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A note on topologically nilpotent Banach algebras

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Abstract. A Banach algebra A is said to be topologically nilpotent if $\sup\{\|x_1 \ldots \ldots x_n\|^{1/n} : x_i \in A, \|x_i\| \le 1\ (1 \le i \le n)\}$ tends to 0 as $n \to \infty$. We continue the study of topologically nilpotent algebras which was started in [2].

A Banach algebra A is said to be *nilpotent* if $A^n = 0$ for some n, i.e. if $x_1 \ldots x_n = 0$ for all $x_1, \ldots, x_n \in A$. An element $x \in A$ is said to be *nilpotent* if $x^n = 0$ for some n. The algebra is said to be *nil* if every element is nilpotent.

By [4], a Banach algebra is nil if and only if it is nilpotent.

The topological versions of these notions were studied in [2] and [3]. Following the notation there, define for a Banach algebra A and a positive integer n the quantities

$$N_A(n) = \{\sup \|x_1 \ldots x_n\|^{1/n} : x_i \in A,\ \|x_i\| = 1\ (i = 1, \ldots, n)\}$$

and

$$S_A(n) = \{\sup \|x^n\|^{1/n} : x \in A,\ \|x\| = 1\}.$$

Note that we do not assume the existence of the unit element in A and, if A has a unit element e, then we do not suppose $\|e\| = 1$ (otherwise these notions would become trivial).

A Banach algebra is said to be *topologically nilpotent* if $\lim_{n\to\infty} N_A(n) = 0$ and *uniformly topologically nil* if $\lim_{n\to\infty} S_A(n) = 0$. These two notions are equivalent for commutative Banach algebras (see [2]).

Our first result shows that this is not true for noncommutative Banach algebras.

We denote by $\mathbb{N}$ the set of all positive integers.

THEOREM 1. *There exists a Banach algebra A such that $N_A(n) = 1$ $(n \in \mathbb{N})$ and $\lim_{n\to\infty} S_A(n) = 0$.*

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Proof. Let S be the free semigroup with generators $x_1, x_2, \ldots$ satisfying the relations $x_i x_j = 0$ $(j \neq i+1)$. Let A be the ℓ^1 algebra over S, i.e. A is the set of all formal series

(1) $$y = \sum_{i<j} \alpha_{i,j} x_i x_{i+1} \ldots x_{j-1}$$

with complex coefficients $\alpha_{i,j}$ such that $\|y\| = \sum_{i<j} |\alpha_{i,j}| < \infty$. With algebraic operations defined in the natural way A becomes a Banach algebra. Clearly $\|x_1 \ldots x_n\| = 1$ for every n so that $N_A(n) = 1$ $(n \in \mathbb{N})$.

Let y be of the form (1) such that $\|y\| = \sum_{i<j} |\alpha_{i,j}| \le 1$. Then

$$\|y^n\| = \Big\| \sum_{i_0<i_1<\ldots<i_n} \alpha_{i_0,i_1}\alpha_{i_1,i_2}\ldots\alpha_{i_{n-1},i_n} x_{i_0}\ldots x_{i_n-1} \Big\| \le \sum_{i_0<i_1<\ldots<i_n} |\alpha_{i_0,i_1}||\alpha_{i_1,i_2}|\ldots|\alpha_{i_{n-1},i_n}|.$$

Further,

$$1 \ge \Big(\sum_{i<j} |\alpha_{i,j}|\Big)^n \ge n! \sum_{i_0<i_1<\ldots<i_n} |\alpha_{i_0,i_1}||\alpha_{i_1,i_2}|\ldots|\alpha_{i_{n-1},i_n}|$$

so that $\|y^n\| \le 1/n!$. Thus $S_A(n) \le (1/n!)^{1/n} \le 3/n \to 0$. (In fact, it is not difficult, but rather tedious, to show that $S_A(n) = 1/n$ and the supremum is attained for $y = n^{-1}\sum_{i=1}^n x_i$.)

Remark 1. In the previous example, consider the element

$$y = \sum_{k=1}^{\infty} \frac{1}{k\ln(k+1)} x_k .$$

Then

$$y^n = \frac{1}{n!\ln 2\ldots\ln(n+1)} x_1\ldots x_n + \ldots,$$

so that

$$\|y^n\| \ge \frac{1}{n!\ln 2\ldots\ln(n+1)} \ge \frac{1}{n!n^2}.$$

If $z = y/\|y\|$ then $\|z\| = 1$ and

$$\|z^n\|^{1/n} \ge \Big(\frac{1}{n!n^2\|y\|^n}\Big)^{1/n} \sim \frac{e}{n\|y\|}$$

as $n \to \infty$. Thus, although our example has $\|x^n\|^{1/n} = O(1/n)$ for each x, it does not have $\|x^n\|^{1/n} = o(1/n)$ for all x.

