[6] F. Cobos and J. Peetre, *Interpolation of compact operators: the multidimensional case*, Proc. London Math. Soc. 63 (1991), 371–400.

[7] M. Cwikel and S. Janson, *Real and complex interpolation methods for finite and infinite families of Banach spaces*, Adv. in Math. 66 (1987), 234–290.

[8] D. L. Fernandez, *Interpolation of 2^n Banach spaces*, Studia Math. 45 (1979), 175–201.

[9] —, *Interpolation of 2^d Banach spaces and the Calderón spaces $X(E)$*, Proc. London Math. Soc. 56 (1988), 143–162.

[10] C. Foiaş and J. L. Lions, *Sur certains théorèmes d'interpolation*, Acta Sci. Math. (Szeged) 22 (1961), 269–282.

[11] S. Janson, P. Nilsson, J. Peetre and M. Zafran, *Notes on Wolff's note on interpolation spaces*, Proc. London Math. Soc. 48 (1984), 283–299.

[12] G. Sparr, *Interpolation of several Banach spaces*, Ann. Mat. Pura Appl. 99 (1974), 247–316.

[13] T. Wolff, *A note on interpolation spaces*, in: Proc. Conf. on Harmonic Analysis, Minneapolis 1981, Lecture Notes in Math. 908, Springer, Berlin 1982, 199–204.

DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD AUTÓNOMA DE MADRID
28049 MADRID, SPAIN

FACULTAD DE MATEMÁTICAS
UNIVERSIDAD DE MURCIA
CAMPUS DE ESPINARDO
30071 MURCIA, SPAIN

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Fréchet spaces of continuous vector-valued functions: Complementability in dual Fréchet spaces and injectivity

by

P. DOMAŃSKI and L. DREWNOWSKI (Poznań)

Abstract. Fréchet spaces of strongly, weakly and weak*-continuous Fréchet space valued functions are considered. Complete solutions are given to the problems of their injectivity or embeddability as complemented subspaces in dual Fréchet spaces.

1. Introduction. There is a famous conjecture that every injective Banach (or Fréchet) space is isomorphic to the space of scalar-valued continuous functions over an extremally disconnected topological space [19, p. 269]. It might seem that there is a chance to find some essentially new examples of injective Fréchet spaces by considering the spaces of vector-valued continuous functions. Unfortunately, as should be clear from the results of the present paper, this is not so, at least in the case of spaces of strongly or weakly continuous functions. (The situation is not so clear, however, for the spaces of weak*-continuous functions.) For Banach spaces of strongly continuous vector functions this was observed earlier by Cembranos [6].

We now briefly describe the contents of our paper.

Let X be a Hausdorff topological space that is locally compact and *hemicompact* (i.e., has a fundamental sequence of compact sets); we will call such spaces *LCH-spaces* for short. Let E be a Fréchet space or, when

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E'-valued functions are considered, a locally convex space whose strong dual E' is Fréchet. In this paper we deal with the Fréchet spaces

$$C(X,E),\quad C(X,E,w)\quad \text{and}\quad C(X,E',w').$$

They consist of all functions $f : X \to E$ (or E') that are continuous with respect to the strong, or weak, or weak* topology in E (or E'), respectively, and each of these spaces is equipped with the topology of uniform convergence on compact sets in X, relative to the *strong topology* in E (or E'). Of course, function spaces of the above types can be considered in a much greater generality, and we do so from time to time.

Our main purpose is to determine when the spaces listed above are injective or, more generally, complemented in dual Fréchet spaces, and we accomplish that in §§5 and 4, respectively. Note that, unlike for the spaces of scalar functions (see Fact (F) in §2), these two problems are not equivalent.

It is clear that if X is a discrete (hence countable) LCH-space, then $C(X) \simeq \omega$ and, in general,

$$C(X,E) = C(X,E,w) \simeq E^{\mathbb{N}},\qquad C(X,E',w') \simeq (E')^{\mathbb{N}}.$$

In this case the questions mentioned above have easy and obvious answers. So, to avoid trivialities, we will assume throughout that X is nondiscrete (in which case it contains an infinite compact subset).

