$ for $x\in[2\varepsilon,1)$ and

$$\int_1^{\infty}\frac{dt}{t}\int_{(f>t)} f\,d\mu=\infty. \tag{27}$$

Then $(f^*)^+=f^+$ and

$$\int_{(f>t)\cup(f^*\circ T>t)} f\,d\mu=\Big(\int_{(f>t)}+\int_{T^{-1}(f>t)}\Big)f\,d\mu=-\int_{(f>t)} f\,d\mu.$$

Hence $\int_1^{\infty}\mu(f^*>t)\,dt<\infty$ and

$$\int_1^{\infty}\frac{dt}{t}\int_{(f^*>t)} f\,d\mu=\infty,\qquad\int_1^{\infty}\frac{dt}{t}\int_{(f>t)\cup(f^*\circ T>t)} f\,d\mu=-\infty.$$

Theorem 1.7 may be used to prove inequality (28) below which yields a theorem of Vakhania and Davis.

THEOREM 1.13. *Let T be a measure-preserving ergodic transformation of a measure space $(X,\mathbb{S},\mu)$, let $f\in L(X)$ and define*

$$f^{\#}(x)=\sup_N\frac{1}{N}\Big|\sum_{m=0}^{N-1} f\circ T^m(x)\Big|.$$

Then

$$2\mu(f^{\#}>t)+\mu(|f|>t)\ge\frac{1}{t}\Big|\int_{(|f|>t)} f\,d\mu\Big| \tag{28}$$

for $t>|E(f)|$.

Proof. Since

$$\int_{(f>t)\cup(f^*\circ T>t)} f\,d\mu=\Big(\int_{(f>t)}+\int_{(f^*\circ T>t)\cap(f<-t)}+\int_{(f^*\circ T>t)\cap(|f|\le t)}\Big)f\,d\mu$$
$$\ge\int_{(f>t)} f\,d\mu+\int_{(f<-t)} f\,d\mu-t\mu(f^*\circ T>t)$$
$$=\int_{(|f|>t)} f\,d\mu-t\mu(f^*>t),$$

from (23) we obtain

$$2\mu(f^*>t)+\mu(f>t)\ge\frac{1}{t}\int_{(|f|>t)} f\,d\mu$$

for $t>\max(E(f),0)$. Applying this to $-f$ we have

$$2\mu((-f)^*>t)+\mu(f<-t)\ge-\frac{1}{t}\int_{(|f|>t)} f\,d\mu$$

for $t>\max(-E(f),0)$. Hence

$$2\max(\mu(f^*>t),\mu((-f)^*>t))+\mu(|f|>t)\ge\frac{1}{t}\Big|\int_{(|f|>t)} f\,d\mu\Big|$$

for $t>|E(f)|$ and, since $f^{\#}=\max(f^*,(-f)^*)$, (28) holds.

COROLLARY 1.14. *Under the assumptions of Theorem* 1.13,

$$2\int_X f^{\#}\,d\mu+\int_X |f|\,d\mu\ge\int_{|E(f)|}^{\infty}\frac{dt}{t}\Big|\int_{(|f|>t)} f\,d\mu\Big|.$$

This corollary contains the known result about the integrability of $f^{\#}$:

THEOREM (Vakhania [15], [16], Davis [2]). *Let T be a measure-preserving ergodic transformation of $(X,\mathbb{S},\mu)$, let $f\in L(X)$ and suppose $f^{\#}\in L(X)$. Then*

(i) *if $\mu(X)<\infty$, then*

$$\int_1^\infty \frac{dt}{t}\Big|\int_{(|f|>t)} f\,d\mu\Big|<\infty;$$

(ii) *if $\mu(X)=\infty$, then*

$$\int_X f\,d\mu=0,\qquad \int_0^\infty \frac{dt}{t}\Big|\int_{(|f|>t)} f\,d\mu\Big|<\infty.$$

THEOREM 1.15. *Let T be a measure-preserving ergodic transformation of a measure space $(X,\mathbb{S},\mu)$, let $f\ge 0$ and suppose $f\in L\log\log(L+3)$. Then the function (4) is the principal part of the distribution function $t\to\mu(f^*>t)$, i.e.*

$$\int_1^\infty \Big|\mu(f^*>t)-\frac{1}{t}\int_{(f>t)} f\,d\mu\Big|\,dt<\infty. \tag{29}$$