It is possible to improve the preceding example to get $\|x^n\|^{1/n} = o(1/n)$, by setting $x_{2^k-1}x_{2^k} = 0$ $(k = 1, 2, \ldots)$, i.e. A is now the set of all formal series

(2) $$y = \sum_{k=0}^{\infty} \sum_{2^k \le i<j \le 2^{k+1}} \alpha_{i,j} x_i \ldots x_{j-1} .$$

Again $N_A(n) = 1$ and $S_A(n) = 1/n$ $(n = 1, 2, \ldots)$ but we have $\|y^n\|^{1/n} = o(1/n)$ for every $y \in A$. Indeed, if y is given by (2) with $\|y\| = \sum_{i<j} |\alpha_{i,j}| = 1$, we have, for $n \ge 2^k$,

$$\|y^n\|^{1/n} \le \Big(\sum_{2^k \le i_1<\ldots<i_n} |\alpha_{i_1,i_2}|\ldots|\alpha_{i_{n-1},i_n}|\Big)^{1/n} \le (1/n!)^{1/n} \sum_{2^k \le i<j} |\alpha_{i,j}| = o(1/n).$$

Remark 2. By the Nagata–Higman theorem [5], every product of 2^n-1 elements of a complex algebra can be expressed as a linear combination of nth powers. When studying topologically nilpotent algebras it is very important to have some estimate of the corresponding coefficients. In [2] the following quantitative version of the Nagata–Higman theorem was proved:

Let A be a complex algebra. Suppose that B is an absolutely convex, multiplicatively closed subset of A (i.e. $\lambda_1 x_1 + \lambda_2 x_2 \in B$ and $x_1 x_2 \in B$ whenever $x_1, x_2 \in B$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ with $|\lambda_1| + |\lambda_2| \le 1$). Denote by P_n the absolutely convex hull of the set of all nth powers of elements of B. Then every product of $2^n - 1$ (or more) elements of B is contained in $n^{\alpha 2^n} \cdot P_n$, where $\alpha = 2.26$.

Denote by K_n the smallest coefficient with this property. By the above mentioned result we have $K_n \le n^{\alpha 2^n}$. The example constructed in Theorem 1 gives a lower bound for K_n: Taking B to be the unit ball in A we have $K_n \ge n^n$. There is still a big gap, however, between the lower and upper bounds.

Proposition 2. *Let A be a Banach algebra. Then the limit $\lim_{n\to\infty} N_A(n)$ exists and $\lim_{n\to\infty} N_A(n) = \inf_{n\in\mathbb{N}} N_A(n)$.*

Proof. We have

$$N_A(m+n)^{m+n} = \sup\{\|x_1\ldots x_{m+n}\| : x_i \in A,\ \|x_i\| = 1\ (i = 1, \ldots, m+n)\} \le N_A(m)^m \cdot N_A(n)^n$$

for every $m, n \in \mathbb{N}$. It is well known that this implies the conclusion.

The limit given in the previous proposition will be denoted by $N_A = \lim_{n\to\infty} N_A(n)$.

Theorem 3. *Let A be a Banach algebra. Then there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset A$ with $\|x_i\| \le 1$ $(i \in \mathbb{N})$ such that $\limsup_{n\to\infty} \|x_1\ldots x_n\|^{1/n} = N_A$.*

Proof. Clearly $\limsup_{n\to\infty}\|x_1\ldots x_n\|^{1/n}\le N_A$ for every sequence $\{x_i\}_{i=1}^\infty\subset A$ with $\|x_i\|\le 1$ $(i\in\mathbb{N})$. Therefore it is sufficient to show that $\limsup_{n\to\infty}\|x_1\ldots x_n\|^{1/n}\ge N_A$ for some sequence $\{x_i\}_{i=1}^\infty$. This is clear if $N_A=0$. In the following we shall suppose that $N_A>0$.

For $r>0$, let $A_r=\{x\in A:\|x\|\le r\}$. We consider the space $X=(A_1)^{\mathbb{N}}$ with the product metric

$$d((x_i),(y_i))=\sum_{i=1}^\infty 2^{-i}\frac{\|x_i-y_i\|}{1+\|x_i-y_i\|}\,. \tag{3}$$

Then (X,d) is a complete metric space. For each $\delta\in(0,1)$ and $k\in\mathbb{N}$, consider the set

$$X_{k,\delta}=\{(x_i)\in X:\|x_1\ldots x_n\|^{1/n}>(1-\delta)N_A \text{ for some } n\ge k\}\,.$$

Then $X_{k,\delta}$ is open in X, since $X_{k,\delta}=\bigcup_{n\ge k}Y_{n,\delta}$, where $Y_{n,\delta}=\{(x_i)\in X:\|x_1\ldots x_n\|^{1/n}>(1-\delta)N_A\}$, which is clearly open in X.