One way of seeing that a space is not injective (or complemented in a dual space) is by exhibiting in it a complemented copy of c_0. In §3 we prove a general result (Theorem 1) on the existence of complemented copies of c_0 in $C(X,E)$ (here X need not be an LCH-space). It plays an important role in the proofs of Theorems 5 and 6. We also give a similar result for $C(X,E,w)$ (Theorem 2).

In §4 we give complete answers to the question of when our spaces can be embedded as complemented subspaces into dual Fréchet spaces: For $C(X,E',w')$ the answer is "if and only if $C(X)$ is injective" (Theorem 3); for $C(X,E,w)$ it is "if and only if $C(X)$ is injective and E is reflexive" (Theorem 4); lastly, for $C(X,E)$ it is "if and only if $C(X)$ is injective and E is a Fréchet–Montel space" (Theorem 5).

Finally, in §5, we consider the problem of injectivity. It turns out that the first two spaces are injective only in "trivial cases", i.e., when $C(X)$ is injective and $E \simeq \omega$ (Theorem 6). As for the third space, we show that if E is barreled and bornological, then $C(X,E',w')$ is injective if and only if both $C(X)$ and E' are injective (Theorem 7). Thus, as far as injectivity is concerned, it is the spaces $C(X,E',w')$ that seem to be the proper vector-valued analogues of the spaces $C(X)$ (comp. [3]–[5]).

It is worth while to note that at least for $C(X,E,w)$ all the above results seem to be new also in the Banach space case.

2. Preliminaries.

In general, our terminology and notation concerning locally convex spaces agrees with [15] and [21]. In what follows, if E is a locally convex space (lcs), then E' always denotes the strong dual $(E',\beta(E',E))$ of E, *operator* means a continuous linear operator, and ω is the space of all scalar sequences with the pointwise convergence topology. If E and F are lcs, then $L_\beta(E,F)$ denotes the space of all operators from E into F equipped with the topology of uniform convergence on bounded sets in E.

If X is a topological space and E is a lcs, then $C(X,E)$ denotes the space of all continuous functions from X to E, with the compact-open topology. We write $C(X)$ when E is the space of scalars. For the definitions of $C(X,E,w)$ and $C(X,E',w')$, see §1. It should be noted that if E is a barreled lcs, then

$$C(X,E',w') \simeq L_\beta(E,C(X))$$

via the isomorphism $J : C(X,E',w') \to L_\beta(E,C(X))$ given by the formula

$$[J(f)(x)](t) := \langle f(t), x\rangle.$$

Moreover, if X is the so-called *k-space* (see [13]) or, in particular, if X is locally compact, then $C(X,E')$ (resp., $C(X,E',w)$) corresponds in this isomorphism to the subspace of $L_\beta(E,C(X))$ consisting of all operators mapping bounded sets into relatively compact sets (resp., relatively weakly compact sets).

We now collect some basic facts concerning injective Fréchet spaces, Fréchet quojections, and dual Fréchet spaces that will be needed in this paper.

A Fréchet space E is called

— *injective* if it is complemented in every Fréchet space containing it;

— a *quojection* if each of its quotients with a continuous norm is a Banach space;

— a *dual Fréchet space* if it is isomorphic to the strong dual of a bornological space.

More information on injective spaces can be found in [9], and for quojections see [1], [9] and [18]. We will freely use the well-known fact that complemented subspaces and countable products of injective Fréchet spaces are again injective. Given a Fréchet space E, let $\widehat{E} := (E',\mathrm{bor})'$, where bor means the associated bornological topology on E'. Of course, every Fréchet space E is canonically embedded in $\widehat{E}$.

FACTS. (A) *Injective Fréchet spaces, and Fréchet $C(X)$ spaces, are quojections* [9, Prop. 1.4(c)].

(B) *A Fréchet space E is isomorphic to a complemented subspace of a dual Fréchet space if and only if E is complemented in $\widehat{E}$* [9, Prop. 1.3].