Proof. Since

$$\begin{aligned}\int_1^\infty \frac{dt}{t}\Big(\int_{(f^*>t)}-\int_{(f>t)}\Big)f\,d\mu&=\int_1^\infty\frac{dt}{t}\int_{(f^*>t)\cap(f\le t)} f\,d\mu\\&=\int_{(f^*>1)\cap(f\le1)} f\,d\mu\int_1^{f^*}\frac{dt}{t}+\int_{(f>1)} f\,d\mu\int_f^{f^*}\frac{dt}{t}\\&=\int_{(f^*>1)\cap(f\le1)} f\log f^*\,d\mu+\int_{(f>1)} f\log\frac{f^*}{f}\,d\mu\end{aligned}$$

and

$$\begin{aligned}\int_{(f^*>1)\cap(f\le1)} f\log f^*\,d\mu&\le\int_{(f^*>1)}\log f^*\,d\mu=\int_0^\infty\mu(f^*>e^t)\,dt\\&\le\int_0^\infty e^{-t}\,dt\int_{(f^*>e^t)} f\,d\mu\le\int_{(f^*>1)} f\,d\mu\int_0^\infty e^{-t}\,dt\\&=\int_{(f^*>1)} f\,d\mu<\infty\end{aligned}$$

(inequality (10) has been used here), we have

$$\int_1^\infty\frac{dt}{t}\Big(\int_{(f^*>t)}-\int_{(f>t)}\Big)f\,d\mu<\infty$$

if and only if

$$\int_{(f>1)} f\log\frac{f^*}{f}\,d\mu<\infty. \tag{30}$$

Hence, taking into account (26), we obtain

$$\int_1^\infty\Big|\mu(f^*>t)-\frac1t\int_{(f>t)} f\,d\mu\Big|\,dt<\infty\Leftrightarrow\int_{(f>1)} f\log\frac{f^*}{f}\,d\mu<\infty. \tag{31}$$

But (30) follows from a theorem of O. Tsereteli (see Theorem 2.1 in [14]): if $\mu(\Omega)<\infty$, $f>0$, $f\in L\log\log(L+3)(\Omega)$, $\varphi\ge f$ and $\mu(\varphi>t)\le K/t$, $K>0$, $t>0$, then $\int_\Omega f\log(\varphi/f)\,d\mu<\infty$ (we have to assume that $\Omega=(f>1)$).

Remark 1.16. For infinite measure spaces one can similarly prove that the function (4) is the principal part of $t\to\mu(f^*>t)$ in the neighbourhood of zero, i.e.

$$\int_0^1\Big|\mu(f^*>t)-\frac1t\int_{(f>t)} f\,d\mu\Big|\,dt<\infty,$$

if $f\ge0$, $f\in L(X)$ and $\int_{(f>0)} f\log\log(1/f+3)\,d\mu<\infty$, using Remark 1.10 and the following theorem of O. Tsereteli (personal communication): if $\mu(X)=\infty$, $f\ge0$, $\int_{(f>0)} f\log\log(1/f+3)\,d\mu<\infty$, $\varphi\ge f$ and $\mu(\varphi>t)\le K/t$, $K>0$, $t>0$, then $\int_{(f>0)} f\log(\varphi/f)\,d\mu<\infty$.

EXAMPLE 1.17. For an arbitrary function $f\ge0$ from $L(X)$, (29) may not hold. Let $\varphi\ge1$ be a decreasing continuous integrable function on $(0,1]$ such that

$$\int_0^1\varphi(x)\log\frac{(1/x)\int_0^x\varphi(y)\,dy}{\varphi(x)}\,dx=\infty$$

(an example of such a function is given e.g. in [14], p. 72). Setting $a_{n,k}=\varphi(1-k/n)$, $n=1,2,\ldots$, $k=0,1,\ldots,n-1$, we have

$$\frac1n\sum_{k=0}^{n-1}a_{n,k}\to\int_0^1\varphi(x)\,dx<\infty,$$

$$\frac1n\sum_{k=0}^{n-1}a_{n,k}\log\frac{\sum_{i=k}^{n-1}a_{n,i}}{(n-k)a_{n,k}}\to\infty\quad\text{as } n\to\infty.$$

Let $(\varepsilon_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty}\varepsilon_n = 1$ and

$$\sum_{n=1}^{\infty}\varepsilon_n\sum_{k=0}^{n-1}a_{n,k}<\infty\,,$$

$$\sum_{n=1}^{\infty}\varepsilon_n\sum_{k=0}^{n-1}a_{n,k}\log\frac{\sum_{i=k}^{n-1}a_{n,i}}{(n-k)a_{n,k}}=\infty\,.$$

Write $\Delta_1=(0,\varepsilon_1]$ and $\Delta_n=(\sum_{k=1}^{n-1}\varepsilon_k,\sum_{k=1}^{n}\varepsilon_k]$, $n=2,3,\ldots$