Our desired result will follow from the statement

$$\bigcap_{k=1}^\infty X_{k,1/k}\ne\emptyset\,,$$

which, in turn, follows from Baire's Category Theorem if we can show that each of the sets $X_{k,\delta}$ is dense in X. Now,

$$X=\overline{\bigcup_{r<1}(A_r)^{\mathbb{N}}}\,,$$

so it suffices to show that if $x=(x_i)\in(A_r)^{\mathbb{N}}$, $k\in\mathbb{N}$, $\delta\in(0,1)$ and $0<\varepsilon<1-r$, then there exists $y\in X_{k,\delta}$ with $d(x,y)<\varepsilon$.

We observe from (3) that if $2^{-m+1}<\varepsilon$ then

$$\|x_i-y_i\|<\varepsilon/2\ (1\le i\le m)\ \Rightarrow\ d(x,y)<\varepsilon\,. \tag{4}$$

Choose such an m and then choose $l\ge\max\{m,k\}$ such that

$$(\varepsilon^m/2^{m+1})^{1/l}\ge(1-\delta)^{1/2}\,.$$

Now find $u_1,\ldots,u_l\in A_1$ with

$$\|u_1\ldots u_l\|^{1/l}\ge(1-\delta)^{1/2}N_A(l)\ge(1-\delta)^{1/2}N_A\,.$$

Let $\varepsilon_r=e^{2\pi ir/m}\varepsilon$ $(1\le r\le m)$ and $\varepsilon_0=0$. Set

$$z_r=\Big(x_1+\frac{\varepsilon_r}{2}u_1\Big)\ldots\Big(x_m+\frac{\varepsilon_r}{2}u_m\Big)u_{m+1}\ldots u_l\quad(r=0,1,\ldots,m)\,.$$

Then

$$\Big\|\sum_{r=1}^m(z_r-z_0)\Big\|=\frac{m\varepsilon^m}{2^m}\|u_1\ldots u_l\|\ge\frac{m\varepsilon^m(1-\delta)^{l/2}N_A^l}{2^m}\,,$$

so that there exists $s\in\{0,\ldots,m\}$ such that

$$\|z_s\|\ge\frac{\varepsilon^m(1-\delta)^{l/2}N_A^l}{2^{m+1}}\ge(1-\delta)^lN_A^l\,.$$

Set

$$y_i=\begin{cases}x_i+(\varepsilon_s/2)u_i & (1\le i\le m),\\ u_i & (m<i\le l),\\ x_i & (i>l).\end{cases}$$

Then $y=(y_i)\in(A_1)^{\mathbb{N}}$ and $d(x,y)<\varepsilon$ by (4). Moreover,

$$\|y_1\ldots y_l\|^{1/l}=\|z_s\|^{1/l}\ge(1-\delta)N_A$$

and $l\ge k$, so $y\in X_{k,\delta}$. This finishes the proof.

Corollary 4. *Let A be a Banach algebra. The following statements are equivalent:*

(1) *A is topologically nilpotent,*

(2) *$\lim_{n\to\infty}\|x_1\ldots x_n\|^{1/n}=0$ for every sequence $\{x_i\}_{i=1}^\infty\subset A$ with $\|x_i\|\le1$ $(i\in\mathbb{N})$.*

Remark 3. The analogous statement for $S_A(n)$ is not true even for commutative Banach algebras.

Consider the free semigroup S with generators $x_1,x_2,\ldots$ satisfying the relations $x_ix_j=0$ $(i\ne j)$ and $x_i^{i+1}=0$ $(i=1,2,\ldots)$. Let A be the ℓ^1 algebra over S, i.e. A is the set of all formal series

$$y=\sum_{i=1}^\infty\sum_{j=1}^i\alpha_{i,j}x_i^j \tag{5}$$

where $\alpha_{i,j}$ are complex numbers and $\|y\|=\sum_{i,j}|\alpha_{i,j}|<\infty$. Then $\|x_n^n\|=\|x_n\|=1$ so that $S_A(n)=1$ for every positive integer n.

On the other hand, $\|y^n\|^{1/n}\to0$ for every $y\in A$. Indeed, if y is given by (5) then

$$\|y^n\|^{1/n}=\Big\|\sum_{i=n}^\infty\Big(\sum_{j=1}^i\alpha_{i,j}x_i^j\Big)^n\Big\|^{1/n}$$
$$\le\Big\|\sum_{i=n}^\infty\sum_{j=1}^i\alpha_{i,j}x_i^j\Big\|=\sum_{i=n}^\infty\sum_{j=1}^i|\alpha_{i,j}|\to0\quad\text{as } n\to\infty\,.$$

We say that two Banach algebras A,B are *isomorphic* if there exists a bijective homomorphism $f:A\to B$ such that both f and f^{-1} are continuous.