(C) *Every operator $S : E \to F$ between Fréchet spaces E and F can be extended to an operator $\widehat{S} : \widehat{E} \to \widehat{F}$.*

(D) *c_0 is not complemented in any dual Fréchet space containing it.*

(E) *No injective Fréchet space contains a complemented copy of c_0.*

(F) *If X is an LCH-space, then $C(X)$ is injective if and only if $C(X)$ is complemented in a dual Fréchet space* [9, Cor. 5.8 and 8.8]).

Note. (D) follows from (B) because $\widehat{c}_0 = l_\infty$, and c_0 is not complemented in l_∞. Since every injective Fréchet space E must be complemented in the dual Fréchet space $\widehat{E}$, (E) is a particular case of (D). (In fact, (E) is obvious: otherwise, c_0 would be injective.)

It should be noted that (B) and (C) remain true for all lcs E, F if we modify the definition of $\widehat{E}$ by taking as bor the bornological topology associated with the so-called equicontinuous bornology in E' and write in (B) "strong dual of a bornological space" instead of "dual Fréchet space".

3. Complemented copies of c_0. The theorem proved below concerns the existence of complemented copies of c_0 in spaces $C(X,E)$, and is an extension of the results of Cembranos [6] and Freniche [14, Cor. 2.5], where E was assumed to be a Banach space. Our proof of this result is based on a similar idea to that in [6] and [14], and makes essential use of the following Josefson–Nissenzweig type theorem for Fréchet spaces, recently obtained by J. Bonet, M. Lindström and M. Valdivia [2].

THEOREM (B-L-V). *A Fréchet space E is Montel if (and only if) every $\sigma(E',E)$-null sequence in E' is also $\beta(E',E)$-null.*

THEOREM 1. *Let X be a completely regular Hausdorff topological space containing an infinite compact set, and let E be a non-Montel Fréchet space. Then $C(X,E)$ contains a complemented copy of c_0.*

Proof. By Theorem (B-L-V), there is an absolutely convex bounded set $B \subset E$ and a weak*-null sequence of functionals $(u_n) \subset E'$ satisfying

$$\sup_{x\in B} |\langle u_n, x\rangle| > 1 \quad \text{for every } n \in \mathbb{N}.$$

Pick a sequence (x_n) in B so that $\langle u_n, x_n\rangle = 1$ $(n \in \mathbb{N})$. Next, let K be an infinite compact subset of X. Then we can find a sequence of continuous functions $\varphi_n : X \to [0,1]$ having pairwise disjoint supports and such that

$$\varphi_n(t_n) = 1 \quad \text{for some } t_n \in K \ (n \in \mathbb{N}).$$

Finally, we define linear maps $J : c_0 \to C(X,E)$ and $P : C(X,E) \to c_0$ by

$$J((a_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} a_n x_n \varphi_n(t) \quad \text{and} \quad P(f) = (\langle u_n, f(t_n)\rangle)_{n\in\mathbb{N}}.$$

These definitions make sense: Since the sequence (x_n) is bounded, the series in the first formula converges uniformly and so defines a continuous function; and $P(f)$ is a null sequence because the weak*-null (hence equicontinuous) sequence (u_n) converges uniformly to zero on the compact set $f(K)$ containing the sequence $(f(t_n))$. It is easily seen that both J and P are continuous and $P \circ J = \mathrm{id}_{c_0}$. It follows that J is an isomorphic embedding and that $J \circ P$ is a projection from $C(X,E)$ onto its subspace $J(c_0) \simeq c_0$. ∎

With the same proof, but using Lemma 1 given below instead of Theorem (B-L-V), one obtains the following analogue of Theorem 1 for $C(X,E,w)$.