Let X be the set $\bigcup_{n=1}^{\infty}\Delta_n\times\{0,1,\ldots,n-1\}$, let μ be the measure whose restriction to the Borel σ-algebra of $\Delta_n\times\{k\}$, $n=1,2,\ldots$, $0\le k<n$, is the Lebesgue measure and let $T:X\to X$ be defined by $T(x,k)=(x,k+1)$ if $x\in\Delta_n$ and $0\le k<n-1$, and $T(x,n-1)=(T'(x),0)$, $x\in\Delta_n$, where T' is an arbitrary Lebesgue measure-preserving and ergodic transformation of $(0,1]$. Then T is μ-measure-preserving and ergodic (see [11], p. 56). Let $f:X\to\mathbb{R}$ be defined by $f(x,k)=a_{n,k}$, $x\in\Delta_n$, $0\le k<n$. Then

$$\int_X f\,d\mu=\sum_{n=1}^{\infty}\varepsilon_n\sum_{k=0}^{n-1}a_{n,k}<\infty$$

and since

$$f^*(x,k)\ge\frac{1}{n-k}\sum_{i=k}^{n-1}a_{n,i}\,,\qquad x\in\Delta_n\,,\ 0\le k<n\,,$$

we have

$$\int_{(f>1)}f\log\frac{f^*}{f}\,d\mu\ge\sum_{n=1}^{\infty}\varepsilon_n\sum_{k=0}^{n-1}a_{n,k}\log\frac{\sum_{i=k}^{n-1}a_{n,i}}{(n-k)a_{n,k}}=\infty\,.$$

Thus, on account of (31), inequality (29) fails for f.

2. Let $y_1,y_2,\ldots$ be a sequence of independent random variables on a probability space $(\Omega,\mathcal{F},P)$. It is well known (see, e.g., [1], I, §1) that from the viewpoint of measure theory it can be identified with the sequence of coordinate functions $\pi_1,\pi_2,\ldots$ on $\mathbb{R}^\infty=\prod_{n=1}^{\infty}\mathbb{R}_n$, $\mathbb{R}_n=\mathbb{R}$, $n=1,2,\ldots$, defined by $\pi_m(x)=x_m$, $x=(x_1,x_2,\ldots)$, $m=1,2,\ldots$, when the measure μ on the σ-algebra $\mathbb{B}^\infty=\bigoplus_{n=1}^{\infty}\mathbb{B}_n$, $\mathbb{B}_n=\mathbb{B}$ ($\mathbb{B}$ is the Borel σ-algebra of $\mathbb{R}$) is defined as the product $\bigoplus_{n=1}^{\infty}\mu_n$, where $\mu_n:\mathbb{B}\to\mathbb{R}$ is the measure $\mu_n(e)=P(y_n^{-1}(e))$, $e\in\mathbb{B}$. (In this case $\pi_1,\pi_2,\ldots$ are independent random variables and for any set of integers $1\le i_1<\ldots<i_n$ the joint distributions of $y_{i_1},\ldots,y_{i_n}$ and $\pi_{i_1},\ldots,\pi_{i_n}$ coincide.) Note that $\pi_m=\pi_1\circ T^m$, $m=1,2,\ldots$, where T is the shift operator defined on $\mathbb{R}^\infty$ by

$$T(x_1,x_2,x_3,\ldots)=(x_2,x_3,\ldots)\,. \tag{32}$$

At the same time, if the random variables have the same distribution, then $\mu_n=\mu_1$, $n=1,2,\ldots$, and T is a μ-measure-preserving ergodic transformation. Consequently, all the theorems proved in §1 are true for this particular situation. Hence they are also true for a sequence of independent equidistributed random variables $y_1,y_2,\ldots$ if $(X,\mathbb{S},\mu)$, f and f^* are everywhere in the assumptions replaced by $(\Omega,\mathcal{F},P)$, y_1 and $y^*=\sup_N(1/N)\sum_{m=1}^{N}y_m$, respectively.

In particular, if $y_1,y_2,\ldots$ is a sequence of independent equidistributed random variables with finite expectation and if $y_1\ge0$, then the principal part of the distribution function $t\to P(y^*>t)$, $t>0$, is $t\to t^{-1}\int_{(y^*>t)}y_1\,dP$. However, in the case under consideration it is possible to obtain much more exact results.

THEOREM 2.1. *In a probability space $(\Omega,\mathcal{F},P)$, let $y_1,y_2,\ldots$ be a sequence of independent equidistributed random variables with finite expectation. Then*

$$\Big|P(y^*>t)-\frac{1}{t}\int_{(y_1>t)}y_1\,dP\Big|\le\frac{2E(|y_1|)}{t^2}\int_{(y_1>t)}y_1\,dP+P(y_1>t) \tag{33}$$

for $t>2E(y_1^+)$.