Theorem 5. *Let A be a Banach algebra.*

(1) *If $N_A=0$ and B is a Banach algebra isomorphic to A then $N_B=0$.*

(2) *If $N_A > 0$ and $r \in (0,1)$ then there exists a Banach algebra B isomorphic to A such that $N_B = r$.*

Proof. (1) Let $(A, \|\cdot\|)$ be a Banach algebra with $N_A = 0$. We may assume without loss of generality that $B = (A, \|\cdot\|')$ where $\|\cdot\|'$ is an algebra norm on A equivalent to the original norm $\|\cdot\|$. This means that there is a constant $K > 0$ such that $K^{-1}\|a\| \le \|a\|' \le K\|a\|$ for every $a \in A$.

Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in A$, $\|x_i\|' \le 1$ $(i = 1, \ldots, n)$. Then

$$\|x_1 \ldots x_n\|' \le K\|x_1 \ldots x_n\| = K^{n+1}\left\|\frac{x_1}{K} \cdots \frac{x_n}{K}\right\| \le K^{n+1} N_A(n)^n$$

and $N_B(n) \le K^{(n+1)/n} N_A(n)$. Hence $N_B = \lim_{n\to\infty} N_B(n) \le K \cdot N_A = 0$.

(2) Let $N_A > 0$ and $r \in (0,1)$. Find $n \in \mathbb{N}$ such that $N_A/N_A(n) \ge r$. Define $M = \{x \in A : \|x\| \le N_A(n)^{-1}\}$. If $x_1, \ldots, x_n \in M$ then $\|x_1 \ldots x_n\| \le 1$. If $k \in \mathbb{N}$, $k = k_1 n + z$ with $k_1, z \in \mathbb{N} \cup \{0\}$, $0 \le z \le n-1$ and $x_1, \ldots, x_k \in M$ then

$$\|x_1 \ldots x_k\| \le \|x_1 \ldots x_n\| \ldots \|x_{(k_1-1)n+1} \ldots x_{k_1 n}\| \cdot \|x_{k_1 n+1}\| \ldots \|x_{k_1 n+z}\| \le N_A(n)^{1-n}.$$

Thus M generates a bounded semigroup and by [1], p. 18, there exists an equivalent algebra norm $\|\cdot\|'$ on A such that $\|x\|' \le 1$ for every $x \in M$. We have

$$K^{-1}\|a\| \le \|a\|' \le K\|a\| \quad (a \in A)$$

for some constant $K > 0$ (in fact, $K = N_A(n)^{1-n}$).

Let $A' = (A, \|\cdot\|')$ and $k \in \mathbb{N}$, $k \ge n$. We find $x_1, \ldots, x_k \in A$, $\|x_i\| \le 1$ $(i = 1, \ldots, k)$ such that $\|x_1 \ldots x_k\| \ge \frac{1}{2} N_A(k)^k$. Then $N_A(n)^{-1} x_i \in M$ so that

$$\|N_A(n)^{-1} x_i\|' \le 1 \quad (i = 1, \ldots, k)$$

and

$$\left\|\frac{x_1}{N_A(n)} \cdots \frac{x_k}{N_A(n)}\right\|' \ge K^{-1}\left\|\frac{x_1}{N_A(n)} \cdots \frac{x_k}{N_A(n)}\right\| \ge \frac{1}{2K} \frac{N_A(k)^k}{N_A(n)^k}.$$

Thus

$$N_{A'}(k) \ge \left(\frac{1}{2K}\right)^{1/k} \frac{N_A(k)}{N_A(n)}$$

and $N_{A'} = \lim_{k\to\infty} N_{A'}(k) \ge N_A/N_A(n) \ge r$.

Set $C = N_{A'}/r$ and consider the norm $\|\cdot\|''$ on A defined by

$$\|a\|'' = C\|a\|' \quad (a \in A).$$

Clearly $\|\cdot\|''$ is equivalent to the original norm $\|\cdot\|$. Let $B = (A, \|\cdot\|'')$ and $k \in \mathbb{N}$. If $x_1, \ldots, x_k \in A$, $\|x_i\|'' \le 1$ $(i = 1, \ldots, k)$ then

$$\|x_1 \ldots x_k\|'' = C\|x_1 \ldots x_k\|' = C^{1-k}\|(Cx_1) \ldots (Cx_k)\|'$$

where $\|Cx_i\|' \le 1$. Thus $N_B(k) = C^{(1-k)/k} N_{A'}(k)$ $(k \in \mathbb{N})$ and $N_B = C^{-1} N_{A'} = r$.

The following problem remains open: if A is a Banach algebra with $N_A > 0$, does there exist a Banach algebra B isomorphic to A with $N_B = 1$?

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