THEOREM 2. *Let X be a completely regular Hausdorff topological space containing an infinite compact set, and let E be a locally convex space containing an isomorphic copy of l_1. Then $C(X,E,w)$ contains a complemented copy of c_0.*

LEMMA 1. *Let E be a lcs containing an isomorphic copy E_0 of l_1. Then there is an equicontinuous sequence $(u_n)_{n\in\mathbb{N}}$ of linear functionals on E which tends uniformly to zero on every weakly compact set in E, but for some bounded set B in E one has*

$$\sup_{x\in B} |\langle u_n, x\rangle| = 1 \quad \textit{for each } n \in \mathbb{N}.$$

Proof. The identity map $i : l_1 \to l_2$ is a 1-absolutely summing operator [15, 20.4.1], hence it factorizes through the Banach space $C(K)$ for some compact space K [15, 19.6.4]. Since its range is reflexive, it in fact factorizes through $C(K)''$. By the injectivity of the latter space, and identifying l_1 with a subspace of l_∞, we can prove that i factorizes through l_∞. Then, since l_∞ is also injective, identifying l_1 with E_0, we can extend i to an operator $j : E \to l_2$ factorizing through l_∞. Composing j with the identity map from l_2 into c_0, we get an operator $T : E \to c_0$ which factorizes through l_∞.

It is well known that every operator from l_∞ to c_0 is weakly compact and maps weakly compact sets to strongly compact sets (see [15, 20.7.8] and [11, Lemma 2.6]). Hence T has these properties as well. From this, using the well-known description of compact sets in c_0, it follows that if e'_n $(n \in \mathbb{N})$ are the coefficient functionals on c_0, then the functionals $u_n := e'_n \circ T$ and B, the isomorphic image in E_0 of the closed unit ball in l_1, are as required. ∎

Remarks. (a) If, in Theorem 1 (resp., Theorem 2), the space X is hemicompact and each compact set K in X is contained in a compact set L so that $L \setminus K$ is infinite, then it can be shown that $C(X,E)$ (resp., $C(X,E,w)$) contains a complemented copy of $c_0^{\mathbb{N}}$.

(b) Only a weaker version of Theorem 1 is needed in the proof of Theorem 6 in §5, namely, for infinite-dimensional Fréchet quojections $E \not\simeq \omega$.

For this reason we include here a simple proof of Theorem (B-L-V) for this special case: We have to find a weak*-null sequence in E' which is not uniformly convergent on some bounded subset of E. By assumption, there is an infinite-dimensional Banach space F and a quotient map $q: E \to F$. By [7], we may assume that there is a bounded set B such that $q(B)$ coincides with the unit ball in F. Now, apply the original Josefson–Nissenzweig Theorem for Banach spaces [8, Ch. XII] to find a normalized weak*-null sequence (v_n) in F', and define $u_n := v_n \circ q$. The sequence (u_n) is obviously weak*-null in E', and $\sup_{x\in B} |\langle u_n, x\rangle| = 1$ for all $n \in \mathbb{N}$.

(c) As we learned from G. Metafune, the celebrated Rosenthal l_1-Theorem [8, Ch. XI] extends to Fréchet spaces: Every bounded sequence in any Fréchet space E has a subsequence that is either weak-Cauchy or equivalent to the unit vector basis of l_1. (This can be easily seen by embedding E in a countable product of Banach spaces and applying the original Rosenthal Theorem.) Using this one can readily verify that, for Fréchet spaces, the existence of a sequence (u_n) as specified in Lemma 1 is in fact equivalent to the containment of an isomorphic copy of l_1. Another characterization of a similar type for Banach spaces is proved in [12, Th. 2].

(d) The factorization of the identity operator $i: l_1 \to c_0$ through l_∞ (shown in the proof of Lemma 1) is certainly well known.

4. Complementability in dual Fréchet spaces. When are the Fréchet spaces of vector-valued functions considered in this paper isomorphic to complemented subspaces of dual Fréchet spaces? In this section we give a complete answer to this question for each of the spaces $C(X,E',w')$, $C(X,E,w)$, and $C(X,E)$ (in this order).

In the proof of our first result here, concerning $C(X,E',w')$, we will need the following lemma. We are grateful to J. Bonet for pointing it out to us. We recall once again our convention that if E is a lcs, then E' denotes the strong dual of E. If Γ is a set, then $l_\infty(\Gamma,E')$ denotes the space of all equicontinuous families $(u_\gamma)_{\gamma\in\Gamma}$ in E', equipped with the topology of uniform convergence on Γ.