Proof. By the reasoning given at the beginning of this section, we may assume that $(\Omega,\mathcal{F},P)=(\mathbb{R}^\infty,\bigoplus_{n=1}^{\infty}\mathbb{B}_n,\mu)$ and $y_1,y_2,\ldots$ are the coordinate functions $\pi_1,\pi_2,\ldots$ For convenience π_1 will be denoted by f. Then $\pi_m=f\circ T^{m-1}$, $m=1,2,\ldots$, where T is the measure-preserving ergodic shift (32) and $\pi^*=f^*$.

If $h\in L(\mathbb{R}^\infty)$ only depends on the first coordinate and $A\in\bigoplus_{n=1}^{\infty}\mathbb{B}_n$, then h and $\mathbf{1}_{T^{-1}(A)}$ are certainly independent and therefore

$$\int_{T^{-1}(A)}h\,d\mu=\int_{\mathbb{R}^\infty}\mathbf{1}_{T^{-1}(A)}h\,d\mu=\mu(A)\int_{\mathbb{R}^\infty}h\,d\mu\,. \tag{34}$$

Taking into account (1), (24) and (34), we have

$$\begin{aligned}\mu(f^*>t)&\le\frac{1}{t}\int_{(f^*>t)}f\,d\mu\le\frac{1}{t}\int_{(f>t)\cup(f^*\circ T>t)}f^+\,d\mu\\&\le\frac{1}{t}\int_{(f>t)}f\,d\mu+\frac{1}{t}\int_{(f^*\circ T>t)}f^+\,d\mu\\&=\frac{1}{t}\int_{(f>t)}f\,d\mu+\frac{1}{t}\mu(f^*>t)\int_{\mathbb{R}^\infty}f^+\,d\mu\,,\quad t>0\,.\end{aligned} \tag{35}$$

If $t > 2\int_{\mathbb{R}^\infty} f^+\,d\mu$, then it follows from (35) that

$$\mu(f^* > t) \le \frac{2}{t} \int\limits_{(f>t)} f\,d\mu\,. \tag{36}$$

Also,

$$\begin{aligned}\int\limits_{(f>t)\cup(f^*\circ T>t)} f\,d\mu &= \int\limits_{(f>t)} f\,d\mu + \int\limits_{(f^*\circ T>t)} \mathbf{1}_{(f\le t)} f\,d\mu \\ &= \int\limits_{(f>t)} f\,d\mu + \mu(f^*>t) \int\limits_{\mathbb{R}^\infty} \mathbf{1}_{(f\le t)} f\,d\mu\,.\end{aligned}$$

Thus, on account of (23), for $t > \max(E(f),0)$ we have

$$\mu(f^* > t) \ge \frac{1}{t} \int\limits_{(f>t)} f\,d\mu - \frac{E(|f|)}{t}\mu(f^*>t) - \mu(f>t)\,. \tag{37}$$

It follows from (35) and (37) that

$$\Big|\mu(f^* > t) - \frac{1}{t} \int\limits_{(f>t)} f\,d\mu\Big| \le \frac{E(|f|)}{t}\mu(f^*>t) + \mu(f>t)\,, \tag{38}$$

for $t > \max(E(f),0)$. Substituting (36) in the right side of (38), we obtain

$$\Big|\mu(f^* > t) - \frac{1}{t} \int\limits_{(f>t)} f\,d\mu\Big| \le \frac{2E(|f|)}{t^2} \int\limits_{(f>t)} f\,d\mu + \mu(f>t)\,.$$

Thus (33) holds and Theorem 2.1 is proved.

COROLLARY 2.2. *Under the assumptions of Theorem* 2.1 *the principal part of the distribution function* $t \to P(y^* > t)$ *is* $t \to (1/t)\int_{(y_1>t)} y_1\,dP$.

Proof. Just note that the right side of (33) as a function of t is integrable in the neighbourhood of infinity. Hence

$$\int\limits_1^\infty \Big|P(y^* > t) - \frac{1}{t} \int\limits_{(y_1>t)} y_1\,dP\Big|\,dt < \infty\,. \tag{39}$$

Inequality (39) contains the known result about the integrability of y^*:

THEOREM (Doob [4], Davis [3], McCabe–Shepp [8]). *Let* $y_1, y_2, \ldots$ *be a sequence of independent equidistributed random variables with finite expectation. Then* $y^* \in L$ *if and only if* $y_1^+ \in L\log^+ L$.

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