LEMMA 2. *Let E be a (DF)-space, and let Γ be any set. Then*

$$l_\infty(\Gamma,E') \simeq F',$$

where $F = l_1(\Gamma)\otimes_\pi E$ is a (DF)-space which is also bornological whenever E is bornological.

Proof. We have

$$l_1(\Gamma)\otimes_\pi E \subset l_1(\Gamma)\widehat{\otimes}_\pi E,$$

where $l_1(\Gamma)\otimes_\pi E$ is (DF) [15, 15.6.2], dense and "large" in $l_1(\Gamma)\widehat{\otimes}_\pi E$; that is, every bounded set in $l_1(\Gamma)\widehat{\otimes}_\pi E$ lies in the closure of a bounded set in $l_1(\Gamma)\otimes_\pi E$ (see [20, 8.3.23(a)]). Hence

$$(l_1(\Gamma)\otimes_\pi E)' = (l_1(\Gamma)\widehat{\otimes}_\pi E)' = L_\beta(E, l_\infty(\Gamma)) = l_\infty(\Gamma,E').$$

If E is also bornological, then so is $l_1(\Gamma)\otimes_\pi E$ (see [15, 15.6.8]). ■

THEOREM 3. *Assume that X is an LCH-space, and that a locally convex space E is barreled, bornological and its strong dual E' is a Fréchet space. Then $C(X,E',w')$ is isomorphic to a complemented subspace of a dual Fréchet space if and only if $C(X)$ is injective.*

Proof. Since $C(X)$ is a complemented subspace of $C(X,E',w')$, the necessity part follows from Fact (F).

Sufficiency. Since E is barreled, we have (see §2)

$$C(X,E',w') \simeq L_\beta(E,C(X)).$$

Let $C(X)$ be injective. Then $C(X)$ is complemented in $l_\infty(\Gamma)^{\mathbb{N}}$ for some set Γ; hence $L_\beta(E,C(X))$ is complemented in

$$L_\beta(E,l_\infty(\Gamma)^{\mathbb{N}}) \simeq L_\beta(E,l_\infty(\Gamma))^{\mathbb{N}} \simeq l_\infty(\Gamma,E')^{\mathbb{N}} \simeq (F')^{\mathbb{N}} \simeq \Big(\bigoplus_{k\in\mathbb{N}} F\Big)',$$

where the last but one isomorphism holds by Lemma 2 with the bornological (DF)-space $F = l_1(\Gamma)\otimes_\pi E$. In consequence, $C(X,E',w')$ is isomorphic to a complemented subspace in a dual Fréchet space. ■

In the necessity part of the proof of our next theorem, concerning $C(X,E,w)$, we apply the method developed by the authors in [10]. We first introduce some additional notation, and prove a lemma.

Given a lcs E, we denote by $\kappa(E,w)$ the space of all E-valued relatively weakly compact sequences $(x_n)_{n\in\mathbb{N}}$, equipped with the topology of uniform convergence on $\mathbb{N}$. If (x_n) is any sequence in E, then for each $n\in\mathbb{N}$ we define

$$\widehat{x}_n := (0,\ldots,0,x_n,0,\ldots) \in \kappa(E,w),$$

where x_n occupies the nth slot. The unit vector sequence in l_∞ is denoted by (e_n).

LEMMA 3. *Let E be an arbitrary complete lcs. If (x_n) is a bounded sequence in E for which there is an operator $T: l_\infty \to \kappa(E,w)$ with*

$$T(e_n) = \widehat{x}_n \quad \textit{for all } n\in\mathbb{N},$$

then $(x_n)\in\kappa(E,w)$.

Proof. We first prove the lemma for $E = l_\infty$. Although in this case the argument is quite similar to that used in the proof of [10, Prop. 2], for the sake of completeness we give the details.

Let P_n be the nth coordinate projection $(z_j) \mapsto \widehat{z}_n$ in $\kappa(l_\infty, w)$, and let R_n denote the nth coordinate projection in l_∞. Since

$$T \circ R_n e_j = P_n \circ T e_j = \widehat{x}_n \quad \text{if } j = n, \quad \text{and } = \widehat{0} \text{ otherwise},$$

all the operators

$$U_n := T \circ R_n - P_n \circ T : l_\infty \to \widehat{X}_n := \{ (z_k) \in \kappa(l_\infty, w) : z_k = 0 \text{ for } k \neq n \} \simeq l_\infty$$

vanish on c_0. By [16, Prop. 4], for every operator $A : l_\infty \to l_\infty$ vanishing on c_0 there is an infinite set $M \subset \mathbb{N}$ with $A|l_\infty(M) = 0$, where $l_\infty(M) := \{(x_k) \in l_\infty : x_k = 0 \text{ if } k \notin M\} \simeq l_\infty$. Applying this result, we can find an infinite set $M \subset \mathbb{N}$ such that

$$U_n|l_\infty(M) = 0 \quad \text{for every } n \in \mathbb{N}.$$

It follows, in particular, that

$$T(1_M) = (y_n)_{n\in\mathbb{N}} \in \kappa(l_\infty, w), \quad \text{where } y_n = x_n \text{ for } n \in M, \text{ and } = 0 \text{ otherwise}.$$

Hence the subsequence $(x_n)_{n\in M}$ is relatively weakly compact, and modifying the above reasoning we can see that every subsequence of $(x_n)_{n\in\mathbb{N}}$ contains a further relatively weakly compact subsequence. By [21, IV.11.2], $(x_n)_{n\in\mathbb{N}}$ belongs to $\kappa(l_\infty, w)$.

Now, let E be an arbitrary complete lcs and let E_0 be the closed linear span of $(x_n)_{n\in\mathbb{N}}$. As E_0 is separable, there is an isomorphic embedding S of E_0 into the product space $(l_\infty)^\Gamma$ for a suitable set Γ. By the completeness of E_0, if $(x_n)_{n\in\mathbb{N}}$ is not relatively weakly compact, then there must exist an l_∞ factor such that the projection of $(Sx_n)_{n\in\mathbb{N}}$ on that factor is not relatively weakly compact. It follows that there exists an operator $U : E_0 \to l_\infty$ such that $(z_n)_{n\in\mathbb{N}} \notin \kappa(l_\infty, w)$, where $z_n := Ux_n$. By the injectivity of l_∞, we can extend U onto the whole E. Define an operator $W : \kappa(E, w) \to \kappa(l_\infty, w)$ by

$$W((y_n)_{n\in\mathbb{N}}) := (Uy_n)_{n\in\mathbb{N}}.$$

Then for the operator $W \circ T : l_\infty \to \kappa(l_\infty, w)$ we have $W \circ T(e_n) = \widehat{z}_n$ for all $n \in \mathbb{N}$. Hence, by the first part of the proof, $(z_n)_{n\in\mathbb{N}} \in \kappa(l_\infty, w)$; a contradiction. ∎

THEOREM 4. *Let X be a nondiscrete LCH-space, and let E be a Fréchet space. Then $C(X, E, w)$ is complemented in a dual Fréchet space if and only if $C(X)$ is injective and E is reflexive.*

Proof. *Necessity.* As $C(X)$ is isomorphic to a complemented subspace of $C(X, E, w)$, it must be injective by Fact (F). Suppose that E is not reflexive so that there is a bounded closed convex set $B \subset E$ which is not weakly compact. Hence, by [21, IV.11.2], there is a sequence $(x_n)_{n\in\mathbb{N}} \subset B$ without weak cluster points. In particular, (x_n) is not relatively weakly compact.

By assumption, X contains an infinite compact set K, hence there is a sequence of continuous functions $\varphi_n : X \to [0,1]$ with disjoint supports and satisfying $\varphi_n(t_n) = 1$ for some $t_n \in K$ $(n \in \mathbb{N})$. We define operators

$$S : c_0 \to C(X, E, w) \quad \text{and} \quad Q : C(X, E, w) \to \kappa(E, w)$$

by

$$S((a_n)_{n\in\mathbb{N}})(t) = \sum_{n=0}^{\infty} a_n x_n \varphi_n(t) \quad \text{and} \quad Q(f) = (f(t_n))_{n\in\mathbb{N}}.$$

The assumption that $C(X, E, w)$ is complemented in a dual Fréchet space and Fact (B) imply that there exists a projection

$$P : C(X, E, w)^\wedge \underset{\text{onto}}{\longrightarrow} C(X, E, w).$$

Now, consider the operator $T := Q \circ P \circ \widehat{S} : l_\infty \to \kappa(E, w)$, where $\widehat{S} : \widehat{c}_0 = l_\infty \to C(X, E, w)^\wedge$ is the extension of S provided by Fact (C). Then, as is easily seen, $T(e_n) = \widehat{x}_n$ for every $n \in \mathbb{N}$; a contradiction in view of Lemma 3.

Sufficiency. Let $C(X)$ be injective and E reflexive. Then $C(X, E, w) = C(X, E'', w')$ and Theorem 3 applies, since E' is barreled and bornological [21, Cor. 1 to IV.5.6 and Cor. 1 to IV.6.6]. ∎

Remark. From the above proof it is clear that the necessity part of Theorem 4 is true for all complete lcs E (see the note at the end of §2). In that case, if $C(X, E, w)$ is complemented in a strong dual of some bornological space, then $C(X)$ is injective and E is semireflexive.

Finally, we can treat the case of $C(X, E)$.

THEOREM 5. *Let X be a nondiscrete LCH-space, and let E be a Fréchet space. Then $C(X, E)$ is isomorphic to a complemented subspace of a dual Fréchet space if and only if $C(X)$ is injective and E is a Fréchet–Montel space.*

Proof. In view of Theorem 1 and Facts (D) and (F), the necessity is obvious.

Sufficiency. Assume that E is a Fréchet–Montel space. Then, since X is a k-space (see [13]) and since weakly compact sets in E are strongly compact, we have $C(X, E) = C(X, E, w)$. Finally, as E is reflexive, we conclude by applying Theorem 4. ∎

5. Injectivity of $C(X, E)$, $C(X, E, w)$ and $C(X, E', w')$. We show here that the injective spaces of the first two types coincide and are, in a sense, trivial, while the injective spaces of the third type exist in much greater variety.

THEOREM 6. *Let X be a nondiscrete LCH-space, and let E be an infinite-dimensional Fréchet space. Then the statements* (1), (2) *and* (3) *below are mutually equivalent.*

(1) *$C(X, E)$ is injective.*

(2) *$C(X, E, w)$ is injective.*

(3) *E is isomorphic to ω and $C(X)$ is injective.*

Proof. It is easy to see that (3) implies both (1) and (2): Indeed, it is obvious that if (3) holds, then $C(X,E) = C(X,E,w) \simeq C(X)^{\mathbb{N}}$, with $C(X)$ injective. Hence $C(X,E)$ itself is injective.

(1)⇒(3). Assume that $C(X,E)$ is injective. Then, since $C(X)$ and E are isomorphic to complemented subspaces in $C(X,E)$, both those spaces are injective, too. On the other hand, by Fact (E), $C(X,E)$ contains no complemented copy of c_0. Hence, by Theorem 1, E must be a Montel space. Now, it is known that ω is the only injective infinite-dimensional Fréchet–Montel space (see [17] or [11]), hence $E \simeq \omega$. (Alternatively, by Fact (A), E is a quojection and, by the weaker version of Theorem 1 mentioned in Remark (b) in §3, $E \simeq \omega$.) Thus condition (3) is satisfied.

(2)⇒(3). As above, assuming that $C(X,E,w)$ is injective, we deduce that both $C(X)$ and E are injective. Moreover, since $C(X,E,w)$ is injective, it is complemented in a dual Fréchet space so that, by Theorem 4, E must be reflexive. But the infinite-dimensional Fréchet space E can be both injective and reflexive only if $E \simeq \omega$ (see [17] or [11]), and so (3) is satisfied. ■

THEOREM 7. *Assume that X is an LCH-space, and that a locally convex space E is barreled and its strong dual E' is a Fréchet space. Then $C(X,E',w')$ is injective if and only if both $C(X)$ and E' are injective.*

Proof. Since $C(X)$ and E' are complemented subspaces of $C(X,E',w')$, the necessity part is obvious.

Sufficiency. As in §2 we see that $C(X,E',w') \simeq L_\beta(E,C(X))$. Now, by [17, Prop. 3.18], if $C(X)$ and E' are injective, so is $L_\beta(E,C(X)) \simeq C(X,E',w')$.

References

[1] S. F. Bellenot and E. Dubinsky, *Fréchet spaces with nuclear Köthe quotients*, Trans. Amer. Math. Soc. 273 (1982), 579–591.

[2] J. Bonet, M. Lindström and M. Valdivia, *Two theorems of Josefson–Nissenzweig type for Fréchet spaces*, preprint, 1991.

[3] M. Cambern and P. Greim, *The bidual of $C(X,E)$*, Proc. Amer. Math. Soc. 85 (1982), 53–58.

[4] —, —, *The dual of a space of vector measures*, Math. Z. 180 (1982), 373–378.

[5] M. Cambern and P. Greim, *Uniqueness of preduals for spaces of continuous vector functions*, Canad. Math. Bull. 31 (1988), 98–103.

[6] P. Cembranos, *$C(K,E)$ contains a complemented copy of c_0*, Proc. Amer. Math. Soc. 91 (1984), 556–558.

[7] S. Dierolf and D. N. Zarnadze, *A note on strictly regular Fréchet spaces*, Arch. Math. (Basel) 42 (1984), 549–556.

[8] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, New York 1984.

[9] P. Domański, *$\mathcal{L}_p$-Spaces and injective locally convex spaces*, Dissertationes Math. 298 (1990).

[10] P. Domański and L. Drewnowski, *Uncomplementability of the spaces of norm continuous functions in some spaces of "weakly" continuous functions*, Studia Math. 97 (1991), 245–251.

[11] P. Domański and A. Ortyński, *Complemented subspaces of products of Banach spaces*, Trans. Amer. Math. Soc. 316 (1989), 215–231.

[12] G. Emmanuele, *A dual characterization of Banach spaces not containing l_1*, Bull. Polish Acad. Sci. Math. 34 (1986), 155–159.

[13] R. Engelking, *General Topology*, Monograf. Mat. 60, PWN, Warszawa 1977.

[14] F. J. Freniche, *Barrelledness of the space of vector-valued and simple functions*, Math. Ann. 267 (1984), 479–486.

[15] H. Jarchow, *Locally Convex Spaces*, Birkhäuser, Stuttgart 1980.

[16] N. J. Kalton, *Spaces of compact operators*, Math. Ann. 208 (1974), 267–278.

[17] G. Metafune and V. B. Moscatelli, *Complemented subspaces of sums and products of Banach spaces*, Ann. Mat. Pura Appl. 159 (1988), 175–190.

[18] —, —, *Quojections and prequojections*, in: Proc. NATO-ASI Workshop on Fréchet spaces, Istanbul, August 1988, T. Terzioğlu (ed.), Kluwer, Dordrecht 1989, 235–254.

[19] A. Pełczyński, *Some aspects of the present theory of Banach spaces*, in: S. Banach, *Oeuvres*, Vol. II, PWN, Warszawa 1979, 218–302.

[20] P. Pérez Carreras and J. Bonet, *Barrelled Locally Convex Spaces*, North-Holland Math. Stud. 131, Elsevier/North-Holland, Amsterdam 1987.

[21] H. H. Schaefer, *Topological Vector Spaces*, Springer, Berlin 1973.

INSTITUTE OF MATHEMATICS
A. MICKIEWICZ UNIVERSITY
MATEJKI 48/49
60-769 POZNAŃ, POLAND

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